Grading: 1, 2, 3, 4, 5 (each 8 pts).

Problem 1: a) Suppose $X_n \to_p X$, $X_n \geq 0$. First, note that if $\liminf_{n\to\infty} E(X_n) = \infty$, we are done since the inequality will obviously hold, so assume $\liminf_{n\to\infty} E(X_n) < \infty$. By the definition of liminf, there exists a subsequence $\{X_{n_m}\}$ with $\lim_{m\to\infty} E(X_{n_m}) = \liminf_{n\to\infty} E(X_n)$. By Theorem 8.1 in the lecture notes, there exists a subsubsequence $\{X_{n_{m(k)}}\}$ with $X_{n_{m(k)}} \to X$ a.s. So, by the original Fatou lemma, $E(X) = E(\liminf_{k\to\infty} X_{n_{m(k)}}) \leq \liminf_{k\to\infty} E(X_{n_{m(k)}}) = \liminf_{n\to\infty} E(X_n)$, as desired. **b)** Say $X_n \to_p X$, $|X_i| \leq Y$ with $E(Y) < \infty$. Let $\{X_{n_m}\}$ be any subsequence of X_n. By Theorem 8.1 in the lecture notes, there exists a subsubsequence $\{X_{n_{m(k)}}\}$ with $X_{n_{m(k)}} \to X$ a.s., so by the original DCT, $E(X_{n_{m(k)}}) \to E(X)$. By Lemma 8.2 in the lecture notes with $y_n = E(X_n)$, we have $E(X_n) \to E(X)$, as desired.

Problem 2: For convenience suppose the sequence starts at $n = 2$. Let $c_n = \log n$, $\beta = 1/\alpha$. Note that

$$P(\frac{\log X_n}{\log n} \geq 1/\alpha) = P(X_n > e^{\log n/\alpha}) = (n^{1/\alpha})^{-\alpha} = \frac{1}{n},$$

so that

$$\sum_{n=2}^{\infty} P(\frac{\log X_n}{\log n} \geq 1/\alpha) = \sum_{n=2}^{\infty} \frac{1}{n},$$

which diverges, so that (using independence) by the second Borel-Cantelli Lemma, $P(\frac{\log X_n}{\log n} \geq 1/\alpha \text{ i.o.}) = 1$, and in particular $\limsup_{n\to\infty} \frac{\log X_n}{\log n} \geq 1/\alpha$ a.s. Also, given $\epsilon > 0$,

$$P(\frac{\log X_n}{\log n} \geq (1/\alpha)(1+\epsilon)) = P(X_n > e^{\frac{1}{\alpha}(1+\epsilon)\log n})$$

$$= (e^{\frac{1}{\alpha}(1+\epsilon)\log n})^{-\alpha} = \frac{1}{n^{1+\epsilon}},$$

so that

$$\sum_{n=2}^{\infty} P(\frac{\log X_n}{\log n} \geq (1/\alpha)(1+\epsilon)) = \sum_{n=2}^{\infty} \frac{1}{n^{1+\epsilon}} < \infty,$$

so that by the first Borel-Cantelli Lemma

$$P(\frac{\log X_n}{\log n} \geq (1/\alpha)(1+\epsilon) \text{ i.o.}) = 0.$$

Because of the arbitrary choice of ϵ, we have

$$\limsup_{n\to\infty} \frac{\log X_n}{\log n} \leq 1/\alpha$$

a.s., and combining this with the previous inequality, we have as desired.

[1] Contributed by Ilya Amburg

Problem 3: a) Claim: $X_n \to_p 0$ iff $p_n \to 0$. For any $0 < \epsilon < 1$, $P(|X_n| > \epsilon) = p_n$. If $X_n \to_p 0$ then clearly $p_n \to 0$, and if $p_n \to 0$ then $P(|X_n| > \epsilon) \to 0$ for all $0 < \epsilon < 1$ (and trivially for all $\epsilon \geq 1$).

b) Claim: $X_n \to 0$ a.s. iff $\sum_{n=1}^{\infty} p_n < \infty$. For the forward implication, suppose $\sum_{n=1}^{\infty} p_n = \infty$. Then by the second Borel-Cantelli Lemma, $P(X_n = 1 \text{ i.o.}) = 1$, so that $X_n \not\to 0$ a.s. For the reverse implication, suppose that $\sum_{n=1}^{\infty} p_n < \infty$. By the first Borel-Cantelli Lemma, $P(X_n = 1 \text{ i.o.}) = 0$, so that $X_n \to 0$ a.s.

Problem 4: (i) Since $P(A_n) \to 0$, we have $P(\cup_{m=n}^{\infty} A_m^c) \geq P(A_n^c) \to 1$ and so $P(A_n^c \text{ i.o.}) = P(\cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m^c) = \lim_{n\to\infty} P(\cup_{m=n}^{\infty} A_m^c) = 1$. Note that if $\omega \in \{A_n^c \text{ i.o}\}$ and $\omega \in \{A_n \text{ i.o.}\}$, then the sequence $\{\mathbb{1}_{A_n}(\omega)\}_{n=1}^{\infty}$ contains infinitely many instances of 0 followed by 1, so that $\omega \in \{A_n^c \cap A_{n+1} \text{ i.o.}\}$. From this, it follows that $\{A_n^c \text{ i.o.}\} \cap \{A_n \text{ i.o}\} \subseteq \{A_n^c \cap A_{n+1} \text{ i.o.}\}$, so that since $P(A_n^c \text{ i.o.}) = 1$, $P(A_n \text{ i.o.}) = P(\{A_n^c \text{ i.o.}\} \cap \{A_n \text{ i.o.}\}) \leq P(A_n^c \cap A_{n+1} \text{ i.o.})$. But by the first Borel-Cantelli Lemma, $P(A_n^c \cap A_{n+1} \text{ i.o.}) = 0$, so that $P(A_n \text{ i.o.}) = 0$, as desired.

(ii) Let $\Omega = (0, 1)$ and P be Lebesgue measure, and set $A_n = (0, 1/n)$. Then $\sum_{n=1}^{\infty} P(A_n) = \infty$ so the first Borel-Cantelli Lemma does not apply. However, each $P(A_n^c \cap A_{n+1}) = P([1/n, 1) \cap (0, 1/(n+1))) = 0$. Thus the result of (i) implies that $P(A_n \text{ i.o.}) = 0$.

Problem 5: We show that for all $c > 0$, $P(X_n/(n \log_2 n) \geq c \text{ i.o.}) = 1$. It follows that $\limsup_{n\to\infty} X_n/(n \log_2 n) \geq c$ a.s. for all c, hence the limsup equals ∞ a.s. For all $r > 0$, $P(X_n \geq 2^r) = P(X_n \geq 2^{\lceil r \rceil}) = \sum_{j=\lceil r \rceil}^{\infty} 2^{-j} = 2^{-\lceil r \rceil + 1} \geq 2^{-r}$. In other words, $P(X_n \geq a) \geq 1/a$ for all $a > 1$. Thus if $c > 0$ is fixed,

$$\sum_{n=2}^{\infty} P(\frac{X_n}{n \log_2 n} \geq c) = \sum_{n=2}^{\infty} P(X_n \geq cn \log_2 n) \geq \sum_{n=2}^{\infty} \frac{1}{cn \log_2 n} = \infty$$

and so $P(X_n/(n \log_2 n) \geq c \text{ i.o.}) = 1$ by the second Borel-Cantelli Lemma. We conclude that $\limsup_{n\to\infty} X_n/(n \log_2 n) = \infty$ a.s. and also $\limsup_{n\to\infty} S_n/(n \log_2 n) = \infty$ a.s.