Math 6710, Fall 2016
Homework 8 Extra Credit Solution

Extra credit (hard). Let $\mathcal{T}$ be the tail σ-field associated with the random variables $X_1, X_2, \ldots$, which are all defined on the sample space Ω. Suppose $A \in \sigma(X_1, X_2, \ldots)$ satisfies the following condition: For all $\omega, \omega' \in \Omega$ such that the sequences $(X_1(\omega), X_2(\omega), \ldots)$ and $(X_1(\omega'), X_2(\omega'), \ldots)$ differ in only finitely many places, either $\omega, \omega' \in A$ or $\omega, \omega' \notin A$. Is it necessarily true that $A \in \mathcal{T}$?

You may consider the following easier (but still hard!) question. Suppose $A \in \sigma(X_1, X_2)$ satisfies the following condition: For all $\omega, \omega' \in \Omega$ such that $X_2(\omega) = X_2(\omega')$, either $\omega, \omega' \in A$ or $\omega, \omega' \notin A$. Is it necessarily true that $A \in \sigma(X_2)$? Keep in mind that the image of a measurable set under a measurable function is not necessarily measurable.

Solution: The easier question is false. Here is a counterexample. Let $\Omega = [0, 1]$ and $\mathcal{F}$ be the collection of Lebesgue-measurable subsets of Ω. Importantly, there are Lebesgue-measurable sets that are not Borel sets. Let $A \in \mathcal{F}$ be one such set. Define $X_1, X_2 : \Omega \to \mathbf{R}$ by $X_1(\omega) = \mathbf{1}_A(\omega)$ and $X_2(\omega) = \omega$. Both X_1 and X_2 are measurable functions from $(\Omega, \mathcal{F})$ to $(\mathbf{R}, \mathcal{B})$. In the case of X_1, the preimage of any Borel set is one of $\varnothing, \Omega, A, A^C$. In the case of X_2, the preimage of any Borel set is its intersection with $[0, 1]$, which is also a Borel set and therefore in $\mathcal{F}$. Since $A = X_1^{-1}(\{1\})$, we have $A \in \sigma(X_1) \subseteq \sigma(X_1, X_2)$. In addition, the condition "For all $\omega, \omega' \in \Omega$ such that $X_2(\omega) = X_2(\omega')$, either $\omega, \omega' \in A$ or $\omega, \omega' \notin A$" is trivially true because the function X_2 is injective. However, $\sigma(X_2) = \{X_2^{-1}(B) : B \in \mathcal{B}\}$ is the collection of Borel subsets of $[0, 1]$, so $A \notin \sigma(X_2)$.

It follows from this construction that the question about $\mathcal{T}$ is also false. Set $X_n = X_2$ for all $n \geq 3$. The same set A as above satisfies the required condition, since the only way that $(X_1(\omega), X_2(\omega), \ldots)$ and $(X_1(\omega'), X_2(\omega'), \ldots)$ can differ in finitely many places is when $\omega = \omega'$. Also, $A \in \sigma(X_1) \subseteq \sigma(X_1, X_2, \ldots)$. For $n \geq 2$, each $\sigma(X_n)$ is the collection of Borel subsets of $[0, 1]$, so $\sigma(X_n, X_{n+1}, \ldots)$ is the same collection of Borel sets. It follows that $\mathcal{T}$ is also the Borel subsets of $[0, 1]$. In particular, $A \notin \mathcal{T}$.