Grading: 1, 2, 3, 4, 5 (each 8 pts).

Problem 1: Let the random variables X_n be iid with $P(X_n = 1) = p_n$, $P(X_n = 0) = 1 - p_n$, with p_n to be determined later. By the result of problem 3 in HW 7, we know that $p_n \to_p 0$ iff $p_n \to 0$, while by the second Borel-Cantelli lemma with $A_n = \{X_n = 1\}$, if $\sum_{n=1}^{\infty} p_n = \infty$, then $P(X_n = 1 \text{ i.o.}) = 1$. Choose $p_n = \frac{1}{n}$. Then clearly $p_n \to 0$, but $\sum_{n=1}^{\infty} p_n = \infty$, so that with probability 1, $X_n = 1$ i.o. and there is a subsequence $N(n)$ such that each $X_{N(n)} = 1$. (On the probability 0 event that $X_n = 1$ only finitely often, $N(n)$ can be defined however we like.)

Problem 2: The result will follow from the strong law of large numbers and the result of a previous HW problem. Using the argument in the solutions to HW 6 problem 1, we can assume WLOG that $E(X_i) = \mu = 0$. Further, by the argument in that same solution, we know that

$$\frac{1}{n}\sum_{k=1}^{n}(X_k - \overline{X}_n)^2 = \left(\frac{1}{n}\sum_{j=1}^{n} X_k^2\right) - \overline{X}_n^2.$$

Since $E(|X_1^2|) = \sigma^2$ is finite, by the strong law of large numbers (Theorem 2.5.6 in Durrett), $\frac{1}{n}\sum_{j=1}^{n} X_k^2 \to \sigma^2$ a.s; further, since we have $\mu = 0$, again by the strong law we have that $\overline{X}_n \to 0$ a.s. Since the function $f(y) = -y^2$ is continuous, because $\overline{X}_n \to 0$ a.s., by the result of Exercise 1.3.3 in Durrett, which has been done for HW, we also have $-\overline{X}_n^2 \to 0$ a.s. Hence, $\frac{1}{n}\sum_{k=1}^{n}(X_k - \overline{X}_n)^2 \to \sigma^2$ a.s., so that (since $\frac{n-1}{n} \to 1$) also $\overline{V}_n = \frac{1}{n-1}\sum_{k=1}^{n}(X_k - \overline{X}_n)^2 \to \sigma^2$ a.s., as desired.

Problem 3: First we check that $Z = \{Y^{-1}(B) : B \in \mathcal{B}^{\mathbb{N}}\}$ is a $\sigma-$algebra. This was shown in class but just in case, here is the argument: **a)** $Y^{-1}(\mathbb{R}^{\mathbb{N}}) = \Omega$, so $\Omega \in Z$. **b)** If $A = Y^{-1}(B) \in Z$ for some $B \in \mathcal{B}^{\mathbb{N}}$, then $A^c = Y^{-1}(B^c) \in Z$. **c)** Say $\{A_i\}_{i=1}^{\infty} \subset Z$. Write $A_i = Y^{-1}(B_i)$ for $B_i \in \mathcal{B}^{\mathbb{N}}$. Then $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} Y^{-1}(B_i) = Y^{-1}(\cup_{i=1}^{\infty} B_i) \in Z$.
Let $\pi_n : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be the nth coordinate map and let $\mathcal{A} = \{\pi_n^{-1}(B) : n \in \mathbb{N}, B \in \mathcal{B}\} \subseteq \mathcal{B}^{\mathbb{N}}$. Elements of $\mathcal{A}$ have the form $\mathbb{R}\times\mathbb{R}\times\cdots\times B\times\mathbb{R}\times\cdots$. We have $Y^{-1}(\mathcal{A}) = \cup_{n=1}^{\infty}\{Y_n^{-1}(B) : B \in \mathcal{B}\} = \cup_{n=1}^{\infty}\sigma(Y_n)$. Clearly $Y^{-1}(\mathcal{A}) \subseteq Y^{-1}(\mathcal{B}^{\mathbb{N}}) = Z$. Also, since $\mathcal{A}$ generates $\mathcal{B}^{\mathbb{N}}$, Proposition 4.1 in the notes (a.k.a. HW 2 problem 6) implies that $Y^{-1}(\mathcal{A})$ generates Z, that is, $\sigma(\cup_{n=1}^{\infty}\sigma(Y_n)) = Z$.

Problem 4: Let A_j be the event that $|S_{m,j}| > 2a$ while $|S_{m,i}| \le 2a$ for all $m < i < j$. Then

$$P\left(\max_{m<j\le n}|S_{m,j}| > 2a\right) = P\left(\bigcup_{j=m+1}^{n} A_j\right) \le \sum_{j=m+1}^{n} P(A_j).$$

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It follows that

$$P\left(\max_{m<j\le n}|S_{m,j}|>2a\right)\min_{m<k\le n}P(|S_{k,n}|\le a)\le\sum_{j=m+1}^{n}P(A_j)P(|S_{j,n}|\le a).$$

Since $A_j\in\sigma(X_{m+1},\dots,X_j)$ and $\{|S_{j,n}|\le a\}\in\sigma(X_{j+1},\dots,X_n)$ are independent,

$$P(A_j)P(|S_{j,n}|\le a)=P(A_j\text{ and }|S_{j,n}|\le a)\le P(A_j\text{ and }|S_{m,n}|>a),$$

where the last inequality is because $|S_{m,j}|>2a$ and $|S_{j,n}|\le a$ together imply that $|S_{m,n}|>a$. Thus,

$$P\left(\max_{m<j\le n}|S_{m,j}|>2a\right)\min_{m<k\le n}P(|S_{k,n}|\le a)\le\sum_{j=m+1}^{n}P(A_j\text{ and }|S_{m,n}|>a)\le P(|S_{m,n}|>a),$$

using that the A_j are disjoint.

Problem 5: Say $S_n\to_p Z$. For any $a>0$ and integer $m>0$, by the result of Problem 4, for all $n>m$,

$$P(\max_{m<j\le n}|S_j-S_m|>2a)\le\frac{P(|S_n-S_m|>a)}{\min_{m<k\le n}P(|S_n-S_k|\le a)}.$$

Since $P(|S_n-S_m|>a)\le P(|S_n-Z|>a/2)+P(|S_m-Z|>a/2)$, we also have that

$$\limsup_{n\to\infty}P(|S_n-S_m|>a)\le\limsup_{n\to\infty}P(|S_n-Z|>a/2)+P(|S_m-Z|>a/2)$$

$$=P(|S_m-Z|>a/2),$$

where we have used $S_n\to_p Z$. Further,

$$\min_{m<k\le n}P(|S_n-S_k|\le a)=1-\max_{m<k\le n}P(|S_n-S_k|>a)$$

$$\ge 1-\max_{m<k\le n}[P(|S_k-Z|>a/2)+P(|S_n-Z|>a/2)]$$

$$\ge 1-2\sup_{l>m}P(|S_l-Z|>a/2).$$

The sequence of events $B_n=\{\max_{m<j\le n}|S_j-S_m|>2a\}$ is increasing in n and its union is $\{\sup_{j>m}|S_j-S_m|>2a\}$, which means that

$$P(\sup_{j>m}|S_j-S_m|>2a)=\lim_{n\to\infty}P(\max_{m<j\le n}|S_j-S_m|>2a)\le\frac{P(|S_m-Z|>a/2)}{1-2\sup_{l>m}P(|S_l-Z|>a/2)}.$$

Since $S_m \to_p Z$, the sequences $P(|S_m - Z| > a/2)$ and $\sup_{l>m} P(|S_l - Z| > a/2)$ converge to 0 as $m \to \infty$, so

$$\lim_{m\to\infty} P(\sup_{j>m} |S_j - S_m| > 2a) = 0.$$

Now note that we have

$$P(\sup_{j,k>m} |S_j - S_k| > 4a) \le P(\sup_{j>m} |S_j - S_m| > 2a),$$

so that $W_m = \sup_{j,k>m} |S_j - S_k| \to 0$ in probability. By the argument used to prove Theorem 2.5.3 in Durrett, $W_m \to 0$ a.s., so that $\{S_n\}$ is Cauchy a.s. and hence S_n must converge almost surely.