Math 6710, Fall 2016
Homework 8

1. Durrett Exercise 2.4.1.

2. Returning to the setup of Question 1 on Homework 6, show that $\overline{V}_n \to \sigma^2$ almost surely. (This makes $\overline{V}_n$ a "strongly consistent estimator" for σ^2.)

3. Let $Y_1, Y_2, \ldots$ be random variables defined on the same sample space Ω. Consider the space $\mathbf{R}^{\mathbf{N}} = \{(y_1, y_2, \ldots) : \text{each } y_n \in \mathbf{R}\}$ with the Borel σ-algebra $\mathcal{B}^{\mathbf{N}}$. For $B \in \mathcal{B}^{\mathbf{N}}$, let $Y^{-1}(B) = \{\omega \in \Omega : (Y_1(\omega), Y_2(\omega), \ldots) \in B\}$. Show that

$$\sigma\left(\bigcup_{n=1}^{\infty} \sigma(Y_n)\right) = \{Y^{-1}(B) : B \in \mathcal{B}^{\mathbf{N}}\}.$$

4. Durrett Exercise 2.5.9. *Hint:* Let A_j be the event that $|S_{m,j}| > 2a$ while $|S_{m,i}| \leq 2a$ for all $m < i < j$. Use this to decompose the left side.

5. Durrett Exercise 2.5.10. *Hint:* Show that $W_M = \sup_{M \leq m < n} |S_{m,n}| \to 0$ in probability as $M \to \infty$.

Extra credit (hard). Let $\mathcal{T}$ be the tail σ-field associated with the random variables $X_1, X_2, \ldots$, which are all defined on the sample space Ω. Suppose $A \in \sigma(X_1, X_2, \ldots)$ satisfies the following condition: For all $\omega, \omega' \in \Omega$ such that the sequences $(X_1(\omega), X_2(\omega), \ldots)$ and $(X_1(\omega'), X_2(\omega'), \ldots)$ differ in only finitely many places, either $\omega, \omega' \in A$ or $\omega, \omega' \notin A$. Is it necessarily true that $A \in \mathcal{T}$?

You may consider the following easier (but still hard!) question. Suppose $A \in \sigma(X_1, X_2)$ satisfies the following condition: For all $\omega, \omega' \in \Omega$ such that $X_2(\omega) = X_2(\omega')$, either $\omega, \omega' \in A$ or $\omega, \omega' \notin A$. Is it necessarily true that $A \in \sigma(X_2)$? Keep in mind that the image of a measurable set under a measurable function is not necessarily measurable.