Grading: 1, 3, 4, 5 (each 8 pts) 2 and 6 (each 4 pts).

Problem 1: **i)** Since $\{X_n\}_{n=1}^{\infty}$ are independent, clearly also $\{\frac{X_n}{n}\}_{n=1}^{\infty}$ are independent, and of course $E(X_n/n) = 0$. Note that $\mathrm{Var}(\frac{X_n}{n}) = \frac{1}{n^2}\mathrm{Var}(X_n) = \frac{\sigma_n^2}{n^2}$, and $\sum_{n=1}^{\infty}\mathrm{Var}(\frac{X_n}{n}) = \sum_{n=1}^{\infty}\frac{\sigma_n^2}{n^2} < \infty$, so that by Theorem 2.5.3 in Durrett, $\sum_{n=1}^{\infty}\frac{X_n}{n}$ converges a.s., and hence by Theorem 10.5 in the lecture notes, $\frac{1}{n}\sum_{m=1}^{n} X_m \to 0$ a.s. **ii)** Let $p_n = \sigma_n^2/n^2 \leq 1$ and define each X_n to equal $\pm n$ with probability $p_n/2$ each and to equal 0 with probability $1-p_n$. Then $E(X_n) = 0$ and $\mathrm{Var}(X_n) = p_n n^2 = \sigma_n^2$. Each $X_n/n \in \{-1, 0, 1\}$. Since $\sum_{n=1}^{\infty} p_n = \infty$, by the second Borel-Cantelli lemma $P(|X_n/n| = 1 \text{ i.o.}) = 1$. Thus, neither X_n/n nor $(X_1 + \cdots + X_n)/n$ can converge to 0 a.s.

Problem 2: For integer $n > 0$, let X_n have density $f_n(x) = (1-\cos(2\pi nx))\mathbb{1}_{(0,1)}(x)$. Clearly since for $x \in (0,1)$ we have $\int_{-\infty}^{x} f_n(y)dy = x - \frac{1}{2\pi n}\sin(2\pi nx)$, $X_n \Rightarrow$ uniform distribution on $(0,1)$. Further, $f_n(x)$ does not converge for any $x \in [0,1]$. (If you are not happy with the work required to prove the last sentence rigorously, define instead $f_n(x) = 0$ when $\lfloor 2^n x\rfloor$ is even and $f_n(x) = 2$ when $\lfloor 2^n x\rfloor$ is odd.)

Problem 3: **i)** We have

$$P(\frac{M_n}{n^{1/\alpha}} \leq y) = P(M_n \leq yn^{1/\alpha}) = F(yn^{1/\alpha})^n = (1-(yn^{1/\alpha})^{-\alpha})^n = (1-\frac{y^{-\alpha}}{n})^n.$$

Note that we have $-\frac{y^{-\alpha}}{n} \to 0$, $n\left(-\frac{y^{-\alpha}}{n}\right) = -y^{-\alpha} = \lambda$, $n \to \infty$, so that by fact 11.1 in the lecture notes, $(1-\frac{y^{-\alpha}}{n})^n \to \exp(\lambda) = \exp(-y^{-\alpha})$, as desired.

ii) We have

$$P(n^{1/\beta}M_n \leq y) = P(M_n \leq y/n^{1/\beta}) = F(y/n^{1/\beta})^n = (1-|y/n^{1/\beta}|^{\beta})^n.$$

Clearly, $-|y/n^{1/\beta}|^{\beta} \to 0$, $n\left(-|y/n^{1/\beta}|^{\beta}\right) = -|y|^{\beta} = \lambda$, $n \to \infty$, so that by fact 11.1 in the lecture notes, $(1-|y/n^{1/\beta}|^{\beta})^n \to \exp(\lambda) = \exp(-|y|^{\beta})$, as desired.

iii) We have

$$P(M_n - \log n \leq y) = P(M_n \leq y + \log n) = F(y+\log n)^n = (1-e^{-(y+\log n)})^n = (1-\frac{1}{n}e^{-y})^n.$$

Clearly, $-\frac{1}{n}e^{-y} \to 0$, $n\left(-\frac{1}{n}e^{-y}\right) = -e^{-y} = \lambda$, $n \to \infty$, so that again by fact 11.1 in the lecture notes, $(1-\frac{1}{n}e^{-y})^n \to \exp(\lambda) = \exp(-e^{-y})$, as desired.

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Problem 4: Let F_n, $1 \leq n \leq \infty$ be the distribution function of the associated X_n. By Theorem 3.2.2 in Durrett, there exist random variables Y_n, $1 \leq n \leq \infty$ having associated distributions F_n with $Y_n \to Y_\infty$ a.s. Since g is continuous, by Exercise 1.3.3 in Durrett (which we have done) also $g(Y_n) \to g(Y_\infty)$ a.s., and hence $g(Y_n) \to g(Y_\infty)$ in probability, so that by Exercise 2.3.6 in Durrett (which we have also done) $\liminf_{n\to\infty} Eg(Y_n) \geq Eg(Y_\infty)$. Since each $X_n =_d Y_n$, also $g(X_n) =_d g(Y_n)$ and so $Eg(X_n) = Eg(Y_n)$ by the change of variables formula, Theorem 1.6.9 in Durrett. Hence $\liminf_{n\to\infty} Eg(X_n) \geq Eg(X_\infty)$, as desired.

For the example of strict inequality, let $g(x) = |x|$ and define X_n by $P(X_n = 0) = 1 - 1/n$, $P(X_n = n) = 1/n$. Each $Eg(X_n) = E[X_n] = 1$, so $\liminf_{n\to\infty} Eg(X_n) = 1$. Meanwhile, if F_n is the distribution function of X_n, then for $x < 0$ each $F_n(x) = 0$ while for $x \geq 0$ each $F_n(x) = 1 - 1/n$ for sufficiently large n. Thus $\lim_{n\to\infty} F_n(x) = 0$ for $x < 0$ and $= 1$ for $x \geq 0$. This means $X_n \Rightarrow X_\infty$ where $P(X_\infty = 0) = 1$, and $Eg(X_\infty) = 0$.

Problem 5: We show that ρ is a metric first. We need **1)** $\rho(F,G) \geq 0$, **2)** $\rho(F,G) = 0$ iff $F = G$, **3)** $\rho(F,G) = \rho(G,F)$, **4)** $\rho(F,H) \leq \rho(F,G) + \rho(G,H)$. To see this, we have

1) $\rho(F,G) \geq 0$ since by definition the infimum is taken over $\epsilon > 0$. (Or, sending $x \to \infty$ shows that the property cannot hold for negative ϵ.)
2) If $F = G$, then by monotonicity of F, $F(x-\epsilon) - \epsilon \leq F(x) \leq F(x+\epsilon) + \epsilon$ for all $\epsilon > 0$, hence $\rho(F,G) = 0$. Conversely, if $\rho(F,G) = 0$ then for any $\epsilon > 0$, $F(x-\epsilon) - \epsilon \leq G(x) \leq F(x+\epsilon) + \epsilon$. Sending $\epsilon \to 0$ we obtain $F(x^-) \leq G(x) \leq F(x^+)$, so $F = G$ at all continuity points of F. If F is discontinuous at x then let $x_n \searrow x$ be a sequence of continuity points of F, so that $G(x^+) = \lim_{n\to\infty} G(x_n) = \lim_{n\to\infty} F(x_n) = F(x^+)$. Since both F and G are distribution functions it follows that $G(x) = G(x^+) = F(x^+) = F(x)$.
3) If $F(x-\epsilon) - \epsilon \leq G(x) \leq F(x+\epsilon) + \epsilon$ for all real x, then $F(x) - \epsilon \leq G(x+\epsilon)$ and $G(x-\epsilon) \leq F(x) + \epsilon$, which implies that $G(x-\epsilon) - \epsilon \leq F(x) \leq G(x+\epsilon) + \epsilon$ for all x. It follows that $\rho(G,F) \leq \rho(F,G)$, and the converse follows by symmetry.
4) To show that $\rho(F,H) \leq \rho(F,G) + \rho(G,H)$ it suffices to check that $\rho(F,H) \leq \rho(F,G) + \rho(G,H) + \delta$ for all $\delta > 0$. Set $a = \rho(F,G) + \delta/2$ and $b = \rho(G,H) + \delta/2$. For all $x \in \mathbb{R}$, $F(x-a) - a \leq G(x) \leq F(x+a) + a$ and $G(x-b) - b \leq H(x) \leq G(x+b) + b$. Hence $F(x-(a+b)) - (a+b) \leq G(x-b) - b \leq H(x) \leq G(x+b) + b \leq F(x+(a+b)) + (a+b)$, so $\rho(F,H) \leq a + b$.

Now we show that $F_n \Rightarrow F$ iff $\rho(F_n, F) \to 0$. If $\rho(F_n, F) \to 0$ then for all $\epsilon > 0$ and all $x \in \mathbb{R}$, $F(x-\epsilon) - \epsilon \leq F_n(x) \leq F(x+\epsilon) + \epsilon$ for sufficiently large n. It follows that $\liminf_{n\to\infty} F_n(x) \geq F(x-\epsilon) - \epsilon$ and $\limsup_{n\to\infty} F_n(x) \leq F(x+\epsilon) + \epsilon$. Since these statements hold for all $\epsilon > 0$,

we can send $\epsilon \to 0$ and get $F(x^-) \le \liminf_{n\to\infty} F_n(x) \le \limsup_{n\to\infty} F_n(x) \le F(x^+)$. If F is continuous at x then $F_n(x) \to F(x)$, showing that $F_n \Rightarrow F$.
The converse statement requires a uniform convergence argument. Suppose $F_n \Rightarrow F$. Fix $\epsilon > 0$ and find continuity points y, y' of F such that $F(y) \le \epsilon$ and $F(y') \ge 1-\epsilon$. Then find a sequence $y = y_0 < y_1 < \cdots < y_k = y'$ such that each $y_{j+1} - y_j \le \epsilon$ and F is continuous at each y_j. For large enough n (say $n \ge N(\epsilon)$) each $|F_n(y_j) - F(y_j)| \le \epsilon$. We will show that $\rho(F_n, F) \le 2\epsilon$ whenever $n \ge N(\epsilon)$. If $x \le y_0$ then

$$F(x-2\epsilon) - 2\epsilon \le F(y_0) - 2\epsilon \le 0 \le F_n(x) \le F_n(y_0) \le F(y_0) + \epsilon \le 2\epsilon \le F(x+2\epsilon) + 2\epsilon.$$

Similarly, if $x \ge y_k$ then

$$F(x-2\epsilon) - 2\epsilon \le 1 - 2\epsilon \le F(y_k) - \epsilon \le F_n(y_k) \le F_n(x) \le 1 \le F(y_k) + 2\epsilon \le F(x+2\epsilon) + 2\epsilon.$$

If $y_j \le x \le y_{j+1}$ then

$$\begin{aligned} F(x-2\epsilon) - 2\epsilon \le F(y_j) - 2\epsilon \le F_n(y_j) - \epsilon \le F_n(x) \\ \le F_n(y_{j+1}) + \epsilon \le F(y_{j+1}) + 2\epsilon \le F(x+2\epsilon) + 2\epsilon. \end{aligned}$$

It follows that $\rho(F_n, F) \le 2\epsilon$ for all $n \ge N(\epsilon)$. By sending $\epsilon \to 0$ we conclude that $\rho(F_n, F) \to 0$.

Problem 6: This could be proved from scratch using an argument similar to the one directly above. We will cite the result of Problem 5 instead of recapitulating the proof. First we show that F is not just continuous but actually uniformly continuous. Fix a positive integer m. For each $1 \le j \le m$, each set $F^{-1}((\frac{j-1}{m}, \frac{j}{m}))$ is a nonempty open interval (open by continuity, nonempty by the Intermediate Value Theorem, interval by monotonicity). Let $\delta = \delta(m)$ be the shortest length of one of these intervals. For all $x < y \in \mathbb{R}$, if $F(y) - F(x) \ge \frac{2}{m}$ then there must be a full interval $(\frac{j-1}{m}, \frac{j}{m})$ in between $F(x)$ and $F(y)$, so $y - x \ge \delta$. In other words, if $|x-y| < \delta$ then $|F(x) - F(y)| < \frac{2}{m}$, proving uniform continuity of F.
Now we are ready to apply Problem 5. Given $\epsilon > 0$ there is $\delta > 0$ such that $|x - y| < \delta$ implies that $|F(x) - F(y)| < \epsilon$. Without loss of generality we take $\delta \le \epsilon$. Since $F_n \Rightarrow F$, $\rho(F_n, F) < \delta$ for large enough n. Thus, for all $x \in \mathbb{R}$,

$$F(x) - 2\epsilon \le F(x) - \epsilon - \delta \le F(x-\delta) - \delta \le F_n(x) \le F(x+\delta) + \delta \le F(x) + \epsilon + \delta \le F(x) + 2\epsilon$$

which means that $\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \le 2\epsilon$. Sending $\epsilon \to 0$ completes the proof.