16. Random Walk Preliminaries

Thus far, we have been primarily interested in the large n behavior of $S_n = \sum_{i=1}^{n} X_i$ where $X_1, X_2, \ldots$ are independent and identically distributed. We now turn our attention to the sequence $S_1, S_2, \ldots$, which we think of as successive states of a *random walk*.

Recall that the existence of an infinite sequence of random variables with specified finite dimensional distributions is ensured by Kolmogorov's extension theorem.

Here the sample space is $\Omega = \mathbb{R}^{\mathbb{N}} = \{(\omega_1, \omega_2, \ldots) : \omega_i \in \mathbb{R}\}$, the σ-algebra is $\mathcal{B}^{\mathbb{N}}$ (which is generated by cylinder sets), and a consistent sequence of distributions gives rise to a unique probability measure with appropriate marginals via the extension theorem. The random variables are the coordinate maps $X_i((\omega_1, \omega_2, \ldots)) = \omega_i$.

If S is a Polish space (i.e. a separable and completely metrizable topological space) and $\mathcal{S}$ is the Borel σ-algebra for S, then this Kolmogorov construction can be carried out with $\Omega = S^{\mathbb{N}}$ and $\mathcal{F} = \mathcal{S}^{\mathbb{N}}$. When the $X_i's$ are independent $(S, \mathcal{S})$-valued random variables with $X_i \sim \mu_i$, the measure P arises from the sequence of product measures $P_n = \mu_1 \times \cdots \times \mu_n$.

We assume in what follows that we are working in $(S^{\mathbb{N}}, \mathcal{S}^{\mathbb{N}}, P)$ and $X_1, X_2, \ldots$ are given by
$X_i\left((\omega_1, \omega_2, \ldots)\right) = \omega_i$.

Recall that Kolmogorov's $0-1$ law showed that if $X_1, X_2, \ldots$ are independent, then the tail field
$\mathcal{T} = \bigcap_{n=1}^{\infty} \sigma(X_n, X_{n+1}, \ldots)$ is trivial in the sense that every $A \in \mathcal{T}$ has $P(A) \in \{0, 1\}$.

Our first main result is another $0-1$ law. We begin with some terminology.

Definition. A *finite permutation* is a bijection $\pi : \mathbb{N} \to \mathbb{N}$ such that $|\{m : \pi(m) \neq m\}| < \infty$.

If π is a finite permutation and $\omega \in S^{\mathbb{N}}$, then we define $\pi\omega$ by $(\pi\omega)_i = \omega_{\pi(i)}$.

Definition. $A \in \mathcal{S}^{\mathbb{N}}$ is *permutable* if $\pi^{-1}A = \{\omega : \pi\omega \in A\}$ is equal to A for any finite permutation π.

In other words, for every $n \in \mathbb{N}$ and every $\pi \in S_n$, we have

$$(X_1, \ldots, X_n, X_{n+1}, \ldots) \in A \iff (X_{\pi(1)}, \ldots, X_{\pi(n)}, X_{n+1}, \ldots) \in A.$$

Proposition 16.1. *The collection of permutable events is a σ-algebra. It is called the* exchangeable σ-algebra *and is denoted $\mathcal{E}$.*

Taking $S = \mathbb{R}$, $S_n = \sum_{i=1}^{n} X_i$, some examples of permutable events are $E = \{S_n \in B_n \text{ i.o.}\}$ and
$F = \{\limsup_{n\to\infty} \frac{S_n}{c_n} \geq 1\}$ for any sequence of Borel sets $\{B_n\}_{n=1}^{\infty}$ and real numbers $\{c_n\}_{n=1}^{\infty}$.

Also, every event in the tail σ-algebra is also in the exchangeable σ-algebra.

We observe that, in general, $E, F \notin \mathcal{T}$, though F is in the tail field if we assume that $c_n \to \infty$.
Similarly, $\{\lim_{n\to\infty} S_n \text{ exists}\}, \{\limsup_{n\to\infty} S_n = \infty\} \in \mathcal{T}$ while, in general, $\{\limsup_{n\to\infty} S_n > 0\} \in \mathcal{E} \setminus \mathcal{T}$.

The proof of the Hewitt-Savage $0-1$ law will make use of the following result.

Lemma 16.1. *For any $I \in \mathcal{S}^{\mathbb{N}}$, there is a sequence of events $I_1, I_2, ...$ such that $I_n \in \sigma(X_1, ..., X_n)$ and $P(I_n \triangle I) \to 0$ where $A \triangle B = (A \setminus B) \cup (B \setminus A)$.*

Proof. $\sigma(X_1, ..., X_n)$ is precisely the sub-σ-algebra of $\mathcal{S}^{\mathbb{N}}$ consisting of the cylinders $\{\omega : (\omega_1, ..., \omega_n) \in B\}$ as B ranges over $\mathcal{S}^n$. Accordingly, $\mathcal{P} = \bigcup_{n=1}^{\infty} \sigma(X_1, ..., X_n)$ is a π-system which generates $\mathcal{S}^{\mathbb{N}}$. The claim follows from Theorem 2.2 upon noting that $\mathcal{L} = \{J \in \mathcal{S}^{\mathbb{N}} : \text{there exist } I_n \in \sigma(X_1, ..., X_n) \text{ with } P(I_n \triangle J) \to 0\}$ is a λ-system containing $\mathcal{P}$. □

Theorem 16.1 (Hewitt-Savage)**.** *If $X_1, X_2, ...$ are i.i.d. and $A \in \mathcal{E}$, then $P(A) \in \{0, 1\}$.*

Proof. As with Kolmogorov's $0-1$ law, we will show that A is independent of itself.

We begin by taking a sequence of events $A_n \in \sigma(X_1, ..., X_n)$ such that $P(A_n \triangle A) \to 0$, which is justified by Lemma 16.1.

Now let π be the finite permutation $\pi(j) = \begin{cases} j+n, & j \leq n \\ j-n, & n < j \leq 2n \\ j, & j > 2n \end{cases}$.

In words, π transposes j and $n+j$ for $j = 1, ..., n$.

Because the coordinates are i.i.d., $P\left(\pi^{-1}\left(A_n \triangle A\right)\right) = P(A_n \triangle A)$, so, setting $A_n' = \pi^{-1} A_n$ and noting that $A \in \mathcal{E}$ implies $\pi^{-1} A = A$, we see that

$$P(A_n \triangle A) = P\left(\pi^{-1}(A_n \triangle A)\right) = P\left(\left(\pi^{-1} A_n\right) \triangle \left(\pi^{-1} A\right)\right) = P(A_n' \triangle A).$$

Thus, since

$$A \triangle (A_n \cap A_n') = (A \setminus A_n) \cup (A \setminus A_n') \cup [(A_n \cap A_n') \setminus A] \subseteq (A_n \triangle A) \cup (A_n' \triangle A),$$

we have

$$P\left(A \triangle (A_n \cap A_n')\right) \leq P(A_n \triangle A) + P(A_n' \triangle A) = 2P(A_n \triangle A) \to 0,$$

hence $P(A_n \cap A_n') \to P(A)$.

As A_n and A_n' are independent by construction and $P(A_n), P(A_n') \to P(A)$, we conclude that

$$P(A)^2 = \lim_{n \to \infty} P(A_n) P(A_n') = \lim_{n \to \infty} P(A_n \cap A_n') = P(A).$$ □

Because $\mathcal{T} \subset \mathcal{E}$, Hewitt-Savage supersedes Kolmogorov in the case of i.i.d. random variables. However, the latter only requires independence, so it can be used in situations where the former cannot. Also, note that in the examples preceding Theorem 16.1, the sequences $E_n = \{S_n \in B_n\}$ and $F_n = \left\{\frac{S_n}{c_n} \geq 1\right\}$ are each dependent, so the Borel-Cantelli lemmas do not imply that E or F is trivial.

A nice application of Theorem 16.1 is

Theorem 16.2. *For a random walk on $\mathbb{R}$, there are only four possibilities, one of which has probability one:*

(1) $S_n = 0$ *for all* n
(2) $S_n \to \infty$
(3) $S_n \to -\infty$
(4) $-\infty = \liminf_{n\to\infty} S_n < \limsup_{n\to\infty} S_n = \infty$

Proof.

Theorem 16.1 implies that $\limsup_{n\to\infty} S_n$ is a constant $c \in [-\infty, \infty]$. Let $S'_n = S_{n+1} - X_1$.

Since $S'_n =_d S_n$, we must have that $c = c - X_1$. If $c \in (-\infty, \infty)$, then it must be the case that $X_1 \equiv 0$, so the first case occurs. Conversely, if X_1 is not identically zero, then $c = \pm\infty$.

Of course, the exact same argument applies to the lim inf, so either 1 holds or $\liminf_{n\to\infty} S_n, \limsup_{n\to\infty} S_n \in \{\pm\infty\}$.

As $\limsup_{n\to\infty} S_n \geq \liminf_{n\to\infty} S_n$, this implies that we are in one of cases 2, 3, or 4. □