18. Recurrence

In this section, we will consider some questions regarding the recurrence behavior of random walk in $\mathbb{R}^d$.

Throughout, we will take $(S, \mathcal{S})$ to be $\mathbb{R}^d$ with the Borel σ-algebra, we will denote the position of the random walk at time n by $S_n = X_1 + ... + X_n$ with $X_1, X_2, ...$ i.i.d., and we will work with the norm $\|x\| = \max_{1 \le i \le d} |x_i|$.

Definition. $x \in \mathbb{R}^d$ is called a *recurrent value* for the random walk S_n if for every $\varepsilon > 0$, $P(\|S_n - x\| < \varepsilon \text{ i.o.}) = 1$. We denote the set of recurrent values by $\mathcal{V}$.

Note that the Hewitt-Savage $0-1$ law implies that if $P(\|S_n - x\| < \varepsilon \text{ i.o.})$ is less than one, then it is zero.

Definition. $x \in \mathbb{R}^d$ is called a *possible value* for S_n if for every $\varepsilon > 0$, there is an $n \in \mathbb{N}$ with $P(\|S_n - x\| < \varepsilon) > 0$. The set of possible values is denoted $\mathcal{U}$.

Clearly the set of recurrent values is contained in the set of possible values. In fact, we have

Theorem 18.1. *The set $\mathcal{V}$ is either $\emptyset$ or a closed subgroup of $\mathbb{R}^d$. In the latter case, $\mathcal{V} = \mathcal{U}$.*

Proof. Suppose that $\mathcal{V} \neq \emptyset$. If $z \in \mathcal{V}^C$, then there is an $\varepsilon > 0$ such that $P(\|S_n - z\| < \varepsilon \text{ i.o.}) = 0$, and thus $B_\varepsilon(z) = \{w \in \mathbb{R}^d : \|w - z\| < \varepsilon\} \subseteq \mathcal{V}^C$. It follows that $\mathcal{V}^C$ is open, so $\mathcal{V}$ is closed.

The rest of the theorem will follow upon showing

$$(*) \text{ If } x \in \mathcal{U} \text{ and } y \in \mathcal{V}, \text{ then } y - x \in \mathcal{V}.$$

This is because $\mathcal{V} \subseteq U$, so for any $v, w \in \mathcal{V}$ taking $x = y = v$ shows that $0 \in \mathcal{V}$; taking $x = v$, $y = 0$ shows that $-v \in \mathcal{V}$; and taking $x = -w$, $y = v$ shows that $v + w \in \mathcal{V}$. It follows that $\mathcal{V} \le \mathbb{R}^d$.
Also, for any $u \in \mathcal{U}$, taking $x = u$, $y = 0$ shows that $-u \in \mathcal{V}$. As $\mathcal{V}$ is a subgroup, this implies that $u \in \mathcal{V}$, and thus $\mathcal{U} \subseteq \mathcal{V}$. Consequently, $\mathcal{U} = \mathcal{V}$.

To prove $(*)$, we note that if $y - x \notin \mathcal{V}$, then there exist $\varepsilon > 0$, $m \in \mathbb{N}$ such that

$$P(\|S_n - (y - x)\| \ge 2\varepsilon \text{ for all } n \ge m) > 0.$$

Also, since $x \in \mathcal{U}$, there is some $k \in \mathbb{N}$ with

$$P(\|S_k - x\| < \varepsilon) > 0.$$

Now for any $n \ge m + k$, $S_n - S_k = X_{k+1} + ... + X_n$ has the same distribution as S_{n-k} and is independent of S_k.
It follows that $\{\|S_n - S_k - (y - x)\| \ge 2\varepsilon \text{ for all } n \ge m + k\}$ and $\{\|S_k - x\| < \varepsilon\}$ are independent and each have positive probability.
Because $\|S_k(\omega) - x\| < \varepsilon$ and $2\varepsilon \le \|S_n(\omega) - S_k(\omega) - (y - x)\| = \|S_n(\omega) - y\| + \|S_n(\omega) - x\|$ implies $\|S_n(\omega) - y\| \ge \varepsilon$, we conclude that

$$P(\|S_n - y\| \ge \varepsilon \text{ for all } n \ge m + k)$$
$$\ge P(\|S_n - S_k - (x - y)\| \ge 2\varepsilon \text{ for all } n \ge m + k) P(\|S_k - x\| < \varepsilon) > 0.$$

But this contradicts $y \in \mathcal{V}$, so we must have $y - x \in \mathcal{V}$. □

When $\mathcal{V} = \emptyset$, the random walk is called *transient*, otherwise it is called *recurrent*.

It follows from Theorem 18.1 that a random walk is recurrent if and only if 0 is a recurrent value.

By definition, a sufficient condition for this to be the case is $P(S_n = 0 \text{ i.o.}) = 1$.

(That is, if 0 is *point recurrent*, then it is *neighborhood recurrent*. The distinction arises when the range of the X_i's is dense, but the two are equivalent for simple random walk.)

By Proposition 17.2, if we set $\tau_0 = 0$ and let $\tau_n = \inf\{m > \tau_{n-1} : S_m = 0\}$ be the *nth* time the walk visits 0, then $P(\tau_n < \infty) = P(\tau_1 < \infty)^n$. From this observation, we arrive at

Theorem 18.2. *For any random walk, the following are equivalent:*

(1) $P(\tau_1 < \infty) = 1$
(2) $P(S_n = 0 \ i.o.) = 1$
(3) $\sum_{n=1}^{\infty} P(S_n = 0) = \infty$

Proof.

If $P(\tau_1 < \infty) = 1$, then $P(\tau_n < \infty) = 1^n = 1$ for all n, hence $P(S_n = 0 \text{ i.o.}) = 1$.

The contrapositive of the first Borel-Cantelli lemma shows that $P(S_n = 0 \text{ i.o.}) = 1$ implies $\sum_{n=1}^{\infty} P(S_n = 0) = \infty$.

Finally, the number of visits to zero can be expressed as $V = \sum_{n=1}^{\infty} 1\{S_n = 0\}$ and as $V = \sum_{n=1}^{\infty} 1\{\tau_n < \infty\}$. Thus if $P(\tau_1 < \infty) < 1$, then

$$\sum_{n=1}^{\infty} P(S_n = 0) = E[V] = \sum_{n=1}^{\infty} P(\tau_n < \infty)$$
$$= \sum_{n=1}^{\infty} P(\tau_1 < \infty)^n = \frac{P(\tau_1 < \infty)}{1 - P(\tau_1 < \infty)} < \infty.$$

It follows that $\sum_{n=1}^{\infty} P(S_n = 0) = \infty$ implies $P(\tau_1 < \infty) = 1$. □

Analogous to the one-dimensional case, we say that $S_n = X_1 + ... + X_n$ defines a simple random walk on $\mathbb{R}^d$ (equivalently, $\mathbb{Z}^d$) if $X_1, X_2, ...$ are i.i.d. with

$$P(X_i = e_j) = P(X_i = -e_j) = \frac{1}{2d}$$

for each of the d standard basis vectors e_j.

We will show that simple random walk is recurrent in dimensions $d = 1, 2$ and transient otherwise.

Essentially, this is because $P(S_n = 0) \approx C_d n^{-\frac{d}{2}}$, which is summable for $d \geq 3$ but not for $d = 1, 2$.

The argument is combinatorial and relies on

Proposition 18.1 (Stirling's formula)**.**

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

in the sense that their ratio approaches 1 *as* $n \to \infty$.

Theorem 18.3. *Simple random walk is recurrent in dimensions one and two.*

Proof. When $d = 1$, $P(S_{2n-1} = 0) = 0$ and

$$P(S_{2n} = 0) = \frac{1}{2^{2n}}\binom{2n}{n} = \frac{1}{2^{2n}}\frac{(2n)!}{(n!)^2}$$
$$\approx \frac{1}{2^{2n}}\frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)^2} = \frac{1}{\sqrt{\pi n}}$$

for all $n \in \mathbb{N}$.

Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{\pi n}}$ diverges, it follows from the limit comparison test that

$$\sum_{n=1}^{\infty} P(S_n = 0) = \sum_{n=1}^{\infty} P(S_{2n} = 0) = \infty,$$

so $P(S_n = 0 \text{ i.o.}) = 1$ and thus S_n is recurrent.

Similarly, when $d = 2$, $P(S_{2n-1} = 0) = 0$ and

$$P(S_{2n} = 0) = \frac{1}{4^{2n}}\sum_{m=0}^{n}\binom{2n}{m, m, n-m, n-m} = \frac{1}{4^{2n}}\sum_{m=0}^{n}\frac{(2n)!}{m!m!(n-m)!(n-m)!}$$
$$= \frac{1}{4^{2n}}\binom{2n}{n}\sum_{m=0}^{n}\binom{n}{m}^2 = \left(\frac{1}{2^{2n}}\right)^2\binom{2n}{n}\sum_{m=0}^{n}\binom{n}{m}\binom{n}{n-m}$$
$$= \left[\frac{1}{2^{2n}}\binom{2n}{n}\right]^2 \approx \frac{1}{\pi n}$$

since we have seen that $\frac{1}{2^{2n}}\binom{2n}{n} \approx \frac{1}{\sqrt{\pi n}}$.

* The identity

$$\sum_{m=0}^{n}\binom{n}{m}\binom{n}{n-m} = \binom{2n}{n}$$

follows by noting that the number of ways to choose a committee of size n from a population of size $2n$ containing n men and n women is the sum over $m = 0, 1, ..., n$ of the number of such committees consisting of m men and $n - m$ women.

Arguing as in the $d = 1$ case shows that S_n is recurrent. □

In contrast to the $d = 1, 2$ cases, we have

Theorem 18.4. *Simple random walk is transient in three or more dimensions.*

Proof. When $d = 3$,

$$P(S_{2n} = 0) = 6^{-2n}\sum_{\substack{n_1,n_2,n_3 \geq 0:\\ n_1+n_2+n_3=n}}\frac{(2n)!}{(n_1!n_2!n_3!)^2} = 2^{-2n}\binom{2n}{n}\sum_{\substack{n_1,n_2,n_3 \geq 0:\\ n_1+n_2+n_3=n}}\left(3^{-n}\frac{n!}{n_1!n_2!n_3!}\right)^2.$$

Now $3^{-n}\frac{n!}{n_1!n_2!n_3!} \geq 0$ for each choice of n_1, n_2, n_3, n, and the multinomial theorem gives

$$\sum_{\substack{n_1,n_2,n_3\geq 0:\\ n_1+n_2+n_3=n}} 3^{-n}\frac{n!}{n_1!n_2!n_3!} = \sum_{\substack{n_1,n_2,n_3\geq 0:\\ n_1+n_2+n_3=n}} \binom{n}{n_1, n_2, n_3}\left(\frac{1}{3}\right)^{n_1}\left(\frac{1}{3}\right)^{n_2}\left(\frac{1}{3}\right)^{n_3} = \left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)^{n} = 1,$$

so

$$\sum_{\substack{n_1,n_2,n_3\geq 0:\\ n_1+n_2+n_3=n}} \left(3^{-n}\frac{n!}{n_1!n_2!n_3!}\right)^2 \leq \left(\max_{\substack{0\leq n_1\leq n_2\leq n_3:\\ n_1+n_2+n_3=n}} 3^{-n}\frac{n!}{n_1!n_2!n_3!}\right) \sum_{\substack{n_1,n_2,n_3\geq 0:\\ n_1+n_2+n_3=n}} 3^{-n}\frac{n!}{n_1!n_2!n_3!}$$

$$= 3^{-n}\max_{\substack{0\leq n_1\leq n_2\leq n_3:\\ n_1+n_2+n_3=n}} \frac{n!}{n_1!n_2!n_3!}.$$

The latter quantity is maximized when $n_1!n_2!n_3!$ is minimized. This happens when n_1, n_2, n_3 are as close as possible: If $n_i < n_j - 1$ for $i < j$, then $n_i!n_j! > \frac{n_i+1}{n_j}n_i!n_j! = (n_i+1)!(n_j-1)!$.

It follows that

$$\max_{\substack{0\leq n_1\leq n_2\leq n_3:\\ n_1+n_2+n_3=n}} \frac{n!}{n_1!n_2!n_3!} \approx \frac{n!}{\left(\left[\frac{n}{3}\right]!\right)^3} \approx \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\left(\sqrt{\frac{2\pi n}{3}}\left(\frac{n}{3e}\right)^{\frac{n}{3}}\right)^3} = \frac{3^{\frac{3}{2}}\left(\frac{n}{e}\right)^n}{2\pi n\left(\frac{n}{3e}\right)^n} \leq \frac{3^n}{n}.$$

Putting all this together and recalling that $\frac{1}{2^{2n}}\binom{2n}{n} \approx \frac{1}{\sqrt{\pi n}}$ shows that

$$P(S_{2n} = 0) = 2^{-2n}\binom{2n}{n}\sum_{\substack{n_1,n_2,n_3\geq 0:\\ n_1+n_2+n_3=n}} \left(3^{-n}\frac{n!}{n_1!n_2!n_3!}\right)^2 = O\left(n^{-\frac{3}{2}}\right),$$

hence $\sum_{n=1}^{\infty} P(S_n = 0) < \infty$ and we conclude that SRW is transient in 3 dimensions.

Transience in higher dimensions follows by letting $T_n = (S_n^1, S_n^2, S_n^3)$ be the projection onto the first three coordinates and letting $N(n) = \inf\{m > N(n-1) : T_m \neq T_{N(n-1)}\}$ to be the *nth* time that the random walker moves in any of the first three coordinates (with the convention that $N(0) = 0$). Then $T_{N(n)}$ is a simple random walk in three dimensions and the probability that $T_{N(n)} = 0$ infinitely often is 0. Since the first three coordinates of S_n are constant between $N(n)$ and $N(n+1)$ and $N(n+1) - N(n)$ is almost surely finite, this implies that S_n is transient. □

In the case of more general random walks on $\mathbb{R}^d$, the Chung-Fuchs theorem says that

- S_n is recurrent in $d = 1$ if $\frac{S_n}{n} \to_p 0$.
- S_n is recurrent in $d = 2$ if $\frac{S_n}{\sqrt{n}} \Rightarrow N(0, \Sigma)$.
- S_n is transient in $d \geq 3$ if it is "truly (at least) three dimensional" (meaning that it does not live an a plane through the origin).

More generally, one can show that a necessary and sufficient condition for recurrence is

$$\int_{(-\delta,\delta)^d} \text{Re}\left(\frac{1}{1-\varphi(y)}\right) dy = \infty$$

for $\delta > 0$ where $\varphi(t) = E\left[e^{it\cdot X_1}\right]$ is the ch.f. of one step in the walk.

We will content ourselves with proofs of the $d = 1, 2$ results. The proof for $d \geq 3$ can be found in Durrett.

We begin with some lemmas which are valid in any dimension. The first is analogous to Theorem 18.2.

Lemma 18.1. *Let $X_1, X_2, ...$ be i.i.d. and take $S_n = \sum_{i=1}^{n} X_i$. Then S_n is recurrent if and only if $\sum_{n=1}^{\infty} P(\|S_n\| < \varepsilon) = \infty$ for every $\varepsilon > 0$.*

Proof.

If S_n is recurrent, then $P(\|S_n\| < \varepsilon \text{ i.o.}) = 1$ for all $\varepsilon > 0$, so the contrapositive of the first Borel-Cantelli lemma implies that $\sum_{n=1}^{\infty} P(\|S_n\| < \varepsilon) = \infty$ for all $\varepsilon > 0$.

For the converse, fix $k \geq 1$ and define $Z_k = \sum_{i=1}^{\infty} 1\left\{\|S_i\| < \varepsilon, \|S_{i+j}\| \geq \varepsilon \text{ for all } j \geq k\right\}$.

Then $Z_k \leq k$ by construction, so

$$\begin{aligned}
k \geq E[Z_k] &= \sum_{i=1}^{\infty} P\left(S_i < \varepsilon, \|S_{i+j}\| \geq \varepsilon \text{ for all } j \geq k\right) \\
&\geq \sum_{i=1}^{\infty} P\left(\|S_i\| < \varepsilon, \|S_{i+j} - S_i\| \geq 2\varepsilon \text{ for all } j \geq k\right) \\
&= \sum_{i=1}^{\infty} P\left(\|S_i\| < \varepsilon\right) P\left(\|S_{i+j} - S_i\| \geq 2\varepsilon \text{ for all } j \geq k\right) \\
&= P\left(\|S_j\| \geq 2\varepsilon \text{ for all } j \geq k\right) \sum_{i=1}^{\infty} P\left(\|S_i\| < \varepsilon\right).
\end{aligned}$$

Thus if $\sum_{i=1}^{\infty} P\left(\|S_i\| < \varepsilon\right) = \infty$, then it must be the case that $P\left(\|S_j\| \geq 2\varepsilon \text{ for all } j \geq k\right) = 0$.

As this is true for all $k \in \mathbb{N}$, we see that $P\left(\|S_j\| < 2\varepsilon \text{ i.o.}\right) = 1$, so, since this holds for all $\varepsilon > 0$, S_n is recurrent. □

Note that one could equivalently take the lower index of summation to be 0 in the preceding theorem since adding or subtracting 1 does not change whether or not the sum diverges to infinity.

Everything that we have done thus far is true for any norm. Our reason for working with the supremum norm is the following result. As all norms on $\mathbb{R}^d$ are equivalent, this choice entails no loss of generality - the definition of neighborhood recurrence is topological.

Lemma 18.2. *For any $m \in \mathbb{N}$, $\varepsilon > 0$,*

$$\sum_{n=0}^{\infty} P\left(\|S_n\| < m\varepsilon\right) \leq (2m)^d \sum_{n=0}^{\infty} P\left(\|S_n\| < \varepsilon\right).$$

Proof. The left hand side gives the expected number of visits to the open cube $(-m\varepsilon, m\varepsilon)^d$. This can be obtained by summing the number of visits to each of the $(2m)^d$ subcubes of side length ε obtained by dividing each side of the cube into $2m$ equal segments. Thus it suffices to show that the expected number of visits to any of these subcubes is at most $\sum_{n=0}^{\infty} P\left(\|S_n\| < \varepsilon\right)$.

To this end, let C be any such side length ε cube in $\mathbb{R}^d$ and let $T = \inf\{n : S_n \in C\}$ be the hitting time for C. If $T = \infty$ then the walk never visits C, while if $T = m$, then on every subsequent visit to C the walk is within ε of S_m, so

$$\begin{aligned}
\sum_{n=0}^{\infty} P(S_n \in C) &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} P(S_n \in C, T = m) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} P(S_n \in C, T = m) \\
&\leq \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} P\left(\|S_n - S_m\| < \varepsilon, T = m\right) \\
&= \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} P\left(\|S_n - S_m\| < \varepsilon\right) P\left(T = m\right) \\
&= \sum_{m=0}^{\infty} P(T = m) \sum_{k=0}^{\infty} P\left(\|S_k\| < \varepsilon\right) \\
&\leq \sum_{k=0}^{\infty} P\left(\|S_k\| < \varepsilon\right).
\end{aligned}$$

□

The preceding lemma shows that establishing convergence/divergence of $\sum_{n=1}^{\infty} P\left(\|S_n\| < \varepsilon\right)$ for a single value of $\varepsilon > 0$ is sufficient for determining transience/recurrence of S_n. In particular, we have

Corollary 18.1. *S_n is recurrent if and only if* $\sum_{n=1}^{\infty} P\left(\|S_n\| < 1\right) = \infty$.

Proof. Lemma 18.1 shows that if S_n is recurrent, then $\sum_{n=1}^{\infty} P(\|S_n\| < 1) = \infty$.
On the other hand, suppose that $\sum_{n=1}^{\infty} P(\|S_n\| < 1) = \infty$ and let $\varepsilon > 0$ be given.
Applying Lemma 18.2 with $m > \varepsilon^{-1}$ yields

$$\infty = \sum_{n=1}^{\infty} P(\|S_n\| < 1) \leq (2m)^d \sum_{n=0}^{\infty} P\left(\|S_n\| < m^{-1}\right) \leq (2m)^d \sum_{n=0}^{\infty} P\left(\|S_n\| < \varepsilon\right),$$

so, since ε was arbitrary, it follows from Lemma 18.1 that S_n is recurrent. □

With the previous results at our disposal, we are able to show

Theorem 18.5. *If $X_1, X_2, \ldots$ are i.i.d. $\mathbb{R}$-valued random variables and $\frac{1}{n}S_n \to_p 0$, then S_n is recurrent.*

Proof. By Corollary 18.1, it suffices to prove that $\sum_{n=1}^{\infty} P(|S_n| < 1) = \infty$.
Lemma 18.2 shows that for any $m \in \mathbb{N}$,

$$\begin{aligned}
\sum_{n=0}^{\infty} P\left(|S_n| < 1\right) &\geq \frac{1}{2m}\sum_{n=0}^{\infty} P\left(|S_n| < m\right) \geq \frac{1}{2m}\sum_{n=0}^{Km} P\left(|S_n| < m\right) \\
&\geq \frac{1}{2m}\sum_{n=0}^{Km} P\left(|S_n| < \frac{n}{K}\right) = \frac{K}{2} \cdot \frac{1}{Km}\sum_{n=0}^{Km} P\left(|S_n| < \frac{n}{K}\right)
\end{aligned}$$

for any $K \in \mathbb{N}$. By hypothesis, $P\left(|S_n| < \frac{n}{K}\right) \to 1$ as $n \to \infty$, so, sending m to ∞ shows that $\sum_{n=0}^{\infty} P\left(|S_n| < 1\right) \geq \frac{K}{2}$, and the proof is complete since K was arbitrary. □

It is worth remarking that if the $X_i's$ have a well-defined expectation $E[X_i] = \mu \neq 0$, then the strong law implies $\frac{S_n}{n} \to \mu$ a.s. In this case, we must have $|S_n| \to \infty$, so the walk is transient. If $E[X_i] = 0$, then the weak law implies $\frac{S_n}{n} \to_p 0$. Thus if the increments have an expectation $\mu = E[X_i] \in [-\infty, \infty]$, then the walk is recurrent if and only if $\mu = 0$.

We now show that, in dimension 2, a random walk is recurrent if a mean 0 central limit theorem holds.

In the case where the limit distribution $N(0, \Sigma)$ is degenerate - that is, the covariance matrix Σ has $\text{rank}(\Sigma) < 2$ - the random walk is either always at 0 or is essentially one-dimensional, in which case recurrence follows from Theorem 18.5. Thus we will assume in what follows that $N(0, \Sigma)$ is nondegenerate and thus has a density with respect to Lebesgue measure on $\mathbb{R}^2$.

Theorem 18.6. *If S_n is a random walk in $\mathbb{R}^2$ and $n^{-\frac{1}{2}}S_n$ converges weakly to a nondegenerate normal distribution, then S_n is recurrent.*

Proof. As before, we need to show that $\sum_{n=1}^{\infty} P(\|S_n\| < 1) = \infty$.

By Lemma 18.2,

$$\sum_{n=0}^{\infty} P\left(\|S_n\| < 1\right) \geq \frac{1}{4m^2} \sum_{n=0}^{\infty} P\left(\|S_n\| < m\right),$$

and we can write

$$\frac{1}{m^2} \sum_{n=0}^{\infty} P\left(\|S_n\| < m\right) = \int_0^{\infty} P\left(\left\|S_{\lfloor m^2\theta \rfloor}\right\| < m\right) d\theta$$

since $\lfloor m^2\theta \rfloor = n$ on the segment $\frac{n}{m^2} \leq \theta < \frac{n+1}{m^2}$ of length $\frac{1}{m^2}$.

Also, letting $n(y)$ denote the limiting normal density, we have

$$P\left(\left\|S_{\lfloor m^2\theta \rfloor}\right\| < m\right) \approx P\left(\frac{\left\|S_{\lfloor m^2\theta \rfloor}\right\|}{m\sqrt{\theta}} < \frac{1}{\sqrt{\theta}}\right) \to \int_{\|y\| < \theta^{-\frac{1}{2}}} n(y)dy$$

as $m \to \infty$, so Fatou's lemma shows that

$$\begin{aligned} 4\sum_{n=0}^{\infty} P\left(\|S_n\| < 1\right) &\geq \liminf_{m\to\infty} \frac{1}{m^2} \sum_{n=0}^{\infty} P\left(\|S_n\| < m\right) \\ &= \liminf_{m\to\infty} \int_0^{\infty} P\left(\left\|S_{\lfloor m^2\theta \rfloor}\right\| < m\right) d\theta \\ &\geq \int_0^{\infty} \left(\int_{\|y\| < \theta^{-\frac{1}{2}}} n(y)dy\right) d\theta. \end{aligned}$$

Since n is positive and continuous at 0, letting $|\cdot|$ denote Lebesgue measure, we have

$$\int_{\|y\| < \theta^{-\frac{1}{2}}} n(y)dy \approx \left|\left\{\|y\| \leq \theta^{-\frac{1}{2}}\right\}\right| n(0) = \frac{4n(0)}{\theta}$$

as $\theta \to \infty$.

It follows that the integral with respect to θ (and thus the sum of interest) diverges, and we conclude that S_n is recurrent. ☐