2. Preliminary Results

At this point, we need to establish some fundamental facts about probability measures and σ-algebras in preparation for a discussion of probability distributions and to reacquaint ourselves with the style of measure theoretic arguments.

Probability Measures.

The following simple facts are extremely useful and will be employed frequently throughout this course.

Theorem 2.1. *Let P be a probability measure on $(\Omega, \mathcal{F})$.*

(i) Complements For any $A \in \mathcal{F}$, $P(A^C) = 1 - P(A)$.

(ii) Monotonicity For any $A, B \in \mathcal{F}$ with $A \subseteq B$, $P(A) \leq P(B)$.

(iii) Subadditivity For any countable collection $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$, $P\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} P(E_i)$.

(iv) Continuity from below If $A_i \nearrow A$ (i.e. $A_1 \subseteq A_2 \subseteq ...$ and $\bigcup_{i=1}^{\infty} A_i = A$), then $\lim_{n\to\infty} P(A_n) = P(A)$.

(v) Continuity from above If $A_i \searrow A = \bigcap_{i=1}^{\infty} A_i$, then $\lim_{n\to\infty} P(A_n) = P(A)$.

Proof.

For (i), $1 = P(\Omega) = P(A \sqcup A^C) = P(A) + P(A^C)$ by countable additivity.

For (ii), $P(B) = P\left(A \sqcup (B \setminus A)\right) = P(A) + P(B \setminus A) \geq P(A)$.

For (iii), we "disjointify" the sets by defining $F_1 = E_1$ and $F_i = E_i \setminus \left(\bigcup_{j=1}^{i-1} E_j\right)$ for $i > 1$, and observe that the $F_i's$ are disjoint and $\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i$ for all $n \in \mathbb{N} \cup \{\infty\}$. Since $F_i \subseteq E_i$ for all i, we have

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = P\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} P(F_i) \leq \sum_{i=1}^{\infty} P(E_i).$$

For (iv), set $B_1 = A_1$ and $B_i = A_i \setminus A_{i-1}$ for $i > 1$, and note that the $B_i's$ are disjoint with $\bigcup_{i=1}^{n} B_i = A_n$ and $\bigcup_{i=1}^{\infty} B_i = A$. Then

$$\begin{aligned} P(A) &= P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i) \\ &= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} B_i\right) = \lim_{n\to\infty} P(A_n). \end{aligned}$$

For (v), if $A_1 \supseteq A_2 \supseteq ...$ and $A = \bigcap_{i=1}^{\infty} A_i$, then $A_1^C \subseteq A_2^C \subseteq ...$ and $A^C = \left(\bigcap_{i=1}^{\infty} A_i\right)^C = \bigcup_{i=1}^{\infty} A_i^C$, so it follows from (i) and (iv) that

$$P(A) = 1 - P(A^C) = 1 - \lim_{n\to\infty} P(A_n^C) = \lim_{n\to\infty} \left(1 - P(A_n^C)\right) = \lim_{n\to\infty} P(A_n). \qquad \square$$

Note that (ii)-(iv) hold for any measure space $(S, \mathcal{G}, \nu)$, (v) is true for arbitrary measure spaces under the assumption that there is some A_i with $\nu(A_i) < \infty$, and (i) holds for all finite measures upon replacing 1 with $\nu(S)$.

Sigma Algebras.

We now review some some basic facts about σ-algebras. Our first observation is

Proposition 2.1. *For any index set I, if $\{\mathcal{F}_i\}_{i\in I}$ is a collection of σ-algebras on Ω, then so is $\cap_{i\in I}\mathcal{F}_i$.*

It follows easily from Proposition 2.1 that for any collection of sets $\mathcal{A} \subseteq 2^{\Omega}$, there is a smallest σ-algebra containing $\mathcal{A}$ - namely, the intersection of all σ-algebras containing $\mathcal{A}$.
This is called the *σ-algebra generated by* $\mathcal{A}$ and is denoted by $\sigma(\mathcal{A})$.

Note that if $\mathcal{F}$ is a σ-algebra and $\mathcal{A} \subseteq \mathcal{F}$, then $\sigma(\mathcal{A}) \subseteq \mathcal{F}$.

An important class of examples are the Borel σ-algebras: If $(X, \mathcal{T})$ is a topological space, then $\mathcal{B}_X = \sigma(\mathcal{T})$ is called the *Borel σ-algebra*. It is worth recalling that for the standard topology on $\mathbb{R}$, the Borel sets are generated by open intervals, closed intervals, half-open intervals, open rays, and closed rays, respectively.

Our main technical result about σ-algebras is Dynkin's π-λ Theorem, an extremely useful result which is often omitted in courses on measure theory. To state the result, we will need the following definitions.

Definition. A collection of sets $\mathcal{P} \subseteq 2^{\Omega}$ is called a *π-system* if it is closed under intersection.

Definition. A collection of sets $\mathcal{L} \subseteq 2^{\Omega}$ is called a *λ-system* if

(1) $\Omega \in \mathcal{L}$
(2) If $A, B \in \mathcal{L}$ and $A \subseteq B$, then $B \setminus A \in \mathcal{L}$
(3) If $A_n \in \mathcal{L}$ with $A_n \nearrow A$, then $A \in \mathcal{L}$

Theorem 2.2 (Dynkin)**.** *If $\mathcal{P}$ is a π-system and $\mathcal{L}$ is a λ-system with $\mathcal{P} \subseteq \mathcal{L}$, then $\sigma(\mathcal{P}) \subseteq \mathcal{L}$.*

Proof. We begin by observing that the intersection of any number of λ-systems is a λ-system, so for any collection $\mathcal{A}$, there is a smallest λ-system $\ell(\mathcal{A})$ containing $\mathcal{A}$. Thus it will suffice to show

a) $\ell(\mathcal{P})$ is a σ-algebra (since then $\sigma(\mathcal{P}) \subseteq \ell(\mathcal{P}) \subseteq \mathcal{L}$).

In fact, as one easily checks that a λ-system which is closed under intersection is a σ-algebra
($A^C = \Omega \setminus A$, $A \cup B = (A^C \cap B^C)^C$, and $\bigcup_{i=1}^{n} A_i \nearrow \bigcup_{i=1}^{\infty} A_i$), we need only to demonstrate

b) $\ell(\mathcal{P})$ is closed under intersection.

To this end, define $\mathcal{G}_A = \{B : A \cap B \in \ell(\mathcal{P})\}$ for any set A. To complete the proof, we will first show

c) $\mathcal{G}_A$ is a λ-system for each $A \in \ell(\mathcal{P})$,

and then prove that **b)** follows from **c)**.

To establish **c)**, let A be an arbitrary member of $\ell(\mathcal{P})$. Then $A = \Omega \cap A \in \ell(\mathcal{P})$, so $\mathcal{G}_A \ni \Omega$. Also, for any $B, C \in \mathcal{G}_A$ with $B \subseteq C$, we have $A \cap (C \setminus B) = (A \cap C) \setminus (A \cap B) \in \ell(\mathcal{P})$ since $A \cap B, A \cap C \in \ell(\mathcal{P})$ and $\ell(\mathcal{P})$ is a λ-system, hence $\mathcal{G}_A$ is closed under subset differences. Finally, for any sequence $\{B_n\}$ in $\mathcal{G}_A$ with $B_n \nearrow B$, we have $(A \cap B_n) \nearrow (A \cap B) \in \ell(\mathcal{P})$, so $\mathcal{G}_A$ is closed under countable increasing unions as well and thus is a λ-system.

It remains only to show that **c)** implies **b)**. To see that this is the case, first note that since $\mathcal{P}$ is a π-system, $\mathcal{P} \subseteq \mathcal{G}_A$ for every $A \in \mathcal{P}$, so it follows from **c)** that $\ell(\mathcal{P}) \subseteq \mathcal{G}_A$ for every $A \in \mathcal{P}$. In particular, for any $A \in \mathcal{P}$, $B \in \ell(\mathcal{P})$, we have $A \cap B \in \ell(\mathcal{P})$. Interchanging A and B yields $A \in \ell(\mathcal{P})$ and $B \in \mathcal{P}$ implies $A \cap B \in \ell(\mathcal{P})$. But this means if $A \in \ell(\mathcal{P})$, then $\mathcal{P} \subseteq \mathcal{G}_A$, and thus **c)** implies that $\ell(\mathcal{P}) \subseteq \mathcal{G}_A$. Therefore, it follows from the definition of $\mathcal{G}_A$ that for any $A, B \in \ell(\mathcal{P})$, $A \cap B \in \ell(\mathcal{P})$. □

It is not especially important to commit the details of this proof to memory, but it worth seeing once and you should definitely know the statement of the theorem. Though it seems a bit obscure upon first encounter, we will use this result in a variety of contexts throughout the course. In typical applications, we show that a property holds on a π-system that we know generates the σ-algebra of interest. We then show that the collection of all sets for which the property holds is a λ-system in order to conclude that the property holds on the entire σ-algebra.

A related result which is probably more familiar to those who have taken measure theory is the monotone class lemma used to prove Fubini-Tonelli.

Definition. A *monotone class* is a collection of subsets which is closed under countable increasing unions and countable decreasing intersections.

Like π-systems, λ-systems, and σ-algebras, the intersection of monotone classes is a monotone class, so it makes sense to talk about the monotone class generated by a collection of subsets.

Lemma 2.1 (Monotone Class Lemma). *If $\mathcal{A}$ is an algebra of subsets, then the monotone class generated by $\mathcal{A}$ is $\sigma(A)$.*

Note that σ-algebras are λ-systems and λ-systems are monotone classes, but the converses need not be true.