20. Law of The Iterated Logarithm

We conclude our investigation of simple random walk in one dimension with a result concerning the magnitude of its fluctuations. Specifically, we will prove

Theorem 20.1 (The Law of the Iterated Logarithm for Simple Random Walk)**.**
If $X_1, X_2, ...$ *are i.i.d. with* $P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}$ *and* $S_n = X_1 + ... + X_n$*, then*

$$P\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\left(\log(n)\right)}} = 1\right) = 1.$$

Though we will content ourselves with the above version, the statement holds more generally for $X_1, X_2, ...$ i.i.d. with $E[X_1] = 0$, $\mathrm{Var}(X_1) = 1$.

In this setting, we can think of the law of the iterated logarithm as treating an intermediate position between the laws of large numbers and the central limit theorem:
The former tell us that $\frac{S_n}{n} \to 0$ in probability and a.s., respectively, and the latter shows that $\frac{S_n}{\sqrt{n}}$ does not converge in either sense.

(Since $P\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} \geq M\right) \in \{0, 1\}$ by either of our $0 - 1$ laws, and the CLT shows that

$$P\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} \geq M\right) \geq \limsup_{n\to\infty} P\left(\frac{S_n}{\sqrt{n}} \geq M\right) = \frac{1}{\sqrt{2\pi}} \int_M^\infty e^{-\frac{x^2}{2}} dx > 0$$

for all $M > 0$, we see that $\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} = \infty$ a.s. This in turn implies that $\liminf_{n\to\infty} \frac{S_n}{\sqrt{n}} = -\infty$ a.s. by symmetry considerations, and the implication follows.)

Since a CLT argument also shows that $\frac{S_n}{\sqrt{2n\log(\log(n))}} \to_p 0$, the LIL can be interpreted as giving the scaling factor for which the almost sure limit and the limit in probability differ.

In order to prove Theorem 20.1, we first establish several lemmas which are of interest in their own right.

Our first is perhaps the easiest example of a *concentration inequality*, which enables one to obtain exponential rather than polynomial decay of certain tail probabilities by augmenting the Chebychev approach with a consideration of moment generating functions. We will only prove the result for the case of SRW, but the basic technique generalizes.

Lemma 20.1 (Bernstein)**.** *Let* $X_1, X_2, ...$ *be i.i.d. with* $P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}$*, and set* $S_n = \sum_{i=1}^n X_i$*. Then for any* $x > 0$*,*

$$P\left(|S_n| \geq x\right) \leq 2e^{-\frac{x^2}{2n}}.$$

Proof. By symmetry,

$$P\left(|S_n| \geq x\right) = P(S_n \geq x) + P(S_n \leq -x) = 2P(S_n \geq x).$$

Thus for any $t > 0$, we have

$$P(|S_n| \geq x) = 2P(S_n \geq x) = 2P\left(e^{tS_n} \geq e^{tx}\right)$$
$$\leq 2e^{-tx}E\left[e^{tS_n}\right] = 2e^{-tx}E\left[\prod_{i=1}^{n} e^{tX_i}\right]$$
$$= 2e^{-tx}E\left[e^{tX_1}\right]^n$$

by Chebychev and the fact that the $X_i's$ are i.i.d.

Now the moment generating function for X_1 satisfies

$$E\left[e^{tX_1}\right] = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh(t)$$
$$= \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{\left(\frac{t^2}{2}\right)^k}{\frac{(2k)!}{2^k}}$$
$$\leq \sum_{k=0}^{\infty} \frac{\left(\frac{t^2}{2}\right)^k}{k!} = e^{\frac{t^2}{2}},$$

so we have

$$P(|S_n| \geq x) \leq 2e^{-tx}E\left[e^{tX_1}\right]^n = 2e^{-tx}e^{\frac{nt^2}{2}}.$$

Taking $t = \frac{x}{n}$ yields

$$P(|S_n| \geq x) \leq 2e^{-\frac{x^2}{n}}e^{\frac{x^2}{2n}} = 2e^{-\frac{x^2}{2n}}.$$ □

Our next lemma concerns the distribution of the maximum absolute values of sums of i.i.d. random variables:

Lemma 20.2 (Lévy)**.** *Let $\xi_1, ...\xi_n$ be i.i.d., and set $S_n = \xi_1 + ...\xi_n$, $M_n = \max_{1 \leq k \leq n} |S_k|$. If there exist $\sigma > 0$, $\delta \in (0,1)$ such that $P\left(|S_k| \geq \frac{\sigma}{2}\right) \leq \delta$ for all $k = 1, ..., n$, then*

$$P(M_n \geq \sigma) \leq \frac{\delta}{1-\delta}.$$

Proof. Let $\tau = \inf\{k \in \mathbb{N} : |S_k| \geq \sigma\}$ be the hitting time of $\mathbb{R} \setminus (-\sigma, \sigma)$.

Then

$$P\left(M_n \geq \sigma, |S_n| < \frac{\sigma}{2}\right) = \sum_{k=1}^{n} P\left(\tau = k, |S_n| < \frac{\sigma}{2}\right)$$
$$\leq \sum_{k=1}^{n} P\left(\tau = k, |S_n - S_k| > \frac{\sigma}{2}\right)$$
$$= \sum_{k=1}^{n} P(\tau = k)\, P\left(|S_n - S_k| > \frac{\sigma}{2}\right)$$
$$= \sum_{k=1}^{n} P(\tau = k)\, P\left(|S_{n-k}| > \frac{\sigma}{2}\right)$$
$$\leq \sum_{k=1}^{n} P(\tau = k)\,\delta = \delta P(M_n \geq \sigma).$$

It follows that

$$P(M_n \geq \sigma) = P\left(M_n \geq \sigma, |S_n| < \frac{\sigma}{2}\right) + P\left(M_n \geq \sigma, |S_n| \geq \frac{\sigma}{2}\right)$$

$$\leq \delta P\left(M_n \geq \sigma\right) + P\left(|S_n| \geq \frac{\sigma}{2}\right) \leq \delta P\left(M_n \geq \sigma\right) + \delta,$$

and thus $(1-\delta)P(M_n \geq \sigma) \leq \delta$. $\square$

Our final ingredient is a lower bound on the upper tail probability for S_n.

Lemma 20.3. *For $k = k(n)$ satisfying $k, k + \frac{n}{k} \in o_{n\to\infty}\left(n^{\frac{2}{3}}\right)$, we have*

$$P(S_n \geq k) \geq C\frac{\sqrt{n}}{k}e^{-\frac{k^2}{2n}}$$

for some $C > 0$.

Proof. We first show that if $k \in o_{n\to\infty}\left(n^{\frac{2}{3}}\right)$ with $n+k$ even, then $P(S_n = k) \approx \sqrt{\frac{2}{\pi n}}e^{-\frac{k^2}{2n}}$. This follows by computing

$$\begin{aligned}
P(S_n = k) = \binom{n}{\frac{n+k}{2}}2^{-n} &\approx \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n 2^{-n}}{\sqrt{\pi\left(n+k\right)}\left(\frac{n+k}{2e}\right)^{\frac{n+k}{2}}\sqrt{\pi\left(n-k\right)}\left(\frac{n-k}{2e}\right)^{\frac{n-k}{2}}} \\
&= \sqrt{\frac{2n}{\pi(n^2-k^2)}}\cdot\frac{n^n}{\left(n+k\right)^{\frac{n+k}{2}}\left(n-k\right)^{\frac{n-k}{2}}} \\
&= \sqrt{\frac{2}{\pi n\left(1-\frac{k^2}{n^2}\right)}}\cdot\frac{1}{\left(1+\frac{k}{n}\right)^{\frac{n+k}{2}}\left(1-\frac{k}{n}\right)^{\frac{n-k}{2}}} \\
&= \sqrt{\frac{2}{\pi n\left(1-\frac{k^2}{n^2}\right)}}\exp\left[-\left(\frac{n+k}{2}\log\left(1+\frac{k}{n}\right)+\frac{n-k}{2}\log\left(1-\frac{k}{n}\right)\right)\right].
\end{aligned}$$

Using the Taylor bound $\log(1+x) = x - \frac{x^2}{2} + o(x^3)$ for $|x| < 1$, we have

$$\begin{aligned}
\frac{n+k}{2}\log\left(1+\frac{k}{n}\right)+\frac{n-k}{2}\log\left(1-\frac{k}{n}\right) &= \frac{n+k}{2}\left(\frac{k}{n}-\frac{k^2}{2n^2}+o\left(\frac{k^3}{n^3}\right)\right) \\
&\quad+\frac{n-k}{2}\left(-\frac{k}{n}-\frac{k^2}{2n^2}+o\left(\frac{k^3}{n^3}\right)\right) \\
&= \frac{k^2}{2n}+o\left(\frac{k^3}{n^2}\right),
\end{aligned}$$

hence

$$\begin{aligned}
P(S_n = k) &\approx \sqrt{\frac{2}{\pi n\left(1-\frac{k^2}{n^2}\right)}}\exp\left[-\left(\frac{n+k}{2}\log\left(1+\frac{k}{n}\right)+\frac{n-k}{2}\log\left(1-\frac{k}{n}\right)\right)\right] \\
&= \sqrt{\frac{2}{\pi n\left(1-\frac{k^2}{n^2}\right)}}\exp\left[-\left(\frac{k^2}{2n}+o\left(\frac{k^3}{n^2}\right)\right)\right] \\
&\approx \sqrt{\frac{2}{\pi n}}e^{-\frac{k^2}{2n}}.
\end{aligned}$$

(One can show that this result holds whenever $k \in o_{n\to\infty}\left(n^{\frac{3}{4}}\right)$ by taking one more term in the expansion of $\log(x)$, but the preceding is fine for our purposes and requires a bit less arithmetic.)
The above estimate implies that there is a $c > 0$ such that $P(S_n = k) \geq \frac{c}{\sqrt{n}}e^{-\frac{k^2}{2n}}$ for all $k \in o_{n\to\infty}\left(n^{\frac{2}{3}}\right)$ with $n+k$ even. (The asymptotic shows that $P(S_n = k) \geq \frac{1}{2\sqrt{n}}e^{-\frac{k^2}{2n}}$ for n sufficiently large, and there are only finitely many other values of $(n, k(n))$ to consider.)

If we further stipulate that $k+\frac{n}{k} \in o_{n\to\infty}\left(n^{\frac{2}{3}}\right)$, then the monotonicity of $f(j)=e^{-\frac{j^2}{2n}}$ gives

$$\begin{aligned}
P\left(S_n \geq k\right) &\geq P\left(k \leq S_n \leq k+\frac{n}{k}\right) \geq \frac{c}{\sqrt{n}} \sum_{j=k}^{\left\lfloor k+\frac{n}{k}\right\rfloor} e^{-\frac{j^2}{2n}} \\
&\geq \frac{c}{\sqrt{n}} \cdot \frac{n}{k} \exp\left[-\frac{1}{2n}\left(k+\frac{n}{k}\right)^2\right] \\
&= \frac{c\sqrt{n}}{k} \exp\left[-\frac{k^2}{2n}-\frac{n}{2k^2}-1\right] \\
&\geq C\frac{\sqrt{n}}{k} e^{-\frac{k^2}{2n}}.
\end{aligned}$$

□

The proof of Theorem 20.1 is now fairly straightforward.

Proof. Denote $\phi(n)=\sqrt{2n\log(\log(n))}$.

We will first show that $P(S_n \geq \phi(n)(1+\varepsilon) \text{ i.o.})=0$ for all $\varepsilon>0$ so that $\limsup_{n\to\infty} \frac{S_n}{\phi(n)} \leq 1$ a.s.

* Throughout the proof, we will consider the values of S_n and $\phi(n)$ along subsequences of the form $n_k=\left[a^k\right]$ for some $a>1$. For the sake of simplicity, we will just treat a^k as if it were always an integer and note that all of the arguments remain valid when we round to the nearest integer in either direction.

Let $\varepsilon>0$ be given. For every $a>1$, Lemma 20.1 implies

$$\begin{aligned}
P\left(S_{a^k} \geq \phi\left(a^k\right)\left(1+\frac{\varepsilon}{2}\right)\right) &\leq \exp\left[-\frac{1}{2a^k}\left(1+\frac{\varepsilon}{2}\right)^2\left(2a^k\log\left(\log\left(a^k\right)\right)\right)\right] \\
&= \exp\left[-\left(1+\frac{\varepsilon}{2}\right)^2\log\left(\log\left(a^k\right)\right)\right]=(k\log(a))^{-\left(1+\frac{\varepsilon}{2}\right)^2},
\end{aligned}$$

hence

$$\sum_{k=1}^{\infty} P\left(S_{a^k} \geq \phi\left(a^k\right)\left(1+\frac{\varepsilon}{2}\right)\right)<\infty,$$

and thus

$$P\left(S_{a^k} \geq \phi\left(a^k\right)\left(1+\frac{\varepsilon}{2}\right) \text{ i.o.}\right)=0$$

by the first Borel-Cantelli lemma.

If we take $a=1+\frac{\varepsilon^2}{32}$, then another application of Lemma 20.1 yields

$$\begin{aligned}
\delta_k :&= \max_{a^k+1\leq i\leq a^{k+1}} P\left(\left|S_i-S_{a^k}\right| \geq \frac{\varepsilon}{4}\phi\left(a^k\right)\right) \\
&\leq \max_{a^k+1\leq i\leq a^{k+1}} 2\exp\left[-\frac{1}{2\left(i-a^k\right)}\frac{\varepsilon^2}{16}\left(2a^k\log\left(\log\left(a^k\right)\right)\right)\right] \\
&= 2\exp\left[-\frac{1}{\left(a^{k+1}-a^k\right)}\frac{\varepsilon^2}{16}\left(a^k\log\left(k\log\left(a\right)\right)\right)\right] \\
&= 2\exp\left[-\frac{1}{\left(a-1\right)}\frac{\varepsilon^2}{16}\left(\log\left(k\log\left(a\right)\right)\right)\right] \\
&= 2\exp\left[-2\left(\log\left(k\log\left(a\right)\right)\right)\right]=\frac{2}{k^2\log(2a)}.
\end{aligned}$$

It follows from Lemma 20.2 with $\sigma_k = \frac{\varepsilon}{2}\phi\left(a^k\right)$ that

$$P\left(\max_{a^k+1\le i\le a^{k+1}} |S_i - S_{a^k}| \ge \frac{\varepsilon}{2}\phi\left(a^k\right)\right) \le \frac{\delta_k}{1-\delta_k} \le \frac{4}{k^2\log(2a)}$$

for $k > \frac{2}{\sqrt{\log(2a)}}$.

Since $\frac{4}{k^2\log(2a)}$ is summable, we have

$$P\left(\max_{a^k+1\le i\le a^{k+1}} |S_i - S_{a^k}| \ge \frac{\varepsilon}{2}\phi\left(a^k\right) \text{ i.o.}\right) = 0.$$

As we have already established that $P\left(S_{a^k} \ge \phi\left(a^k\right)\left(1+\frac{\varepsilon}{2}\right) \text{ i.o.}\right) = 0$ and ϕ is an increasing function, we conclude that

$$P(S_n \ge \phi(n)(1+\varepsilon) \text{ i.o.}) = 0.$$

For the other direction, we begin by noting that for any $a > 1$ we have

$$\lim_{k\to\infty} \frac{\phi\left(a^k\right)}{\phi\left(a^{k-1}\right)} = \sqrt{a}, \qquad \lim_{k\to\infty} \frac{\phi\left(a^k\right)}{\phi\left(a^k - a^{k-1}\right)} = \sqrt{\frac{a}{a-1}}.$$

Thus, given $\varepsilon > 0$, we can take a large enough that $\sqrt{\frac{a}{a-1}} < 1 + \frac{\varepsilon}{2}$ and $\sqrt{a} > \frac{4+\varepsilon}{2\varepsilon}$, hence

$$\left(1-\frac{\varepsilon}{2}\right)\phi\left(a^k - a^{k-1}\right) \ge \left(1-\frac{\varepsilon}{4}\right)\phi\left(a^k\right),$$
$$\left(1+\frac{\varepsilon}{4}\right)\phi\left(a^{k-1}\right) \le \frac{\varepsilon}{2}\phi\left(a^k\right)$$

for all large k.

We now observe that the random variables $R_k = S_{a^k} - S_{a^{k-1}}$, $k \in \mathbb{N}$, are independent with $R_k =_d S_{a^{k-1}(a-1)}$.

Applying Lemma 20.3with the above choice of a gives,

$$\begin{aligned}
P\left(R_k \ge \left(1-\frac{\varepsilon}{2}\right)\phi\left(a^k\right)\right) &= P\left(S_{a^{k-1}(a-1)} \ge \left(1-\frac{\varepsilon}{2}\right)\phi\left(a^k\right)\right) \\
&\ge P\left(S_{a^{k-1}(a-1)} \ge \left(1-\frac{\varepsilon}{4}\right)\phi\left(a^k\right)\right) \\
&\ge P\left(S_{a^{k-1}(a-1)} \ge \left(1-\frac{\varepsilon}{2}\right)\phi\left(a^k - a^{k-1}\right)\right) \\
&\ge C\frac{\sqrt{a^{k-1}(a-1)}}{\left(1-\frac{\varepsilon}{2}\right)\sqrt{2\left(a^k-a^{k-1}\right)\log\left(\log\left(a^k-a^{k-1}\right)\right)}} \\
&\qquad \times \exp\left[-\frac{2\left(a^k-a^{k-1}\right)\log\left(\log\left(a^k-a^{k-1}\right)\right)\left(1-\frac{\varepsilon}{2}\right)^2}{2a^{k-1}(a-1)}\right] \\
&= \frac{C'}{\sqrt{\log\left(\log\left(a^{k-1}(a-1)\right)\right)}} \cdot \log\left(a^{k-1}(a-1)\right)^{-\left(1-\frac{\varepsilon}{2}\right)^2} \\
&\ge \frac{C''}{k^{(1-\frac{\varepsilon}{2})^2}},
\end{aligned}$$

so $P\left(R_k \ge \left(1-\frac{\varepsilon}{2}\right)\phi\left(a^k\right) \text{ i.o.}\right) = 1$ by the second Borel-Cantelli lemma.

We have already shown that for all $\eta > 0$, $a > 1$
$$P\left(S_{a^k} \geq \phi\left(a^k\right)\left(1+\frac{\eta}{2}\right) \text{ i.o.}\right) = 0,$$
so the symmetry of the increments yields
$$\begin{aligned} P\left(S_{a^{k-1}} \leq -\left(1+\frac{\varepsilon}{4}\right)\phi\left(a^{k-1}\right) \text{ i.o.}\right) &= P\left(-S_{a^{k-1}} \geq \left(1+\frac{\varepsilon}{4}\right)\phi\left(a^{k-1}\right) \text{ i.o.}\right) \\ &= P\left(S_{a^{k-1}} \geq \left(1+\frac{\varepsilon}{4}\right)\phi\left(a^{k-1}\right) \text{ i.o.}\right) = 0. \end{aligned}$$
It follows that
$$\begin{aligned} S_{a^k} &= R_k + S_{a^{k-1}} \geq \left(1-\frac{\varepsilon}{2}\right)\phi\left(a^k\right) - \left(1+\frac{\varepsilon}{4}\right)\phi\left(a^{k-1}\right) \\ &\geq \left(1-\frac{\varepsilon}{2}\right)\phi\left(a^k\right) - \frac{\varepsilon}{2}\phi\left(a^k\right) = (1-\varepsilon)\,\phi\left(a^k\right) \end{aligned}$$
for infinitely many k with full probability, and the proof is complete. □