Math 6710, Fall 2016
Notes supplement: Tightness

This is another supplement to the lecture notes, covering Theorem 11.6 but going into slightly more detail.

Theorem. *Suppose the sequence of distribution functions F_n converges vaguely to the nondecreasing right-continuous function F. Then F is a distribution function (that is, $F(x) \to 0$ as $x \to -\infty$ and $F(x) \to 1$ as $x \to \infty$) if and only if the sequence F_n is tight.*

Proof. Backward direction: Fix $\varepsilon > 0$. By tightness of F_n, there is a constant M_ε such that

$$\limsup_{n\to\infty} \Big[1 - F_n(M_\varepsilon) + F_n(-M_\varepsilon)\Big] \le \varepsilon.$$

Let $r < -M_\varepsilon$ and $s > M_\varepsilon$ be continuity points of F. Then $F_n(r) \to F(r)$ and $F_n(s) \to F(s)$, so

$$1 - F(s) + F(r) = \lim_{n\to\infty} \Big[1 - F_n(s) + F_n(r)\Big] \le \varepsilon.$$

It follows that

$$\lim_{x\to\infty} (1 - F(x)) + \lim_{x\to-\infty} F(x) \le \varepsilon.$$

Sending $\varepsilon \to 0$, we conclude that F is a distribution function.

Forward direction: Suppose F_n is not tight. Then there exists $\varepsilon > 0$ such that for every positive integer m,

$$\limsup_{n\to\infty} \Big[1 - F_n(m) + F_n(-m)\Big] > \varepsilon.$$

It follows that we can choose a subsequence $F_{n(m)}$ such that each

$$1 - F_{n(m)}(m) + F_{n(m)}(-m) > \varepsilon.$$

The subsequence $F_{n(m)}$ still converges vaguely to F. Let $r < 0 < s$ be continuity points of F, so that

$$1 - F(s) + F(r) = \lim_{m\to\infty} \Big[1 - F_{n(m)}(s) + F_{n(m)}(r)\Big] \ge \liminf_{m\to\infty} \Big[1 - F_{n(m)}(m) + F_{n(m)}(-m)\Big] \ge \varepsilon.$$

Sending $r \to -\infty$ and $s \to \infty$ along continuity points of F shows that F cannot be a distribution function. □

Theorem (Theorem 11.6 in notes)**.** *A sequence of distribution functions is tight if and only if every subsequential vague limit is a distribution function.*

Proof. Forward direction: If F_n is tight and the subsequence $F_{n(m)}$ converges vaguely to F, then since $F_{n(m)}$ is also tight, the previous theorem implies that F is a distribution function.

Backward direction: Suppose F_n is not tight. As in the previous proof, we can find $\varepsilon > 0$ and a subsequence $F_{n(m)}$ such that each

$$1 - F_{n(m)}(m) + F_{n(m)}(-m) > \varepsilon.$$

In particular, no sub-subsequence $F_{n(m_k)}$ of $F_{n(m)}$ is tight. Helly's Selection Theorem implies that some sub-subsequence has a vague limit, which by the previous theorem cannot be a distribution function. □