Math 6710, Fall 2016
Notes supplement: Weak convergence

This supplement to the lecture notes proves two results about the relationship between weak convergence and convergence in probability. Recall that if $X_1, X_2, \ldots$ and X are random variables, $X_n \to_p X$ means that X_n converges to X in probability, while $X_n \Rightarrow X$ means that X_n converges weakly to X.

Proposition. *Let $X_1, X_2, \ldots$ be defined on the same probability space. If $c \in \mathbf{R}$ is fixed, then $X_n \Rightarrow c$ if and only if $X_n \to_p c$.*

Proof. First, $X_n \Rightarrow c$ if and only if $P(X_n \leq x)$ converges to 0 for all $x < c$ and to 1 for all $x > c$. Equivalently, for every $\varepsilon > 0$, $P(X_n \leq c - \varepsilon) \to 0$ and $P(X_n \geq c + \varepsilon) \to 0$. This is the same as saying that $P(|X_n - c| \geq \varepsilon) \to 0$ for every $\varepsilon > 0$, which is the definition of convergence in probability of X_n to c. $\square$

In general, if $X_1, X_2, \ldots$ and X are defined on the same probability space, convergence in probability of X_n to X is a stronger condition than weak convergence. For a simple example where $X_n \Rightarrow X$ but $X_n \not\to_p X$, let Y be a Uniform($[0,1]$) random variable. Set $X_n = Y$ for all n and set $X = 1 - Y$. The distributions of the X_n and X are identical, so $X_n \Rightarrow X$, but $X_n(\omega)$ is never particularly close to $X(\omega)$ (except when both are $1/2$).

Proposition. *Let $X_1, X_2, \ldots$ and X be defined on the same probability space. If $X_n \to_p X$, then $X_n \Rightarrow X$.*

Proof. Fix $x \in \mathbf{R}$ at which the distribution function of X is continuous, so $P(X = x) = 0$. We want that $P(X_n \leq x) \to P(X \leq x)$. For all $\varepsilon > 0$, if $X > x + \varepsilon$ and $|X_n - X| \leq \varepsilon$ then $X_n > x$. Thus, by a union bound,

$$P(X_n \leq x) \leq P(X \leq x + \varepsilon) + P(|X_n - X| > \varepsilon).$$

Take the limsup as $n \to \infty$. The last term tends to zero because $X_n \to_p X$:

$$\limsup_{n\to\infty} P(X_n \leq x) \leq P(X \leq x + \varepsilon).$$

Since this holds for all $\varepsilon > 0$, by continuity from above we have

$$\limsup_{n\to\infty} P(X_n \leq x) \leq P(X \leq x).$$

A similar union bound gives that for all $\varepsilon > 0$,

$$P(X_n > x) \leq P(X \geq x - \varepsilon) + P(|X_n - X| > \varepsilon).$$

Taking the limsup as $n \to \infty$ and then sending ε to zero,

$$\limsup_{n\to\infty} P(X_n > x) \leq P(X \geq x).$$

Subtracting both sides from 1 gives

$$\liminf_{n\to\infty} P(X_n \leq x) \geq P(X < x) = P(X \leq x)$$

where the last equality is due to the continuity of the distribution function of X at x. In conclusion,

$$\lim_{n\to\infty} P(X_n \leq x) = P(X \leq x)$$

as desired. $\square$