

ACYLINDRICALLY HYPERBOLIC GROUPS AND THEIR QUASI-ISOMETRICALLY EMBEDDED SUBGROUPS

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ABSTRACT. We abstract the notion of an A/QI triple from a number of examples in geometric group theory. Such a triple (G, X, H) consists of a group G acting on a Gromov hyperbolic space X , acylindrically along a finitely generated subgroup H which is quasi-isometrically embedded by the action. Examples include strongly quasi-convex subgroups of relatively hyperbolic groups, convex cocompact subgroups of mapping class groups, many known convex cocompact subgroups of $\text{Out}(F_n)$, and groups generated by powers of independent loxodromic WPD elements of a group acting on a Gromov hyperbolic space. We initiate the study of intersection and combination properties of A/QI triples. Under the additional hypothesis that G is finitely generated, we use a method of Sisto to show that H is stable. We apply theorems of Kapovich–Rafi and Dowdall–Taylor to analyze the Gromov boundary of an associated cone-off. We close with some examples and questions.

1. INTRODUCTION

There are two important threads in the study of group actions on hyperbolic spaces which we bring together in this paper. One is the topic of quasi-convex subgroups of hyperbolic groups. Gromov introduced these in [Gro87] as the geometrically natural subgroups of a hyperbolic group, and they have been intensively studied since then. Without pretending to give a full literature review, we mention in particular some previous work closely related to what we do here. Gitik–Mitra–Rips–Sageev study finiteness and intersection properties of quasi-convex subgroups and show that they have finite height and width [GMRS98]. Bowditch shows that cone-offs of hyperbolic spaces by families of quasi-convex subspaces are themselves hyperbolic [Bow12]. Other important early papers related to our current work are [Sho91, Swe01].

The other thread is that of acylindrical (and weakly properly discontinuous) actions of groups on hyperbolic spaces. Acylindrical actions on *trees* were studied first by Sela [Sel97]. Bowditch formulated the acylindricity condition for actions on hyperbolic spaces in [Bow08], showing in particular that the action of the mapping class group of a surface on the curve complex is acylindrical. Osin in [Osi16] shows that the *existence* of an acylindrical action of a group G on some hyperbolic space is equivalent to the existence of a hyperbolically embedded subgroup, or of a weakly properly discontinuous element for the action of G on some possibly different hyperbolic space, clarifying the existence of an extremely natural class of *acylindrically hyperbolic groups*. For such a group, Abbott–Balasubramanya–Osin explore the poset of acylindrical actions on hyperbolic spaces in [ABO19]; this poset is further explored in [Abb16, ABB⁺17, Bal20, Val21].

In this paper we are interested in understanding actions of a group on hyperbolic spaces which are not necessarily acylindrical, but satisfy the weaker condition of being acylindrical along a finitely generated subgroup which is assumed to be quasi-isometrically embedded by the action.

`def:acyl`

Definition 1.1. Suppose (X, d) is a Gromov hyperbolic geodesic metric space, and $Y \subseteq X$. An isometric action $G \curvearrowright X$ is *acylindrical along* Y if for every $\varepsilon \geq 0$, there are $R = R(\varepsilon, Y) > 0, M = M(\varepsilon, Y) > 0$ so that, whenever $x, y \in Y$ satisfy $d(x, y) \geq R$,

`eq:acyldef`

$$(1) \quad \#\{g \in G \mid d(gx, x) \leq \varepsilon \text{ and } d(gy, y) \leq \varepsilon\} \leq M.$$

While the constants R, M are sometimes called the constants of acylindricity of the action, we reserve this terminology for the constants in an alternate characterization of acylindricity which is easier to check in our context (see Definition 3.1). If $G \curvearrowright X$ is acylindrical along Y , then it is acylindrical along W for any W which is finite Hausdorff distance from Y . It is also acylindrical along any translate gY . This notion interpolates between an acylindrical action and an action with a loxodromic WPD element (see Definition 4.6), in the following sense. The usual notion of an acylindrical action $G \curvearrowright X$ is one which is acylindrical along the entire space X , while an action $G \curvearrowright X$ with a loxodromic WPD element g is acylindrical along any axis for g .

If $H < G$, we say that $G \curvearrowright X$ is *acylindrical along* H if it is acylindrical along Hx for some (equivalently any) $x \in X$. The main new definition of the current paper is the following:

Definition 1.2. An *A/QI triple* (G, X, H) consists of a Gromov hyperbolic space X and an action $G \curvearrowright X$ which is acylindrical along the finitely generated infinite subgroup H , and so that $x \mapsto hx$ gives a quasi-isometric embedding of H into X .

Whenever this final condition is satisfied, we say that H is *quasi-isometrically embedded by the action*.

For example, if G is hyperbolic, $H < G$ is quasi-convex, and Γ is a Cayley graph of G , then (G, Γ, H) is an A/QI triple. More interesting motivating examples are convex cocompact subgroups of mapping class groups. In this case, if H is a convex cocompact subgroup of a mapping class group $\text{Mod}(\Sigma)$ of a non-exceptional surface Σ , and \mathcal{C} is the curve complex of Σ , then $(\text{Mod}(\Sigma), \mathcal{C}, H)$ is an A/QI triple. In this situation, the action of G on \mathcal{C} is a genuine acylindrical action. Other motivating examples include certain convex cocompact subgroups H of $\text{Out}(F_n)$, the outer automorphism group of the free group of rank n . Indeed for many subgroups H , if X is the free factor complex, the cyclic splitting complex, or the free splitting complex, then $(\text{Out}(F_n), X, H)$ is an A/QI triple (see Example 7.1). None of these actions $\text{Out}(F_n) \curvearrowright X$ are known to be acylindrical, and in the case where X is the free splitting complex, it is known that the action is not acylindrical. That these examples are A/QI triples is an application of a general construction of A/QI triples from an action of a group on a hyperbolic space with a loxodromic WPD element (Corollary 4.8). We use this construction to give additional examples of A/QI triples in the Cremona group and FC-type Artin groups (see Examples 7.2 and 7.3), neither of which have known *natural* acylindrical actions on hyperbolic spaces.

rmk:sisto

Remark 1.3. In [Sis16], Sisto gives a more flexible definition of an action being acylindrical along a subspace Y , omitting the number M and replacing the inequality “ $\leq M$ ” in (1) by “ $< \infty$ ”. We will refer to an action satisfying this weaker condition as being *weakly acylindrical along* Y . In general an action can be weakly acylindrical along a subspace without being acylindrical along that subspace. However if X is hyperbolic, Y is a quasi-convex subspace, and $d_X|_Y$ is a proper metric, then any action which is weakly acylindrical along Y is acylindrical along Y . In particular, if $Y = Hx$ and H is quasi-isometrically embedded by the action, then the action is acylindrical along Y if and only if it is weakly acylindrical along Y .

In this paper, we study three aspects of A/QI triples (G, X, H) : intersection and combination properties, stability of the subgroup H , and the boundary of a certain coned-off space of X constructed from (G, X, H) .

Motivated by the fact that the intersection of two quasi-convex subgroups of a hyperbolic group is quasi-convex, we first consider the intersection of A/QI triples. We show that if (G, X, H) and (G, X, K) are A/QI triples, then $(G, X, H \cap K)$ is also an A/QI triple (Proposition 4.1). We also prove the following combination theorem for A/QI triples. (Here we use the notation $\Lambda(H)$ for the *limit set* of H ; see Definition 2.11.)

thmt@combination@data

thhh@combination

Theorem 1.4. *Let G act on a hyperbolic space X , and suppose H_1, \dots, H_k are subgroups of G such that (G, X, H_i) is an A/QI triple for each $i = 1, \dots, k$ and $\Lambda(H_i) \cap \Lambda(H_j) = \emptyset$ for all $i \neq j$. Then there exists a finite collection of elements $\mathcal{S} \subset \cup_{i=1}^k H_i \setminus \{1\}$ such that for any family of subgroups $\mathcal{K} = \{K_i\}_{i=1}^k$ with K_i quasi-isometrically embedded in H_i and satisfying $K_i \cap \mathcal{S} = \emptyset$ for all i , we have $K = \langle \bigcup \mathcal{K} \rangle \cong K_1 * K_2 * \dots * K_k$ and (G, X, K) is an A/QI triple.*

Our first finiteness result is that for any A/QI triple (G, X, H) , the subgroup H has *finite height* in G (see Proposition 4.11). Finite height is a weakening of malnormality which has proved extremely useful in studying residual finiteness and cubulability of (relatively) hyperbolic groups. See for example [AGM09, Ago13, HW14].

Restricting our attention to A/QI triples (G, X, H) where G is finitely generated, we obtain a stronger result.

thmt@stable@data

thhh@stable

Theorem 1.5. *Let (G, X, H) be an A/QI triple, and suppose that G is finitely generated. Then H is stable in G .*

Stability is a strong quasi-convexity condition introduced by Durham–Taylor [DT15] (see Definition 5.1). By work of Antolín–Mj–Sisto–Taylor [AMST19], stability implies finite height as well as the related properties of *finite width* and *bounded packing*. Our arguments here are based on those of Sisto in [Sis16], where he proves a stability result for hyperbolically embedded subgroups. Note that in our setting, the subgroup H may not be hyperbolically embedded in G . In particular, hyperbolically embedded subgroups have height at most one, whereas subgroups that are part of an A/QI triple often have height greater than one.

For our investigation of the boundary, we return to the setting of a general A/QI triple (G, X, H) . For such a triple there is an associated action $G \curvearrowright \hat{X}$, where \hat{X} is a hyperbolic space obtained from X by equivariantly coning off the translates of an H -orbit. (In case X is the Cayley graph of G , this is the *coned-off* or *electrified*

Cayley graph of (G, H) studied, for example, in [Bow12, Far98].) We characterize the boundary of the cone-off \hat{X} as follows.

thmt@aqiboundary@data

thhh@aqiboundary

Theorem 1.6. *Suppose (G, X, H) is an A/QI triple and fix $x \in X$. Let $\mathcal{H} = \{gHx \mid gH \in G/H\}$. Let \hat{X} be the cone-off of X with respect to \mathcal{H} . Then $\partial\hat{X}$ is homeomorphic to the subspace of ∂X obtained by deleting the limit sets of all the conjugates of H .*

In [Ham16], Hamenstädt proves an analogous result to Theorem 1.6 in the case that the hyperbolic space X is strongly hyperbolic (as a metric space) relative to a collection of subspaces \mathcal{H} , without requiring the existence of a group action.

We use this description of $\partial\hat{X}$ to understand the geometry of the action $G \curvearrowright \hat{X}$. In Corollary 6.15, we classify the elliptic, loxodromic, and parabolic elements for this action. In Proposition 6.16, we describe the loxodromic WPD elements of $G \curvearrowright \hat{X}$; in particular we can show (Corollary 6.17) if $G \curvearrowright X$ is a WPD action, then so is $G \curvearrowright \hat{X}$.

Outline. In Section 2, we recall some basic facts and fix some notation. In Section 3 we give an alternative characterization of acylindricity along a subspace. The following three sections can be read independently. In Section 4, we prove some properties of A/QI triples. In particular, we give short arguments establishing certain intersection properties of A/QI triples (see Propositions 4.1 and 4.11). We also prove Theorem 1.4. In Section 5, we recall the definition of stable subgroups and prove Theorem 1.5. In Section 6, we characterize the boundary of the coned-off space \hat{X} by proving Theorem 1.6, along with the associated results about the action $G \curvearrowright \hat{X}$. Finally, in Section 7, we use the combination theorem from Section 4 to describe some further examples of A/QI triples. Finally we pose several questions in Section 8.

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sec:prelim

2. PRELIMINARIES

In this section we recall some basic notions and fix some notation regarding Gromov products and the boundary of a hyperbolic space. For more detail see, for example, [BH99, III.H].

Definition 2.1. Let (X, d_X) be a hyperbolic metric space with basepoint x_0 . For any $x, y \in X$, the *Gromov product of x and y* is

$$(x \mid y)_{x_0} = \frac{1}{2} (d_X(x_0, x) + d_X(x_0, y) - d_X(x, y)).$$

The following observation can be found, for example in Bridson–Haefliger [BH99, p. 410].

Lemma 2.2. *For any δ -hyperbolic space X and any $x, y \in X$, we have*

$$|(x | y)_{x_0} - d_X(x_0, [x, y])| \leq \delta,$$

where $[x, y]$ is any geodesic from x to y and x_0 is a fixed basepoint.

A sequence of points (x_k) in X converges to infinity if $\lim_{n,k \rightarrow \infty} (x_k | x_n)_{x_0} = \infty$. An equivalence relation \sim on sequences $(x_k), (y_k)$ in X which converge to infinity is defined as follows: $(x_k) \sim (y_k)$ if and only if $\lim_{k \rightarrow \infty} (x_k | y_k)_{x_0} = \infty$ if and only if $\lim_{n,k \rightarrow \infty} (x_k | y_n)_{x_0} = \infty$.

Definition 2.3. The Gromov boundary ∂X of a hyperbolic metric space X is the set of equivalence classes of sequences in X which converge to infinity.

The Gromov product extends to $\bar{X} = X \cup \partial X$ by the formula

eq:gprodinfy

$$(2) \quad (x | y)_{x_0} = \sup \{ \liminf_{i,j \rightarrow \infty} (x_i | y_j)_{x_0} \mid x_i \rightarrow x, y_i \rightarrow y, \text{ and } x_i, y_i \in X \},$$

for all $x, y \in \bar{X}$. The union \bar{X} is topologized by declaring that a sequence (x_n) in \bar{X} converges to a point $x \in \partial X$ if and only if

$$\lim_{n \rightarrow \infty} (x_n | x)_{x_0} = \infty.$$

Although convergence to infinity, equivalence, and the Gromov boundary are defined using the basepoint x_0 , they are actually independent of that basepoint.

The boundary can also be thought of in terms of quasi-geodesic rays, which are quasi-isometric embeddings of $[0, \infty)$. To simplify notation we will bundle together all the constants describing the quality of such a quasi-isometric embedding (and hence of such a quasi-geodesic) into a single number.

def:quasi-geodesic

Definition 2.4. Let $\tau \geq 1$. A map $f: (X, d_X) \rightarrow (Y, d_Y)$ of metric spaces is a τ -quasi-isometric embedding if for all $x, y \in X$, we have $\tau^{-1}d_X(x, y) - \tau \leq d_Y(f(x), f(y)) \leq \tau d_X(x, y) + \tau$.

A τ -quasi-geodesic in X is a τ -quasi-isometric embedding of an interval into X , that is, a map $\gamma: I \rightarrow X$ where $I \subseteq \mathbb{R}$ is connected and so that for all $s, t \in I$, $\tau^{-1}|s - t| - \tau \leq d_X(\gamma(s), \gamma(t)) \leq \tau|s - t| + \tau$.

If the map $\gamma: I \rightarrow X$ is continuous, then we denote the length of the quasi-geodesic γ in X by $\ell_X(\gamma)$. Given a discrete path (list of points) $\sigma = x_0, \dots, x_k$ in a metric space X , we define its length to be $\ell^0(\sigma) = \sum_{i=0}^{k-1} d_X(x_i, x_{i+1})$.

It is occasionally convenient to assume that a quasi-geodesic γ is *tame* in the sense of [BH99, III.H.1.11]. When γ is tame, the map $\gamma: I \rightarrow X$ is continuous, and we additionally have $\ell_X(\gamma|_{[s,t]}) \leq \tau d_X(\gamma(s), \gamma(t)) + \tau$ for all $s, t \in I$. Quasi-geodesics of uniform quality are uniformly closely fellow-traveled by tame quasi-geodesics (see [BH99, III.H.1.11]). In this paper we will *not* assume quasi-geodesics are tame unless we explicitly state otherwise.

The following ‘‘Morse’’ property of quasi-geodesics in hyperbolic spaces is well-known (see e.g. [BH99, III.H.1.7]).

thm:stability

Theorem 2.5. *For all $\delta \geq 0, \tau \geq 1$, there is an $M = M(\tau, \delta) \geq 0$ satisfying the following. Let X be a δ -hyperbolic space, and let γ_1, γ_2 be τ -quasi-geodesics with the same endpoints in $X \cup \partial X$. Then the Hausdorff distance between γ_1 and γ_2 is at most M .*

Definition 2.6. Any number M as in Theorem 2.5 will be called a *Morse constant* for the parameters τ, δ .

The following lemma follows from the 2δ -slimness of geodesic quadrilaterals in a δ -hyperbolic space.

lem:unifslimquads

Lemma 2.7. *Let X be a δ -hyperbolic space, let a, b, a', b' be points of X , let γ be a geodesic from a to b , and let γ' be a geodesic from a' to b' . Let $\varepsilon \geq \max\{d_X(a, a'), d_X(b, b')\}$ and suppose that $d_X(a, b) > 2\varepsilon + 2\delta$. Fix points x, y on γ that are distance exactly ε from a and b , respectively. If x', y' are the closest points on γ' to x and y , respectively, then the Hausdorff distance between the subsegment of γ between x and y , and the subsegment of γ' between x' and y' is at most 6δ .*

Proof. We choose geodesics $[a, a']$ and $[b, b']$. Our choice of points x, y on γ ensures that $d_X(x, x') \leq 4\delta$ and $d_X(y, y') \leq 4\delta$. To see this, notice that there is a point z on $[a, a'] \cup \gamma' \cup [b, b']$ at distance at most 2δ from x . If z lies on γ' , then we are done. The lower bound on $d_X(a, b)$ implies that z lies on $[a, a']$. We have $d_X(z, a') = d_X(a, a') - d_X(z, a) \leq \varepsilon - (\varepsilon - 2\delta) = 2\delta$. In this case $d_X(a', x) \leq 4\delta$. An analogous argument shows this holds for y . The result then follows from the 2δ -slimness of the geodesic quadrilateral with vertices x, y, x' , and y' . \square

Definition 2.8. Let $\kappa \geq 0$. A subset Y of a metric space X is κ -*quasi-convex* if every geodesic in X connecting points of Y is contained in the κ -neighborhood of Y .

Given a quasi-geodesic ray γ in X starting at x_0 , we obtain an equivalence class $[\gamma] \in \partial X$ by $[\gamma] := [(a_i)]$ for any sequence of points a_i on γ with $\lim_{i \rightarrow \infty} d_X(x_0, a_i) = \infty$.

rem:visibleboundary

Remark 2.9. By [KB02, Remark 2.16], there is a constant τ_0 depending only on the hyperbolicity constant of X such that given any point $\xi \in \partial X$, there is a τ_0 -quasi-geodesic ray γ_ξ in X starting at x_0 such that $[\gamma_\xi] = \xi$.

The Gromov product at infinity controls the Gromov product between points on quasi-geodesics, in the following sense.

lem:gprodcontrol

Lemma 2.10. *For any $\tau \geq 1$, $\delta \geq 0$ there is some $E \geq 0$ so that the following holds.*

Let X be δ -hyperbolic, $x_0 \in X$, and $\alpha, \beta \in \partial X$. Let γ_α and γ_β be τ -quasi-geodesics starting at x_0 and tending to α, β respectively. Then for all $s, t \geq 0$,

$$(\gamma_\alpha(s) \mid \gamma_\beta(t))_{x_0} \leq (\alpha \mid \beta)_{x_0} + E.$$

Proof. We fix some $s, t \geq 0$. From the definition of Gromov product at infinity (2), we see

$$(\alpha \mid \beta)_{x_0} \geq \liminf_{i, j \rightarrow \infty} (\gamma_\alpha(i) \mid \gamma_\beta(j))_{x_0}.$$

Choosing $S \geq s, T \geq t$ sufficiently large, we can assume

$$(\alpha \mid \beta)_{x_0} \geq (\gamma_\alpha(S) \mid \gamma_\beta(T))_{x_0} - 1$$

$$\text{eq:gprodineq} \quad (3) \quad = \frac{1}{2} (d_X(x_0, \gamma_\alpha(S)) + d_X(x_0, \gamma_\beta(T)) - d_X(\gamma_\alpha(S), \gamma_\beta(T)) - 1$$

Let M be a Morse constant for parameters τ and δ . By the Morse property of quasi-geodesics we have $d_X(x_0, \gamma_\alpha(S)) \geq d_X(x_0, \gamma_\alpha(s)) + d_X(\gamma_\alpha(s), \gamma_\alpha(S)) - 2M$ and $d_X(x_0, \gamma_\beta(T)) \geq d_X(x_0, \gamma_\beta(t)) + d_X(\gamma_\beta(t), \gamma_\beta(T)) - 2M$. By the triangle inequality,

$$d_X(\gamma_\alpha(S), \gamma_\beta(T)) \leq d_X(\gamma_\alpha(S), \gamma_\alpha(s)) + d_X(\gamma_\beta(T), \gamma_\beta(t)) + d_X(\gamma_\alpha(s), \gamma_\beta(t)).$$

These three inequalities together with (3) yield

$$(\alpha | \beta)_{x_0} \geq (\gamma_\alpha(s) | \gamma_\beta(t))_{x_0} - (2M + 1),$$

establishing the lemma. \square

Given a group acting by isometries on a δ -hyperbolic space, we define certain subsets of the boundary which arise as limits of sequences of elements from an orbit of a fixed base point.

`def:limitset`

Definition 2.11. For Y a subset of a δ -hyperbolic space X , we write $\Lambda(Y)$ for the *limit set* of Y . This is

$$\Lambda(Y) := \{p \in \partial X \mid p = \lim_{n \rightarrow \infty} y_n \text{ for some sequence } y_n \in Y\}.$$

If we are given a group G acting by isometries on X , we define the limit set $\Lambda(T)$ of a subset T of G to be $\Lambda(\{tx_0 \mid t \in T\})$ where x_0 is some base point. (The limit set does not depend on this base point.)

`sec:newacyl`

3. AN ALTERNATIVE CHARACTERIZATION OF ACYLINDRICITY

When a subgroup $H < G$ is quasi-isometrically embedded by an action $G \curvearrowright X$ on a hyperbolic space, there is an alternate characterization of acylindricity of $G \curvearrowright X$ along H . In practice, this alternate characterization is easier to check.

`def:newacyl`

Definition 3.1. Let G act by isometries on a hyperbolic metric space X , and let H be a subgroup that is quasi-isometrically embedded by the action. The action of G on X is *acylindrical along H* if for some (equivalently any) H -cobounded $Y \subset X$ the following condition holds: For any $\varepsilon \geq 0$, there exist constants $D = D(\varepsilon, Y)$ and $N = N(\varepsilon, Y)$ such that any collection of distinct cosets $\{g_i H, \dots, g_k H\}$ such that

$$\text{diam} \left(\bigcap_{i=1}^k \mathcal{N}_\varepsilon(g_i Y) \right) > D$$

has cardinality at most N .

For a fixed $\varepsilon > 0$ and H -cobounded $Y \subset X$, we call the constants $D(\varepsilon, Y), N(\varepsilon, Y)$ the *constants of acylindricity* of the action of G on X along H .

The following theorem shows that this definition is equivalent to Definition 1.1.

`thm:newacyl`

Theorem 3.2. *Let G act by isometries on a hyperbolic space X , and let H be a subgroup that is quasi-isometrically embedded by the action. Let Y be an H -cobounded subset of X . Then the action of G is acylindrical along H in the sense of Definition 1.1 if and only if it is acylindrical along H in the sense of Definition 3.1.*

Proof. Let $x \in X$, and fix $Y = Hx$.

We prove the backward direction first and assume that the action of G on X is acylindrical along H in the sense of Definition 3.1. Fix $\varepsilon \geq 0$, and let $D = D(2\varepsilon, Y)$ and $N = N(2\varepsilon, Y)$. Since H is quasi-isometrically embedded by the action, there exists a constant $P > 0$ such that there are at most P elements $h \in H$ such that $d_X(x, hx) \leq 2\varepsilon$.

Fix $h_1, h_2 \in H$ satisfying $d_X(h_1x, h_2x) > D$, and let

$$\Delta = \{g \in G \mid d_X(h_1x, gh_1x) \leq \varepsilon \text{ and } d_X(h_2x, gh_2x) \leq \varepsilon\}.$$

We claim that $\#\Delta \leq NP$. Towards a contradiction, assume this is not the case, so that there exist distinct $\{g_1, \dots, g_k\} \subset \Delta$ with $k > NP$. Then

$$\text{diam} \left(\bigcap_{i=1}^k \mathcal{N}_\varepsilon(g_i Hx) \right) > D.$$

By assumption, the number of *distinct cosets* in $\{g_1H, \dots, g_kH\}$ is at most N . Rearranging the list, we can therefore assume that the first $(P+1)$ cosets are the same. In other words $g_j^{-1}g_1 \in H$ for all $j \leq P+1$. Since the g_j are all in Δ we have $d_X(h_1x, g_jh_1x) \leq \varepsilon$ for each j . Let $\hat{g}_j = h_1^{-1}g_j^{-1}g_1h_1$, and note that $\hat{g}_j \in H$ for $j \leq P+1$. Moreover

eq:hatbound

$$(4) \quad d_X(x, \hat{g}_jx) = d_X(g_1h_1x, g_jh_1x) \leq 2\varepsilon.$$

There are only P elements of H satisfying (4), so we have $\hat{g}_j = \hat{g}_i$ for some $i < j \leq P+1$. But this implies $g_j = g_i$, which is a contradiction. We have therefore shown the action is acylindrical along Y in the sense of Definition 1.1, with $R = D$ and $M = NP$.

We now prove the forward direction and assume the action is acylindrical along H in the sense of Definition 1.1. Let $\varepsilon > 0$, let δ be a hyperbolicity constant for X , and suppose H is τ -quasi-isometrically embedded by the orbit map. The Morse property of quasi-geodesics in hyperbolic spaces implies that there is a constant λ depending only on τ and δ' such that the orbit Hx is λ -quasi-convex in X . Let $R = R(\varepsilon')$, $M = M(\varepsilon')$, be the constants provided by Definition 1.1, where

$$\varepsilon' = 4(\varepsilon + \delta + \lambda).$$

We will show that Definition 3.1 holds with constants

$$D = D(\varepsilon, Y) = R + \varepsilon' + \varepsilon, \quad N = N(\varepsilon, Y) = MB + 1,$$

where B is the number of elements h of H with $d_X(hx, x) \leq R + \varepsilon'$.

Suppose that g_0H, \dots, g_nH are distinct cosets with

$$\text{diam} \left(\bigcap_{i=0}^n \mathcal{N}_\varepsilon(g_i Hx) \right) > D.$$

We may translate this intersection so that it contains points close to x and to some $kx \in Hx$ which is distance at least $D - \varepsilon$ from the identity. More specifically, we can assume the following, possibly after changing the coset representatives g_i :

- (1) $g_0 = 1$;
- (2) $d_X(g_i x, x) \leq 2\varepsilon$ for each i ; and
- (3) there is some $k \in H$ so that $d_X(x, kx) > D - \varepsilon$ and so that for each i there is some $k_i \in H$ so that $d_X(g_i k_i x, kx) \leq 2\varepsilon$.

Let a lie on a geodesic from x to kx so that $d_X(x, a) = R + 2(\delta + \lambda + \varepsilon)$. By quasi-convexity of Hx , there is some $h_0 \in H$ with $d_X(h_0x, a) \leq \lambda$.

Let $i \leq n$. Since $d_X(a, \{x, kx\}) \geq \varepsilon'/2 > 2\varepsilon + 2\delta$, the 2δ -slimness of quadrilaterals implies that there is some a_i on a geodesic from $g_i x$ to $g_i k_i x$ so that $d_X(a, a_i) \leq 2\delta$. Again using quasi-convexity, there is some $h_i \in H$ with $d_X(g_i h_i x, a_i) \leq \lambda$. For each i , we have

$$d_X(h_0x, g_i h_i x) \leq 2\lambda + 2\delta.$$

Using the triangle inequality, for each i we also have

$$\begin{aligned} d_X(x, h_i x) &= d_X(g_i x, g_i h_i x) \\ &\geq d_X(x, a) - (d_X(x, g_i x) + d_X(a, a_i) + d_X(a_i, g_i h_i x)) \\ &\geq R + 2(\delta + \lambda + \varepsilon) - (2\varepsilon + 2\delta + \lambda) \\ &= R + \lambda, \end{aligned}$$

and

$$\begin{aligned} d_X(g_i x, g_i h_i x) &\leq d_X(x, a) + d_X(x, g_i x) + d_X(a, a_i) + d_X(a_i, g_i h_i x) \\ &\leq R + 2(\delta + \lambda + \varepsilon) + 2\varepsilon + 2\delta + \lambda \\ &= R + 4\varepsilon + 4\delta + 3\lambda. \end{aligned}$$

Putting these together, we obtain

eq:distinterval

$$(5) \quad d_X(x, h_i x) \in [R + \lambda, R + 4\varepsilon + 4\delta + 3\lambda].$$

There are at most B elements of H satisfying (5), so it remains to fix some such element h_i and bound the number of g_j associated to it. But for any two such g_j, g_k , we have $d_X(g_j x, g_k x) \leq \varepsilon'$ and $d_X(g_j h_i x, g_k h_i x) \leq \varepsilon'$. Acylindricity tells us there are at most $M = M(\varepsilon')$ such elements, so over all possible h , we have at most MB such. Thus the action is acylindrical along H in the sense of Definition 3.1. \square

def:many subgroups

Definition 3.3. Let \mathcal{H} be a collection of subgroups of G , and suppose $G \curvearrowright X$ is an action on a hyperbolic space so that each $H \in \mathcal{H}$ is quasi-isometrically embedded by the action. Suppose there is a uniformly quasi-convex collection of subspaces $\{Y_H \mid H \in \mathcal{H}\}$ so that H acts coboundedly on Y_H for each $H \in \mathcal{H}$. Then the action is *acylindrical along \mathcal{H}* if for all $\varepsilon \geq 0$ there exist constants $D = D(\varepsilon, \{Y_H\}_{H \in \mathcal{H}})$ and $N = N(\varepsilon, \{Y_H\}_{H \in \mathcal{H}})$ such that any collection $\mathcal{S} \subseteq \{gY_H \mid H \in \mathcal{H}, g \in G\}$ satisfying

$$\text{diam} \left(\bigcap_{C \in \mathcal{S}} \mathcal{N}_\varepsilon(C) \right) > D$$

has cardinality at most N .

Using this characterization of acylindricity along a collection of quasi-isometrically embedded subgroups, we can generalize the notion of an A/QI triple to allow a collection of subgroups.

Definition 3.4. An *A/QI triple* (G, X, \mathcal{H}) consists of a Gromov hyperbolic space X which is acylindrical along the collection \mathcal{H} of infinite subgroups of G in the sense of Definition 3.3.

The following proposition shows that the A/QI condition for *finite* collections of subgroups reduces to the A/QI condition for a single subgroup.

prop:finitecollection

Proposition 3.5. Let $\mathcal{H} = \{H_1, \dots, H_n\}$ be a finite collection of infinite subgroups of G . Then (G, X, \mathcal{H}) is an A/QI triple if and only if (G, X, H_i) is an A/QI triple for all $1 \leq i \leq n$.

Proof. This follows immediately from the definition of an A/QI triple and the fact that \mathcal{H} is a finite collection of subgroups. \square

4. INTERSECTIONS, COMBINATIONS, AND FINITE HEIGHT

sec:finiteheight

In this section, we establish basic properties of A/QI triples. We *do not* assume in this section that G is finitely generated.

4.1. Intersections. The first result of this section implies, in particular, that if (G, X, A) and (G, X, B) are A/QI triples for the same action $G \curvearrowright X$, then $(G, X, A \cap B)$ is an A/QI triple.

Proposition 4.1. *Suppose (G, X, A) is an A/QI triple and $B < G$ is an infinite subgroup quasi-isometrically embedded by the action. Then $(G, X, A \cap B)$ is an A/QI triple.*

Proof. Let δ be a constant of hyperbolicity for X , and let $x \in X$. Since A and B are quasi-isometrically embedded by the action, there is some $\lambda > 0$ so that the sets Ax and Bx are λ -quasi-convex. There is also a constant M so that any geodesic with endpoints in Bx lies Hausdorff distance at most M from the image of a geodesic in some fixed Cayley graph for B .

Let $C = A \cap B$. Since Ax and Bx are proper subspaces of X , Cx is also a proper subspace of X . Thus to prove C is quasi-isometrically embedded by the orbit, it suffices to show that Cx is quasi-convex in X . We suppose by contradiction that it is not. There is therefore a sequence of elements $\{c_i\}$ of C so that geodesics γ_i from x to $c_i x$ travel arbitrarily far away from Cx . In other words, there are points $y_i \in \gamma_i$ with $\lim_{i \rightarrow \infty} d_X(y_i, Cx) = \infty$. Since Ax and Bx are λ -quasi-convex, there are, for each i , $a_i \in A$ and $b_i \in B$ so that both $d_X(a_i x, y_i)$ and $d_X(b_i x, y_i)$ are at most λ . Translating by b_i^{-1} we obtain a sequence of geodesics $b_i^{-1} \gamma_i$ joining $b_i^{-1} x$ to $b_i^{-1} c_i x$. The geodesic $b_i^{-1} \gamma_i$ is Hausdorff distance at most M from some quasi-geodesic σ_i which is the image of a geodesic $\tilde{\sigma}_i$ joining b_i^{-1} to $b_i^{-1} c_i$ in a Cayley graph for B . Passing to a subsequence we may assume the geodesics $\tilde{\sigma}_i$ converge to a bi-infinite geodesic $\tilde{\sigma}$ in the Cayley graph for B , whose image in X is a bi-infinite quasi-geodesic σ .

Let $\varepsilon = M + \delta + 2\lambda$, and let $D = \max\{\varepsilon + 1, D(\varepsilon, Ax)\}$, where $D(\varepsilon, Ax)$ is the constant from Theorem 3.2, applied with $H = A$. Fix a point p on the quasi-geodesic σ so that $d_X(x, p) \geq D$ and so that p is within M of the subsegment of $b_i^{-1} \gamma_i$ joining $b_i^{-1} y_i$ to $b_i^{-1} c_i x$ for all but finitely many i . See Figure 1.

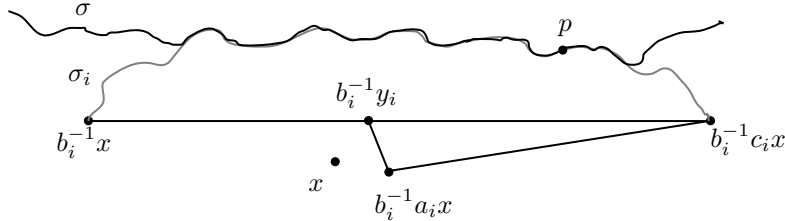


FIGURE 1. The point p is on the bi-infinite quasi-geodesic σ , at distance D from x . The index i is assumed to be large enough that σ_i contains p .

fig:intersect

Discarding finitely many i , we may assume that p is within M of all these subsegments. Let p'_i be a nearest point to p on the subsegment of $b_i^{-1} \gamma_i$ joining $b_i^{-1} y_i$ to $b_i^{-1} c_i x$, and consider a geodesic triangle with one side equal to this subsegment,

and with the third corner equal to $b_i^{-1}a_ix$. Let p_i be a point within δ of p'_i on one of the other two sides of this triangle. We have $d_X(p, p_i) \leq M + \delta$. If p_i lay on the side joining $b_i^{-1}a_ix$ to $b_i^{-1}y_i$, this would imply $d_X(p, x) \leq M + 2\lambda + \delta$, a contradiction, since $d_X(p, x) \geq \varepsilon + 1$. Thus p_i lies on a geodesic joining $b_i^{-1}a_ix$ to $b_i^{-1}c_ix$. Quasi-convexity of Ax gives us an $\hat{a}_i \in A$ so that $b_i^{-1}\hat{a}_ix$ is at most λ from p_i . Thus we have $d_X(b_i^{-1}\hat{a}_ix, p) \leq M + \delta + \lambda < \varepsilon$ and $d_X(b_i^{-1}a_ix, x) \leq 2\lambda < \varepsilon$. In particular, x and p are in $N_\varepsilon(b_i^{-1}Ax)$ for all i , so we have

$$\text{diam} \left(\bigcap_i \mathcal{N}_\varepsilon(b_i^{-1}Ax) \right) \geq D.$$

By Theorem 3.2, infinitely many of the cosets $b_i^{-1}A$ coincide. Discarding terms and relabeling we may assume that $b_i^{-1}A = b_0^{-1}A$ for all i . In particular we have $b_i b_0^{-1} \in C$ for every i . But

$$d_X(b_i b_0^{-1}x, y_i) \leq d_X(b_i b_0^{-1}x, b_ix) + d_X(b_ix, y_i) \leq d_X(x, b_0x) + \lambda,$$

so $d_X(b_i b_0^{-1}x, y_i)$ is bounded independently of i . This contradicts our choice of y_i , showing that in fact Cx is quasi-convex.

The orbit Cx is a subset of Ax , along which G acts acylindrically, so G acts acylindrically along Cx , and (G, X, C) is an A/QI triple as desired. \square

4.2. A combination theorem. In this subsection we prove Theorem 1.4, which we restate for the convenience of the reader.

Theorem 1.4. *Let G act on a hyperbolic space X , and suppose H_1, \dots, H_k are subgroups of G such that (G, X, H_i) is an A/QI triple for each $i = 1, \dots, k$ and $\Lambda(H_i) \cap \Lambda(H_j) = \emptyset$ for all $i \neq j$. Then there exists a finite collection of elements $\mathcal{S} \subset \cup_{i=1}^k H_i \setminus \{1\}$ such that for any family of subgroups $\mathcal{K} = \{K_i\}_{i=1}^k$ with K_i quasi-isometrically embedded in H_i and satisfying $K_i \cap \mathcal{S} = \emptyset$ for all i , we have $K = \langle \bigcup \mathcal{K} \rangle \cong K_1 * K_2 * \dots * K_k$ and (G, X, K) is an A/QI triple.*

We note that by Proposition 3.5, the assumption that each (G, X, H_i) is an A/QI triple is equivalent to assuming that $(G, X, \{H_1, \dots, H_k\})$ is an A/QI triple.

Before proving the theorem, we need some preliminary lemmas. We say that a path q in a metric space X is a *piecewise quasi-geodesic* if q can be expressed as a concatenation $q_1 \cdot \dots \cdot q_n$ where each q_i is a continuous quasi-geodesic path. The following lemma gives conditions under which a piecewise quasi-geodesic is a quasi-geodesic (with constants independent of the number of quasi-geodesic segments being concatenated).

lem:brokenqgeod

Lemma 4.2 ([Min05, Lemma 4.2]). *Let x_0, x_1, \dots, x_n be points in a δ -hyperbolic space X and let q_i be a continuous τ -quasi-geodesic from x_{i-1} to x_i . Then for any $C_0 \geq 14\delta$ and for $C_1 = 12(C_0 + \delta) + \tau + 1$, if $\ell_X(q_i) \geq \tau C_1$ and $(x_{i-1} | x_{i+1})_{x_i} \leq C_0$ for all i , then the concatenation $q_1 \cdot \dots \cdot q_n$ is a $\max\{4\tau, \frac{5}{2}M + C_1\}$ -quasi-geodesic, where M is the Morse constant for the parameters τ, δ .*

Let G, X , and H_1, \dots, H_k be as in the statement of Theorem 1.4. We fix the following data.

Let T_i be a finite generating set for H_i . Let $\delta > 1$ be a hyperbolicity constant which suffices both for X and for the Cayley graphs $\text{Cay}(H_i, T_i)$. We fix a basepoint $x \in X$ and let $\pi: G \rightarrow X$ be the orbit map defined by $g \mapsto gx$. Since each (G, X, H_i) is an A/QI triple, there is a constant τ such that each $H_i x$ is τ -quasi-isometrically

embedded in X . Let E be the constant from Lemma 2.10 with this choice of τ and δ .

Since $\Lambda(H_i) \cap \Lambda(H_j) = \emptyset$ for all $i \neq j$, there is a constant κ such that

$$\text{eqn:Hkappa} \quad (6) \quad (\xi \mid \xi')_x < \kappa$$

for all $\xi \in \Lambda(H_i)$ and $\xi' \in \Lambda(H_j)$.

Let

$$\eta = \max\{d_X(x, tx) \mid i = 1, \dots, k \text{ and } t \in T_i\},$$

which is well-defined since $\#T_i < \infty$ for each i .

For each i and each $t \in T_i$, fix a geodesic $[x, tx]$ in X . Since each $H_i x$ is τ -quasi-isometrically embedded in X , for any i and any geodesic word $t_1 \dots t_\ell$ where $t_j \in T_i$, the concatenation

$$[x, t_1 x] \cdot [t_1 x, t_1 t_2 x] \cdot \dots \cdot [t_1 \dots t_{\ell-1} x, t_1 \dots t_\ell x]$$

is a τ -quasi-geodesic in X . Let M be the Morse constant for parameters τ and δ .

For the remainder of the subsection, the following data is fixed: the sets T_i , the basepoint x , the map π , the constants δ , τ , E , κ , η , and M , and the geodesics $[x, tx]$.

Given any collection of subgroups $\mathcal{K} = \{K_i \leq H_i\}$, define a metric graph $\Gamma(\mathcal{K})$ as follows.

def:Gamma

Definition 4.3. Let $\text{Cay}(H_i, T_i)$ be the Cayley graph of H_i with respect to the finite generating set T_i . We will build $\Gamma(\mathcal{K})$ inductively. At the base stage, we begin with a single Cayley graph $\text{Cay}(H_1, T_1)$, and identify K_1 with the orbit of the identity under K_1 . At each point in $K_1 \subset \text{Cay}(H_1, T_1)$, attach a disjoint copy of $\text{Cay}(H_j, T_j)$ for each $j \neq 1$ along the identity $1 \in \text{Cay}(H_j, T_j)$. That is, for each $h \in K_1$ and each $j \neq 1$, we identify the identity 1 in a disjoint copy of $\text{Cay}(H_j, T_j)$ with the vertex $h \in \text{Cay}(H_1, T_1)$. At each subsequent step, we repeat the same process for each new Cayley graph $\text{Cay}(H_j, T_j)$, by attaching a disjoint copy of $\text{Cay}(H_i, T_i)$ for each $i \neq j$ to each point in $K_j \subset \text{Cay}(H_j, T_j)$ along the identity $1 \in \text{Cay}(H_i, T_i)$.

The resulting graph $\Gamma(\mathcal{K})$ is the smallest subgraph of $\text{Cay}(H_1 * \dots * H_k, \cup_{i=1}^k T_i)$ which is invariant under the action of $K_1 * \dots * K_k \leq H_1 * \dots * H_k$ and contains $\text{Cay}(H_i, T_i)$ for each i .

Vertices in $\Gamma(\mathcal{K})$ come in two types: those that lie in the orbit of the identity under K_i in some copy of $\text{Cay}(H_i, T_i)$, and those that do not. We call vertices of the first type *wedge vertices*.

Each edge in $\Gamma(\mathcal{K})$ is an edge of some $\text{Cay}(H_i, T_i)$ and so is labeled by an element $t \in T_i$. We define the length of such an edge to be $d_X(x, tx)$.

Define a map $\psi_{\mathcal{K}}: \Gamma(\mathcal{K}) \rightarrow X$ by sending each edge whose initial vertex is the identity and whose label is some $t \in T_i$ to the geodesic $[x, tx]$ chosen above, and extending equivariantly by the action of $K_1 * \dots * K_k$ on $\Gamma(\mathcal{K})$. By definition, $\psi_{\mathcal{K}}$ restricts to an isometry on each edge. The metric graph $\Gamma(\mathcal{K})$ and the map $\psi_{\mathcal{K}}$ depend on the choice of subgroups \mathcal{K} . However, we will show that for certain choices of subgroups, the map $\psi_{\mathcal{K}}$ is a uniform quasi-isometric embedding.

We also fix constants

$$C_0 = \kappa + E + 14\delta + \frac{3}{2}(2\tau\delta + \tau)$$

$$C_1 = 12(C_0 + \delta) + \tau + 1$$

$$\mu = \max\{4\tau, \frac{5}{2}M + C_1\} + \tau C_1$$

and let $\mathcal{S}_1 = \{h \in \bigcup_{i=1}^k H_i \setminus \{1\} \mid d_X(x, hx) \leq \tau C_1\}$.

claim:qiemb

Lemma 4.4. *Let $\mathcal{K} = \{K_i \leq H_i\}$ be a family of quasi-isometrically embedded subgroups satisfying*

$$K_i \cap \mathcal{S}_1 = \emptyset,$$

and let $\Gamma(\mathcal{K})$ be the graph from Definition 4.3 with this choice of subgroups K_i . The map $\psi_{\mathcal{K}}: \Gamma(\mathcal{K}) \rightarrow X$ is a μ -quasi-isometric embedding which takes every geodesic in $\Gamma(\mathcal{K})$ to a tame μ -quasi-geodesic.

Proof. For ease of notation, we write Γ for $\Gamma(\mathcal{K})$ and ψ for $\psi_{\mathcal{K}}$. A geodesic edge path in Γ is either completely contained in a copy of $\text{Cay}(H_i, T_i)$ for some i or passes through multiple Cayley graphs. In the first case, the image of the path will be a μ -quasi-geodesic because each $\text{Cay}(H_i, T_i)$ is τ -quasi-isometrically embedded in X and $\mu > \tau$.

Thus we focus on geodesic edge paths in Γ which pass through multiple Cayley graphs. Note that such a path σ can only change Cayley graphs at wedge vertices, which are necessarily in the orbit of the identity under some K_i . Thus σ is the concatenation of geodesic edge paths

eqn:geodpath

$$(7) \quad \sigma_1 \cdot \sigma_2 \cdot \dots \cdot \sigma_p,$$

where each σ_ℓ is a maximal geodesic edge path contained in some $\text{Cay}(H_{i_\ell}, T_{i_\ell})$ with $H_{i_\ell} \neq H_{i_{\ell+1}}$. Note that the terminal vertex of σ_1 is a wedge vertex, the initial vertex of σ_p is a wedge vertex, and, if $2 \leq \ell \leq p-1$, then both endpoints of σ_ℓ are wedge vertices. Moreover, for $2 \leq \ell \leq p-1$, the label of each σ_ℓ is an element of K_{i_ℓ} by construction.

Our main tool for understanding the image of such paths in X will be Lemma 4.2, applied with the constants C_0, C_1 defined just before the statement of the lemma.

For each i and each $h \in K_i \setminus \{1\}$, we have

eqn:Hedgelen

$$(8) \quad d_X(x, hx) > \tau C_1,$$

by our choice of subgroups K_i .

Fix some ℓ . By translating by an element of G we may assume the image of the terminal point of σ_ℓ (and therefore the image of the initial point of $\sigma_{\ell+1}$) is x . Let z_1 be the initial point of σ_ℓ and z_2 the terminal point of $\sigma_{\ell+1}$. Let $y_1 = \psi(z_1)$ and $y_2 = \psi(z_2)$. We have $z_1 \in H_\ell$, and $z_2 \in H_{\ell+1}$. Since H_ℓ is an infinite δ -hyperbolic group, there is some geodesic ray r_1 in H_ℓ starting at the identity and passing within 2δ of z_1 (see for example [BM91, Lemma 3.1]). Similarly there is a geodesic ray r_2 in $H_{\ell+1}$ starting at the identity and passing within 2δ of z_2 . For $i \in \{1, 2\}$ the image $\psi \circ r_i$ is a τ -quasi-geodesic in X , starting at x and passing through some point y'_i within $2\tau\delta + \tau$ of y_i . These quasi-geodesic rays limit to points $\xi_i \in \Lambda(H_\ell)$

and $\xi_2 \in \Lambda(H_{\ell+1})$, respectively. We find (using Lemma 2.10 in the second line)

$$\begin{aligned} (y_1 | y_2)_x &\leq (y'_1 | y'_2)_x + \frac{3}{2}(2\tau\delta + \tau) \\ &\leq (\xi_1 | \xi_2)_x + E + \frac{3}{2}(2\tau\delta + \tau) \\ &\leq \kappa + E + \frac{3}{2}(2\tau\delta + \tau) < C_0. \end{aligned}$$

Combining this bound with the inequality 8 and Lemma 4.2 we see that if σ_1 and σ_p are empty, then σ is mapped by ψ to a $\max\{4\tau, \frac{5}{2}M + C_1\}$ -quasi-geodesic (and hence a μ -quasi-geodesic, since $\mu = \max\{4\tau, \frac{5}{2}M + C_1\} + \tau C_1$).

Suppose $\sigma_p = \emptyset$, while $\sigma_1 \neq \emptyset$, and suppose the initial vertex of σ_1 is an arbitrary vertex a of $\text{Cay}(H_{i_1}, T_{i_1})$ and the terminal vertex is a wedge vertex b . If $d_X(ax, bx) > \tau C_1$ then the argument above shows that σ is mapped by ψ to a $(\mu - \tau C_1)$ -quasi-geodesic. On the other hand, if $d_X(ax, bx) \leq \tau C_1$, then σ is mapped by ψ to a μ -quasi-geodesic in X . A similar argument holds when $\sigma_2 \neq \emptyset$ and when both σ_1 and σ_2 are non-empty. In any case, σ is mapped by ψ to a μ -quasi-geodesic in X .

Since ψ restricted to any geodesic in Γ gives a (tame) μ -quasi-geodesic, the entire map ψ is a μ -quasi-isometric embedding. \square

cor:QIFreeProd

Corollary 4.5. *Under the assumptions of Lemma 4.4 the subgroup $K = \langle \bigcup \mathcal{K} \rangle$ is isomorphic to the free product $K_1 * \cdots * K_k$ and is quasi-isometrically embedded by the action $G \curvearrowright X$.*

Proof. Let $\phi: K_1 * \cdots * K_k \rightarrow K$ be the canonical map. It is clearly surjective. To see that it is injective, suppose f is a non-trivial element of $K_1 * \cdots * K_k$. If $f \in K_i$ for some i then $\phi(f) \neq 1$, by our choice of K_i , so f must have a normal form involving at least two factors. It is then straightforward to see that f has a bi-infinite geodesic axis in Γ . By Lemma 4.4, this bi-infinite geodesic has image a bi-infinite μ -quasi-geodesic in X , on which $\langle \phi(f) \rangle$ acts co-compactly. In particular, $\phi(f)$ is non-trivial.

The free product $K_1 * \cdots * K_k$ is quasi-isometrically embedded by its action on $\Gamma_{\mathcal{K}}$, since each K_i is undistorted in H_i . Again using Lemma 4.4 this implies that the orbit map $k \mapsto kx$ quasi-isometrically embeds K in X . \square

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. We continue to use the data fixed after Lemma 4.2. Let M' be the Morse constant for parameters μ and δ , and fix

$$\varepsilon' = 2M' + M + 10\delta.$$

The action of G of X is assumed to be acylindrical along $\{H_1, \dots, H_k\}$. For each H_i , let Y_i be $\psi(\text{Cay}(H_i, T_i))$, and let $D = D(\varepsilon', \{Y_1, \dots, Y_k\})$ and $N = N(\varepsilon', \{Y_1, \dots, Y_k\})$ be the constants of acylindricity of this action, as described in Definition 3.3.

Fix a constant

eqn:C

$$(9) \quad C > \max\{\mu(2D + 8\delta + 2M' + 1), \tau C_1\},$$

and let

$$\mathcal{S} = \{h \in \bigcup_{i=1}^k H_i \setminus \{1\} \mid d_X(x, hx) \leq C\}.$$

Since the subgroups H_i are uniformly quasi-isometrically embedded in X by the orbit map, there is a constant P so that for any i and any $h \in H_i$, the number of elements $h' \in H_i$ so that $d_X(hx, h'x) \leq \frac{C}{\mu} + 2M + 2M' + 12\delta - 1$ is at most P .

Let $\mathcal{K} = \{K_i \leq H_i\}$ be a family of quasi-isometrically embedded subgroups satisfying

$$K_i \cap \mathcal{S} = \emptyset,$$

and let $\Gamma = \Gamma_{\mathcal{K}}$ be the graph from Definition 4.3 with this choice of subgroups K_i . Since $C > \tau C_1$, we have $\mathcal{S}_1 \subset \mathcal{S}$. Corollary 4.5 therefore implies that $K \cong K_1 * \cdots * K_k$ and that K is quasi-isometrically embedded by the action of G on X .

It remains to show that G acts acylindrically along K . Fix $\varepsilon > 0$, and let

$$R = \frac{C}{\mu} - 1 + 2\varepsilon.$$

Let $\Delta = \{gK\}$ be a collection of distinct cosets satisfying

$$\text{diam} \left(\bigcap_{gK \in \Delta} \mathcal{N}_\varepsilon(gKx) \right) > R.$$

We will give a uniform bound on $\#\Delta$; namely we will show $\#\Delta \leq 2NP$. To do so, we will show that if $\#\Delta > 2NP$, we contradict the acylindricity of the action of G on X along $\{H_1, \dots, H_k\}$.

Let γ be a geodesic segment in X of length at least R whose endpoints are contained in the ε -neighborhood of gKx for all $gK \in \Delta$. We consider an arbitrary $gK \in \Delta$ but for each such coset fix a representative g . By Lemma 4.4, there is a tame μ -quasi-geodesic β_g in $g\psi(\Gamma)$ with endpoints within ε of the endpoints of γ .

Let γ' be an arbitrary subgeodesic of γ with endpoints at distance at least ε from the endpoints of γ and whose length is exactly $\frac{C}{\mu} - 1 - 2M' - 8\delta$. By our choice of R , such a subgeodesic exists. By Lemma 2.7, there is a subsegment ξ_g of the geodesic joining the endpoints of β_g which has endpoints within 4δ of those of γ' and which is Hausdorff distance at most 6δ from γ' . The segment ξ_g is Hausdorff distance at most $2M' + 2\delta$ from a subpath β'_g of β_g whose endpoints are within M' of those of ξ_g . To see this, note that we can find points on β_g at distance at most M' from the endpoints of ξ_g by the Morse property of quasi-geodesics. Now consider the geodesic quadrilateral formed by these two points and the endpoints of ξ_g ; this quadrilateral is 2δ -slim, so the Hausdorff distance between ξ_g and the opposite side is at most $2\delta + M'$. Finally, the opposite side is at Hausdorff distance at most M' from the subpath β'_g of β_g connecting its endpoints by the Morse property of quasi-geodesics.

In particular, the Hausdorff distance between γ' and β'_g is at most $2M' + 8\delta$ and the endpoints of β'_g are at distance at most $M' + 4\delta$ from those of γ' . See Figure 2.

Each quasi-geodesic β_g is a translate by g of the image of a geodesic σ_g in Γ of the form

$$\sigma_g = \sigma_{g,1} \cdot \cdots \cdot \sigma_{g,p_g}$$

as in (7).

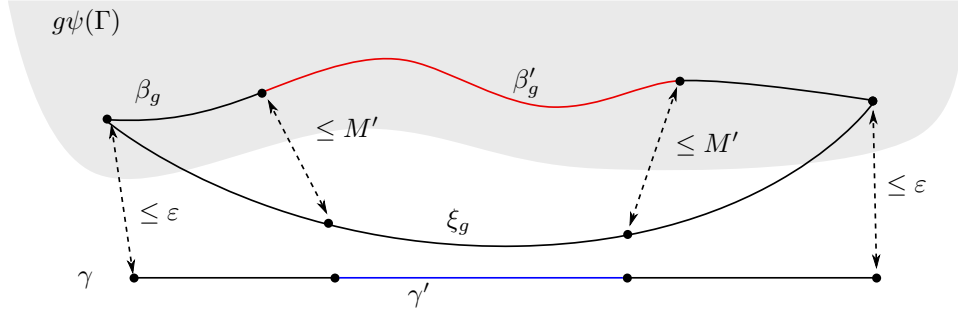
FIGURE 2. Finding the subpath β'_g in the proof of Theorem 1.4.

fig:combthm

We have

$$\frac{C}{\mu} - 4M' - 16\delta - 1 \leq \ell_X(\beta'_g) \leq \mu \left(\frac{C}{\mu} - 1 \right) + \mu = C.$$

There may be the image of a wedge vertex in β'_g for some g , but since the image of a geodesic between wedge vertices is a path of length at least C , there can be at most one such vertex in β'_g . Suppose there is such a vertex w in β'_g , dividing β'_g into two τ -quasi-geodesics. Let z be a closest point on γ' to w . Then $d_X(z, w) \leq 2M' + 8\delta$. We also have that the initial points of γ' and β'_g are at distance at most $M' + 4\delta$. Therefore the initial subgeodesic γ'' of γ' ending at z and a geodesic β''_g from the initial point of β'_g to w are at Hausdorff distance at most $2M' + 10\delta$. Moreover, the Hausdorff distance between the geodesic β''_g and the corresponding subpath of β'_g is at most M , since that subpath is a τ -quasi-geodesic. It follows that γ'' is Hausdorff distance at most $2M' + M + 10\delta$ from the corresponding initial subpath of β'_g . An analogous argument shows that the terminal subpath of γ' starting at z is at Hausdorff distance $2M' + M + 10\delta$ from the terminal subpath of β'_g starting at w .

Therefore, for each g as above there exists $\sigma_{g,j(g)}$ such that either the initial or terminal half of γ' is contained in the $(2M' + M + 10\delta)$ -neighborhood of $g\psi(\sigma_{g,j(g)})$, regardless of the number of images of wedge vertices appearing in β'_g . Let y_g be the initial vertex of the subpath of $g\psi(\sigma_{g,j(g)})$ contained in β'_g . Then y_g is either a wedge vertex or the initial vertex of β'_g , and lies in some translate of $Y_{j(g)}$. (Recall that $Y_i = \psi(\text{Cay}(H_i, T_i))$.) The previous paragraph shows that

eqn:distyg

$$(10) \quad d_X(y_g, \gamma') \leq 2M' + M + 10\delta.$$

If $\#\Delta > 2NP$, then there exists a subcollection $\Delta' \subseteq \Delta$ with $\#\Delta' > NP$ such that the $(2M' + M + 10\delta)$ -neighborhood of $g\psi(\sigma_{g,j(g)})$ contains (without loss of generality) the initial half of γ' for all $gK \in \Delta'$.

However, $\ell_X(\gamma') = \frac{C}{\mu} - 2M' - 8\delta - 1 > 2D$ by (9), and so, recalling that $\varepsilon' = 2M' + M + 10\delta$, we have

eqn:Delta'D

$$(11) \quad \text{diam} \left(\bigcap_{gK \in \Delta'} \mathcal{N}_{\varepsilon'}(g\psi(\sigma_{g,j(g)})) \right) > D.$$

For each g , there is some $k_g \in K$ so that $g\psi(\sigma_{g,j(g)})$ is contained in $gk_g Y_{j(g)}$. Equation (11) shows that the diameter of the intersection of the ε' -neighborhoods

of the image of these Cayley graphs in X is larger than D . In order to contradict the acylindricity of the action of G on X along $\{H_1, \dots, H_k\}$, we must show that at least N of these paths $g\psi(\sigma_{g,j_g})$ lie in translates of these Cayley graphs corresponding to *distinct* elements of $\bigsqcup G/H_i$.

Since $\ell(\gamma') = \frac{C}{\mu} - 2M' - 8\delta - 1$, it follows from (10) that for distinct $gK, g'K \in \Delta'$, we have

$$d_X(y_g, y_{g'}) \leq 2(2M' + M + 10\delta) + \frac{C}{\mu} - 2M' - 8\delta - 1 = \frac{C}{\mu} + 2M + 2M' + 12\delta - 1.$$

By our choice of constant P , there must be a further subcollection $\Delta'' \subseteq \Delta'$ with $\#\Delta'' > N$ so that for distinct $gK, g'K \in \Delta''$, either $j(g) \neq j(g')$ or the vertices y_g and $y_{g'}$, and therefore $g\psi(\sigma_{g,j(g)})$ and $g'\psi(\sigma_{g',j(g)})$, lie in translates of $Y_{j(g)}$ corresponding to distinct elements of $\bigsqcup G/H_i$. In either case, this contradicts the acylindricity of the action of G on X along $\{H_1, \dots, H_k\}$. We conclude $\#\Delta \leq 2NP$, and so the action of G on X is acylindrical along K . This completes the proof of the theorem. \square

Collections of cyclic subgroups generated by WPD elements (see below) give an important special case of Theorem 1.4.

def:WPD

Definition 4.6. Let G act on a hyperbolic space X , and let $g \in G$ act as a loxodromic isometry of X . We say g is a *WPD element of G (with respect to X)* if for every $x \in X$ and every $\varepsilon \geq 0$, there exists $N > 0$ such that the set

$$\{\gamma \in G \mid d(x, \gamma x) \leq \varepsilon \text{ and } d_X(g^N x, \gamma g^N x) \leq \varepsilon\}$$

is finite. The action of G on X is a *WPD action* if G is not virtually cyclic, G contains at least one element which is loxodromic with respect to the action on X , and every loxodromic element is a WPD element.

Suppose g is a loxodromic WPD element with respect to an action of G on a hyperbolic space. Since g is loxodromic, the subgroup $\langle g \rangle$ is quasi-isometrically embedded by the orbit map. Moreover, the definition of a WPD element immediately implies that the action of G is acylindrical along $\langle g \rangle$. Recall that loxodromic isometries of a hyperbolic space are *independent* if their fixed point sets at infinity are disjoint. The following is clear from the definitions.

Lemma 4.7. *If a group G acts on a hyperbolic space X , then given any finite collection h_1, \dots, h_k of independent loxodromic WPD elements with respect to this action, $(G, X, \{\langle h_1 \rangle, \dots, \langle h_k \rangle\})$ is an A/QI triple.*

The following corollary is then immediate from Theorem 1.4.

prop:WPDtoAQI

Corollary 4.8. *Let G act on a hyperbolic space X , and suppose h_1, \dots, h_k are independent loxodromic WPD elements with respect to this action. There exists a constant $D \geq 1$ such that for any $d_1, \dots, d_k \geq D$, if $H = \langle h_1^{d_1}, \dots, h_k^{d_k} \rangle$, then (G, X, H) is an A/QI triple. In particular, every element in H is a loxodromic WPD element with respect to the action of G on X .*

rem:notHE

Remark 4.9. Let G, h_1, \dots, h_k be as in the corollary. For each i let E_i be the maximal elementary subgroup containing h_i . Each E_i is hyperbolically embedded in G , as is any subcollection of $\{E_1, \dots, E_k\}$ obtained by choosing one subgroup per conjugacy class [DGO17, Theorem 6.8]. However, the subgroup H in the conclusion of the corollary is typically not hyperbolically embedded, since it will not be almost malnormal if for some i either $E_i \neq \langle h_i \rangle$ or $d_i > 1$.

4.3. Finiteness of height. We now turn our attention to the height of subgroups in A/QI triples. We recall the definition.

Definition 4.10 (Height). Let G be a group and $H < G$. If H is finite the height of H is 0. Otherwise the height of H in G is the largest n so that there are distinct cosets $\{g_1H, \dots, g_nH\}$ so that the intersection $\bigcap g_iHg_i^{-1}$ is infinite. If there is no largest n we say the height is infinite.

Quasi-convex subgroups of hyperbolic groups have finite height [GMRS98], as do full relatively quasi-convex subgroups of relatively hyperbolic groups [HW09]. Finiteness of height for convex cocompact subgroups of mapping class groups and $\text{Out}(\mathbb{F}_n)$ was established in [DM17], and, more generally, for stable and even strongly quasi-convex subgroups of finitely generated groups in [AMST19, Tra19]. (Stable subgroups are discussed in more detail in Section 5.)

`prop:finiteheight`

Proposition 4.11. *Suppose (G, X, H) is an A/QI triple. Then H has finite height.*

Proof. Choose $\delta \geq 1$ so that X is δ -hyperbolic. Then it is not hard to show that every loxodromic isometry of X has a quasi-axis which is a 2δ -quasi-geodesic.

Fix $x_0 \in X$ and $Q > 0$ so that Hx_0 is Q -quasi-convex and so that any 2δ -quasi-geodesic with endpoints in $Hx_0 \cup \Lambda(Hx_0)$ is contained in a Q -neighborhood of Hx_0 .

Since $G \curvearrowright X$ is acylindrical along H , Theorem 3.2 yields constants $D, N > 0$ so that any collection of distinct cosets $\{g_1H, \dots, g_kH\}$ satisfying

$$\text{diam} \left(\bigcap_{i=1}^k \mathcal{N}_Q(g_iHx_0) \right) > D$$

has cardinality at most N .

Suppose that $\{g_iH\}_{i \in I}$ is a collection of distinct cosets of G so that the corresponding intersection of conjugates $K = \bigcap_{i \in I} g_iHg_i^{-1}$ is infinite. We show H has finite height by giving a bound on the cardinality of I in terms of the quantities already specified.

The intersection K is an infinite subgroup of the hyperbolic group H , so it contains an infinite order element w (see [GdlH90, Ch. 8, Cor. 36]). Since H is quasi-isometrically embedded by the action, w acts loxodromically on X . Replacing w by a power if necessary, w has a 2δ -quasi-geodesic axis γ with endpoints in $\Lambda(g_iH)$ for each i .

By our assumptions on Q , the axis γ is contained in the Q -neighborhood of g_iHx_0 for each i . In particular, this implies that

$$\text{diam} \left(\bigcap_{i \in I} \mathcal{N}_Q(g_iHx_0) \right) > D.$$

By acylindricity, we therefore have $\#I \leq N$. □

5. STABILITY

`sec:stability`

In this section we prove Theorem 1.5. We first recall the definition of a *stable subgroup* and then restate the theorem for convenience.

`def:stable`

Definition 5.1. Let G be a finitely generated group. A finitely generated subgroup H of G is *stable* if it is undistorted and for some (equivalently, any) finite generating

set S of G and for every $\tau \geq 1$, there exists $L = L(S, \tau)$ such that whenever γ_1, γ_2 are τ -quasi-geodesics in $\text{Cay}(G, S)$ with the same endpoints in H , the Hausdorff distance between γ_1 and γ_2 is at most L .

Theorem 1.5. *Let (G, X, H) be an A/QI triple, and suppose that G is finitely generated. Then H is stable in G .*

Known examples of stable subgroups include convex cocompact subgroups of mapping class groups [DT15] and $\text{Out}(F_n)$ [ADT17], cyclic subgroups generated by a loxodromic WPD element in an acylindrically hyperbolic group [Sis16], subgroups of relatively hyperbolic groups which are quasi-isometrically embedded into the coned-off Cayley graph by the orbit map [ADT17], and subgroups of hierarchically hyperbolic groups which are quasi-isometrically embedded into the top-level curve by the orbit map [ABB⁺17].

def:sqc

Definition 5.2. Let G be a finitely generated group. A finitely generated subgroup H of G is *strongly quasi-convex* if for some (equivalently, any) finite generating set S of G and for every $\tau \geq 1$, there exists $L = L(S, \tau)$ such that every τ -quasi-geodesic in $\text{Cay}(G, S)$ with endpoints in H is contained in the L -neighborhood of H .

We will use the following observation of Tran to prove stability.

obs:tran

Theorem 5.3 ([Tra19, Theorem 4.8]). *A subgroup of a finitely generated group is stable if and only if it is hyperbolic and strongly quasi-convex in G .*

In [Sis16], Sisto investigates quasi-convexity properties of hyperbolically embedded subgroups of acylindrically hyperbolic groups and shows that they are strongly quasi-convex. The arguments in our proof of Theorem 1.5 are based on those of Sisto. However, we work in a slightly different setting. Sisto works under the assumption that X is a Cayley graph of G with respect to an infinite generating set. By a Milnor–Schwartz argument, if G acts coboundedly on X , then this is equivalent to our situation where X is a metric space. However, we do not assume that the action of G on X is cobounded. Also, Sisto does not assume that X is a hyperbolic metric space, rather assuming that X is hyperbolic relative to the translates of $\pi(H)$. This assumption is neither weaker nor stronger than ours—our space X is assumed to be hyperbolic, but might not be hyperbolic *relative* to the translates of $\pi(H)$, since these could have unbounded coarse intersection.

5.1. An axiomatic approach to Sisto’s proof. Though Sisto’s setting differs from ours, there are a few features in common, and these are all which are needed to execute Sisto’s strategy. For this subsection we fix a geodesic space X , an isometric action $G \curvearrowright X$, and a subgroup $H < G$, so that both G and H are finitely generated. We fix a basepoint $x_0 \in X$, and let $\pi: G \rightarrow X$ be the orbit map $\pi(g) = gx_0$. The following axioms will be assumed for the rest of the subsection.

ax:qie

(S1) The restriction of π to H is a quasi-isometric embedding.

ax:cl

(S2) There is a nearest-point projection map $\rho: X \rightarrow \pi(H)$ satisfying the following: For all $a_0 > 0$, there is an $a > 0$ so that if $d_X(x, y) \leq a_0$, then $d_X(\rho(x), \rho(y)) \leq a$.

ax:contracting

(S3) There is a constant $C > 0$ so that $d_X(\rho(x), \rho(y)) \geq C$ implies

$$d_X(x, y) \geq \max\{d_X(x, \pi(H)), d_X(y, \pi(H))\}.$$

ax:wa

(S4) $G \curvearrowright X$ is weakly acylindrical along $\pi(H)$.

rem:axiomshold

Remark 5.4. For Sisto, X is a Cayley graph of G which is not necessarily hyperbolic, but is hyperbolic *relative to* a hyperbolically embedded family \mathcal{H} including the subgroup H . Dahmani–Guirardel–Osin show (see the argument for [DGO17, Theorem 4.31]) that in this case H is a Lipschitz quasi-retract of X ; in particular H is quasi-isometrically embedded in X (Axiom (S1)). Also in Sisto’s setting, Axioms (S2) and (S3) follow from [Sis16, Lemma 2.3, parts (2) and (3)]. Axiom (S4) is [Sis16, Lemma 3.2].

In our setting of an A/QI triple (G, X, H) , H is not necessarily part of a hyperbolically embedded collection, but rather the axioms (S1) and (S4) are part of the definition. Axioms (S2) and (S3) are straightforward exercises in Gromov hyperbolic geometry, and use only that $\pi(H)$ is quasi-convex in the hyperbolic space X .

Since H and G are finitely generated, we may fix finite generating sets $S \subset T$ of H and G respectively, and consider the Cayley graph $\text{Cay}(H, S)$ as a subset of the Cayley graph $\text{Cay}(G, T)$ in the natural way. We use d_G and d_X to denote the metrics on $\text{Cay}(G, T)$ and X , respectively, and $\ell_G(\alpha)$ and $\ell_X(\beta)$ to denote the length of a path α in G (more precisely, in $\text{Cay}(G, T)$) and a path β in X , respectively.

For the rest of the section we fix a constant $\kappa \geq 1$ so that $d_X(x_0, gx_0) \leq \kappa$ for every $g \in T$.

We use $B_r^X(x)$ to denote the ball in X about $x \in X$ with radius r (that is, r is measured with respect to d_X). We use $N_K^G(Y)$ and $N_K^X(Z)$ to denote the closed K -neighborhood in G of $Y \subset \text{Cay}(G, T)$ and the closed K -neighborhood in X of $Z \subset X$, respectively.

Definition 5.5. Two subsets A, B of a metric space Y are said to be *geometrically separated* if, for all $D > 0$, the diameter of $N_D(A) \cap B$ is finite.

The first lemma is [Sis16, Lemma 3.3], slightly corrected. It shows that preimages of balls of X which are far apart are geometrically separated in $\Gamma(G, T)$, and only uses Axiom (S4).

lem:sisto3.3

Lemma 5.6 ([Sis16, Lemma 3.3]). *Let G act on the geodesic space X , weakly acylindrically along the subspace Y . Let $x_0 \in X$, and let π denote the orbit map $g \mapsto gx_0$. For each $r \geq 0$, there is an $R_1 \geq 0$ so that if $x, y \in Y$ satisfy $d_X(x, y) \geq R_1$, then $\pi^{-1}(B_r^X(x))$ and $\pi^{-1}(B_r^X(y))$ are geometrically separated.*

Proof. Let $r \geq 0$. By Axiom (S4), G acts weakly acylindrically along Y . In particular there is some R_1 so that for any $x, y \in Y$ with $d_X(x, y) \geq R_1$ we have

$$\#\{g \mid d_X(gx, x), d_X(gy, y) \leq 2r\} < \infty.$$

We therefore fix some $x, y \in Y$ so that $d_X(x, y) \geq R_1$, and let $A = \pi^{-1}(B_r^X(x))$ and $B = \pi^{-1}(B_r^X(y))$.

Claim. *For a fixed $g \in G$, $\#\{a \in A \mid ag \in B\} < \infty$.*

Given the claim, we argue by contradiction. Suppose for some $D > 0$, the intersection $N_D^G(A) \cap B$ is unbounded. There are thus infinitely many pairs of vertices (a_i, b_i) with $a_i \in A$, $b_i \in B$, and $d_G(a_i, b_i) \leq D$. Thus all the elements $a_i^{-1}b_i$ lie in $B_D^G(1)$, which is finite. Passing to a subsequence, the difference $a_i^{-1}b_i$ is

a constant element g . For this g , the set $\{a \in A \mid ag \in B\}$ is infinite, contradicting the claim.

Proof of Claim. Let $\Omega = \{a \in A \mid ag \in B\}$, and fix some $a_0 \in \Omega$. We want to show Ω is finite.

For any $a \in \Omega$, we have

$$d_X(x, aa_0^{-1}x) \leq d_X(x, ax_0) + d_X(ax_0, aa_0^{-1}x) = d_X(x, ax_0) + d_X(a_0x_0, x) \leq 2r.$$

Similarly we have

$$d_X(y, aa_0^{-1}y) \leq d_X(y, agx_0) + d_X(agx_0, aa_0^{-1}y) = d_X(y, agx_0) + d_X(a_0gx_0, y) \leq 2r.$$

Since $d_X(x, y) \geq R_1$ and $x, y \in Y$, the number of possibilities for aa_0^{-1} (and hence for a) is finite. \square

\square

The next lemma uses both Axiom (S4) and Axiom (S1) (in the form of the fact that $\pi|_H$ is proper).

lem:geomsep

Lemma 5.7 ([Sis16, Lemma 3.4]). *For each $r \geq 0$, there is an $R > 0$ and a function $B: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ so that for any $x, y \in H$ with $d_X(\pi(x), \pi(y)) \geq R$ we have, for any $D \geq 0$,*

$$N_D^G(\pi^{-1}(B_r^X(\pi(x)))) \cap \pi^{-1}(B_r^X(\pi(y))) \subseteq N_{B(D)}^G(H).$$

Proof. We note that the orbit map $\pi: G \rightarrow X$ is κ -Lipschitz. Fix some $r \geq 0$. By equivariance it suffices to consider $x = 1, y = h \in H$.

By Lemma 5.6, there is an $R = R_1(2r) \geq 0$ so that if $h \in H$ satisfies $d_X(x_0, \pi(h)) \geq R$, then the sets $\pi^{-1}(B_{2r}^X(x_0))$ and $\pi^{-1}(B_{2r}^X(\pi(h)))$ are geometrically separated.

Now let $D > 0$. If $d_X(x_0, \pi(h)) > \kappa D + 2r$, then the sets $N_D^G(\pi^{-1}(B_r^X(x_0)))$ and $\pi^{-1}(B_r^X(\pi(h)))$ are disjoint. Since $\pi|_H$ is proper (Axiom (S1)), the set

$$\Psi = \{h \in H \mid R \leq d_X(x_0, \pi(h)) \leq \kappa D + 2r\}$$

is finite. For each $h \in \Psi$, Lemma 5.6 tells us that the diameter

$$d(h) = \text{diam } N_D^G(\pi^{-1}(B_{2r}^X(x_0))) \cap \pi^{-1}(B_{2r}^X(\pi(h)))$$

is finite. We let $B(D) = \max\{d(h) \mid h \in \Psi\}$. \square

Let α be a tame τ -quasi-geodesic edge-path in the Cayley graph of G with respect to T . The path α passes through a sequence of group elements a_1, \dots, a_n . By $\pi(\alpha)$, we mean the discrete path $\pi(a_1), \dots, \pi(a_n)$. Given a discrete path $\gamma = x_0, x_1, \dots, x_n$ in X , recall that we define the length of γ to be

$$\ell^0(\gamma) := \sum_{i=0}^{n-1} d_X(x_i, x_{i+1}).$$

If all the summands $d_X(x_i, x_{i+1})$ are at most r , we say that γ is an r -coarse path. Since $\kappa = \max\{d_X(x_0, tx_0) \mid t \in T\}$, the discrete path $\pi(\alpha)$ is a κ -coarse path. In particular we have

eq:lengthbound

$$(12) \quad \ell_G(\alpha) \geq \ell^0(\pi(\alpha))/\kappa.$$

By choosing geodesics in X between x_0 and sx_0 for each $s \in S$, we extend the map $\pi|_H$ to an H -equivariant map

$$\hat{\pi}: \text{Cay}(H, S) \rightarrow X,$$

The image $\Gamma = \hat{\pi}(\text{Cay}(H, S))$ is an H -cocompact, embedded, connected subset which contains the discrete set $\pi(H)$. Fix $\lambda \geq 1$ so that $\hat{\pi}$ is a λ -quasi-isometric embedding.

The map $\hat{\pi}$ and the constant λ will be fixed for the rest of the section.

For $h \in H$ and $0 \leq r_1 \leq r_2$, we define the following subspace of Γ , which we refer to as an *annulus*:

$$A(h, r_1, r_2) = \Gamma \cap \left(B_{r_2}^X(hx_0) \setminus \overset{\circ}{B}_{r_1}^X(hx_0) \right).$$

Since Γ is connected, contains x_0 , and has infinite diameter, $A(h, r_1, r_2)$ is always non-empty.

claim:stable

Proposition 5.8 ([Sis16, Proposition 4.1]). *For any $L \geq 0$ there exists a $K \geq 0$ such that the following holds. For any $h, h' \in H$ with $d_G(h, h') > K$ and any $0 < r_1 < r_2 < d_X(hx_0, h'x_0)$, if α is a path in G from h to h' and $\alpha \cap N_K^G(\hat{\pi}^{-1}(A(h, r_1, r_2))) = \emptyset$, then $\ell_G(\alpha) \geq L(r_2 - r_1)$.*

Proof. We may as well suppose $L \geq 1$. Recall that $\kappa \geq 1$ is a Lipschitz constant for the orbit map π . By equivariance, we may consider just paths α joining 1 to h for $h \in H$. The map π is only defined on group elements, so $\pi(\alpha)$ is a κ -coarse path in X . We will obtain our lower bound on $\ell_G(\alpha)$ by analyzing how $\pi(\alpha)$ interacts with X -neighborhoods of regularly spaced sub-annuli of $A(h, r_1, r_2)$. If it misses enough of these, then $\pi(\alpha)$ is already quite long, and thus so is α . If on the other hand it meets many such neighborhoods, we will see that it meets many pairs of such neighborhoods at controlled distances from each other. When this happens we will be able to use Lemma 5.7 to bound $\ell_G(\alpha)$ from below.

Recall that $\rho: X \rightarrow \pi(H)$ is a nearest point projection map. By Axiom (S2), there is an a so that ρ takes every κ -coarse path in X to an a -coarse path of points in $\pi(H)$.

We now fix several more constants. Some of these will only be used much later in the proof. Let $D_1 = C + a$, where C is the constant from Axiom (S3). Let $C_2 \geq 4000LD_1\kappa + a$. Also fix an integer $R \geq \kappa$ so that Lemma 5.7 holds with $r = C_2 + \kappa/2$. We set $D_2 = 100C_2RL$, and let $B = B(D_2)$, where $B(\cdot)$ is the function from Lemma 5.7. Finally, let

$$B' = \lambda(\kappa B + C_2) + \lambda^2$$

and let

$$K = \max\{1000C_2RL, B + B'\}.$$

We now assume that h satisfies $d_G(1, h) > K$ and choose $r_1 < r_2$ between 0 and $d_X(x_0, hx_0)$. Let α be a path in $\text{Cay}(G, T)$ satisfying

$$\alpha \cap N_K^G(\hat{\pi}^{-1}(A(1, r_1, r_2))) = \emptyset.$$

We must bound the length of α from below.

There are two cases, depending on whether $r_2 - r_1 \geq 1000C_2R$ or not. In the case $r_2 - r_1 < 1000C_2R$, we argue as follows. By assumption, $d_G(1, h) > K$, and so $\ell_G(\alpha) > K$. Since $K \geq 1000C_2RL$, it follows that $\ell_G(\alpha) \geq 1000C_2RL > (r_2 - r_1)L$, completing the proof in this case.

Now suppose $r_2 - r_1 \geq 1000C_2R$.

Let $n = \lfloor (r_2 - r_1)/(10C_2) \rfloor$. Let $r_1 = d_0, d_1, \dots, d_n = r_2$ be evenly spaced real numbers, and notice that $d_{i+1} - d_i \geq 10C_2$. Consider the following annuli and neighborhoods of annuli:

$$A_i = A(1, d_i, d_i + C_2) \quad \text{and} \quad T_i = N_{C_2}^X(A_i).$$

c1:1

Claim 1. *Suppose the number of indices i such that $\pi(\alpha) \cap T_i = \emptyset$ is greater than $(r_2 - r_1)/(1000C_2)$. Then $\ell_G(\alpha) \geq L(r_2 - r_1)$.*

Proof of claim. At least $(r_2 - r_1)/(1000C_2)$ of the indices satisfying $\pi(\alpha) \cap T_i = \emptyset$ are strictly less than n . Fixing one such index i , we will find a subpath β_i of $\pi(\alpha)$ with some definite length. These will be disjoint subpaths and the sum of their lengths will give the desired lower bound on $\ell_G(\alpha)$.

For a fixed i , let x_i be the last point on $\pi(\alpha)$ so that $d_X(x_0, \rho(x_i)) < d_i$, and let y_i be the first point after x_i on $\pi(\alpha)$ so that $d_X(x_0, \rho(y_i)) > d_i + C_2$. Note that such a y_i always exists; otherwise we would have $d_X(x_0, h.x_0) \leq d_i + C_2$ which could only happen for $i = n$, contradicting our assumption on i . Let $\beta_i = \pi(\alpha)|_{[x_i, y_i]}$. We will bound the length of this subpath from below.

Subclaim. *There are points q_1, \dots, q_m on β_i satisfying:*

- (a) $m \geq C_2/D_1$
- (b) $d_X(\rho(q_j), \rho(q_{j+1})) \geq C$
- (c) $\rho(q_j) \in A_i$.

q:many

q:spaced

q:nearsphere

Proof. Since $\rho(\pi(\alpha))$ is an a -coarse path, the set of distances $\{d_X(x_0, \rho(q)) \mid q \in \beta_i\}$ is $\frac{a}{2}$ -dense in the interval $[d_X(x_0, \rho(x_i)), d_X(x_0, \rho(y_i))] \supset [d_i, d_i + C_2]$. In particular, there is some q_1 in β_i so that $d_X(x_0, \rho(q_1)) \in [d_i, d_i + a]$. If we have chosen q_j , we can proceed inductively. The set of distances $\{d_X(x_0, \rho(q)) \mid q \in \beta_i|_{[q_j, y_i]}\}$ is $\frac{a}{2}$ -dense in the interval $[d_X(x_0, \rho(q_j)), d_i + C_2]$. As long as this interval contains $[d_X(x_0, \rho(q_j)) + C, d_X(x_0, \rho(q_j)) + C + a]$, we can find $q_{j+1} \in \beta_i$ so $d_X(x_0, \rho(q_{j+1}))$ lies in that interval. The triangle inequality ensures item (b) holds. Since $d_X(x_0, \rho(q_{j+1})) \in [d_i, d_i + C_2]$, item (c) also holds for q_{j+1} . If we fail to find such a q_{j+1} this can only be because $j + 1 > C_2/D_1$. (Recall $D_1 = C + a$.) \square

It follows from (b) and Axiom (S3) that $d_X(q_j, q_{j+1}) \geq d_X(q_j, \pi(H))$. Since $q_j \notin T_i$, (c) implies that $d_X(q_j, \pi(H)) \geq C_2/2$. Combining this with (a), we conclude that

$$\ell^0(\beta_i) \geq \sum_{j=1}^{m-1} d_X(q_j, q_{j+1}) \geq \frac{C_2}{2} \left(\frac{C_2}{D_1} - 1 \right) \geq \frac{C_2^2}{4D_1} \geq 1000C_2L\kappa.$$

Since $d_{i+1} - d_i \geq 10C_2$, these subpaths β_i must be disjoint for distinct i . In particular, if the set of $i < n$ such that $\pi(\alpha) \cap T_i = \emptyset$ is at least $(r_2 - r_1)/(1000C_2)$, then

$$\ell^0(\pi(\alpha)) \geq \frac{r_2 - r_1}{1000C_2} \cdot 1000C_2L\kappa \geq L\kappa(r_2 - r_1).$$

Applying (12) we obtain $\ell_G(\alpha) \geq L(r_2 - r_1)$, as desired. \square

Let $I_0 \subset \{1, \dots, n\}$ be the set of indices so that $\pi(\alpha) \cap T_i \neq \emptyset$. Claim 1 implies that $\#I_0 \geq n - (r_2 - r_1)/(1000C_2) \geq (1 - 2/100)n$. Subdividing the integers modulo R there must exist $0 \leq s \leq R - 1$ such that the residue class $s + R\mathbb{Z}$ meets I_0 in a set of cardinality at least $(1 - \frac{2}{100})\frac{n}{R}$. A counting argument shows that there is a set $I \subset I_0 \cap (s + R\mathbb{Z})$ of cardinality at least $(1 - \frac{4}{100})\frac{n}{R} \geq \frac{r_2 - r_1}{100C_2R}$ so that for every $i \in I$ we have both $\pi(\alpha) \cap T_i \neq \emptyset$ and $\pi(\alpha) \cap T_{i+R} \neq \emptyset$.

For each $i \in I \cup (R+I)$, choose a point $a_i \in \pi^{-1}(T_i) \cap \alpha$. For distinct $i, i' \in I$ we note that the subpaths $\alpha|_{[a_i, a_{i+R}]}$ and $\alpha|_{[a_{i'}, a_{i'+R}]}$ contribute separately to $\ell_G(\alpha)$.

c1:2 **Claim 2.** *For each $i \in I$, we have $d_G(a_i, a_{i+R}) \geq 100C_2RL$.*

Assuming Claim 2, as there are at least $(r_2 - r_1)/(100C_2R)$ such pairs of points a_j, a_{j+R} and $\ell_G(\alpha) \geq \sum_j d_G(a_j, a_{j+R})$, we have $\ell_G(\alpha) \geq L(r_2 - r_1)$, as desired. Thus it remains to prove the claim.

Proof of claim. We assume for a contradiction that $d_G(a_i, a_{i+R}) < 100C_2RL$ for some $i \in I$. We will show that this forces the path α to enter the K -neighborhood of $\hat{\pi}^{-1}(A(h, r_1, r_2))$, contradicting our basic assumption on α .

For $k \in \{i, i+R\}$, let $p_k = \pi(a_k) \in T_k$. There is some point $b_k \in A_k$ so that $d_X(p_k, b_k) \leq C_2$, and some $c_k \in \pi(H)$ with $d_X(b_k, c_k) \leq \kappa/2$. We have $d_X(p_k, c_k) \leq C_2 + \kappa/2$. Moreover

eq:cjdist (13)
$$d_X(c_i, c_{i+R}) \geq 10RC_2 - C_2 - \kappa > R,$$

so we may apply Lemma 5.7 with c_{i+R} and c_i taking the place of $\pi(x)$ and $\pi(y)$, respectively. Accordingly, let $x \in H \cap \pi^{-1}(c_{i+R})$ and let $y \in H \cap \pi^{-1}(c_i)$. We have $a_i \in \pi^{-1}(B_{C_2+\kappa/2}^X(\pi(y)))$, since $d_X(p_i, c_i) \leq C_2 + \kappa/2$. Recalling that $D_2 = 100C_2RL$, we also have $a_i \in N_{D_2}(\pi^{-1}(B_{C_2+\kappa/2}^X(\pi(x))))$ since we have assumed $d_G(a_i, a_{i+R}) < D_2$. Now Lemma 5.7 implies that there is some $h'' \in H$ within $B = B(D_2)$ of a_i . (In X we have $d_X(\pi(h''), p_i) \leq \kappa B$.)

Now we note that $b_i \in A_i \subset A(1, r_1, r_2)$. Thus

$$d_G(a_i, \hat{\pi}^{-1}(A(1, r_1, r_2))) \leq d_G(a_i, \hat{\pi}^{-1}(b_i)).$$

We will see this is less than K . Indeed

$$\begin{aligned} d_G(a_i, \hat{\pi}^{-1}(b_i)) &\leq d_G(a_i, h'') + d_G(h'', \hat{\pi}^{-1}(b_i)) \\ &\leq B + d_G(h'', \hat{\pi}^{-1}(b_i)). \end{aligned}$$

Now

$$\begin{aligned} d_X(\pi(h''), b_i) &\leq d_X(\pi(h''), p_i) + d_X(p_i, b_i) \\ &\leq \kappa B + C_2, \end{aligned}$$

and since $\hat{\pi}$ is a λ -quasi-isometric embedding, this implies

$$d_G(h'', \hat{\pi}^{-1}(b_i)) \leq \lambda(\kappa B + C_2) + \lambda^2 = B'.$$

We thus have $d_G(a_i, \hat{\pi}^{-1}(b_i)) \leq B + B' = K$, as desired. This contradiction proves the claim. \square

\square

We now use Proposition 5.8 to prove strong quasi-convexity (Definition 5.2). By Tran's Theorem 5.3, a subgroup is stable if and only if it is both hyperbolic and strongly quasi-convex.

thm:axioms_imply_sqc **Theorem 5.9.** *Let G be a finitely generated group acting on a geodesic space X , and let $H < G$ be finitely generated. If (G, X, H) satisfy the axioms (S1)–(S4), then H is strongly quasi-convex in G .*

Proof. In order to show strong quasi-convexity it suffices to consider tame quasi-geodesics (see the discussion after Definition 2.4); in a graph, these tame quasi-geodesics can be taken to be edge-paths. So we assume α is a tame τ -quasi-geodesic edge-path in $\text{Cay}(G, T)$ from $h_1 \in H$ to $h_2 \in H$. We will find a neighborhood of H containing α and only depending on the value of τ . Recall that $\hat{\pi}$ is a λ -quasi-isometric embedding.

Fix $L = \lambda(\tau + 1)$, and let K be as in Proposition 5.8 for this choice of L . Suppose that α is not contained in $N_K^G(H)$. Let x, y be vertices of α so that the subpath α' of α from x to y intersects $N_K^G(H)$ only at its endpoints. Since α is a tame τ -quasi-geodesic, it follows that

$$\boxed{\text{eqn:qgeo}} \quad (14) \quad \ell_G(\alpha') \leq \tau d_G(x, y) + \tau.$$

We will bound $\ell_G(\alpha')$ by a constant M depending only on λ and τ , which will imply that $\alpha \subset N_{K+M}^G(H)$, as desired. Appealing to (14), it suffices to bound $d_G(x, y)$. We claim:

$$\boxed{\text{eq:dxybound}} \quad (15) \quad d_G(x, y) \leq \max \{4\lambda^2 K + 3\lambda^2 + 2K, \tau + 4L\lambda K + 3L\lambda + 2K + 2L\lambda^{-1}K\}.$$

Consider the path β' obtained by concatenating (in the appropriate order) α' with geodesics of length at most K connecting x and y to points $h, h' \in H$, respectively.

We divide into two cases, depending on whether or not

$$\boxed{\text{eq:dichot}} \quad (16) \quad d_G(h, h') > 4\lambda^2 K + 3\lambda^2.$$

If (16) does *not* hold, then since x, y are distance at most K from h, h' respectively,

$$d_G(x, y) \leq 4\lambda^2 K + 3\lambda^2 + 2K,$$

establishing (15) in this case.

So we suppose that (16) holds. Set $r_1 = 2\lambda K + \lambda$ and $r_2 = d_X(\hat{\pi}(h), \hat{\pi}(h')) - (2\lambda K + \lambda)$. From (16) and the fact that $\hat{\pi}$ is a λ -quasi-isometric embedding, we obtain $r_1 < r_2$, so $A = A(h, r_1, r_2)$ is nonempty. Moreover, r_1 and r_2 are chosen so that $d_G(\{h, h'\}, \hat{\pi}^{-1}(A)) \geq 2K$. In particular the initial and terminal segments of β' connecting α' to H lie outside $N_K^G(\hat{\pi}^{-1}(A))$. Since $\hat{\pi}^{-1}(A)$ is contained entirely within $\text{Cay}(H, S)$, the path α' also lies outside $N_K^G(\hat{\pi}^{-1}(A))$. Thus we have

$$\beta' \cap N_K^G(\hat{\pi}^{-1}(A(h, r_1, r_2))) = \emptyset.$$

Now Proposition 5.8 yields $\ell_G(\beta') \geq L(r_1 - r_2)$. We have

$$\begin{aligned} \ell_G(\alpha') &\geq L(r_2 - r_1) - 2K \\ &\geq L(d_X(\hat{\pi}(h), \hat{\pi}(h')) - (4\lambda K + 2\lambda)) - 2K \\ &\geq L(\lambda^{-1}d_G(h, h') - \lambda) - (4L\lambda K + 2L\lambda + 2K) \\ &\geq (\tau + 1)d_G(x, y) - (4L\lambda K + 3L\lambda + 2K + 2L\lambda^{-1}K). \end{aligned}$$

Now (14) tells us that $\ell_G(\alpha')$ is at most $\tau d_G(x, y) + \tau$, so (subtracting $\tau d_G(x, y)$ from both sides and rearranging),

$$d_G(x, y) \leq \tau + 4L\lambda K + 3L\lambda + 2K + 2L\lambda^{-1}K,$$

verifying (15) in this case. \square

5.2. Application to A/QI triples. Returning to our main setting, we have already noted in Remark 5.4 that if (G, X, H) is an A/QI triple, all the axioms (S1)–(S4) hold. From this we easily derive Theorem 1.5.

Proof of Theorem 1.5. Since H quasi-isometrically embeds into the hyperbolic space X , it must be hyperbolic. By Tran’s Theorem 5.3, it therefore suffices to show that H is strongly quasi-convex, which is the content of Theorem 5.9. \square

The work of Antolín–Mj–Sisto–Taylor [AMST19, Theorem 1.1] (cf. [Tra19, Theorem 1.2(3)]) gives the following corollary:

Corollary 5.10. *Let (G, X, H) be an A/QI triple, and suppose that G is finitely generated. Then H has finite height, finite width, and bounded packing.*

sec:coneoff

6. THE BOUNDARY OF A CONE-OFF

Let X be a graph, and let \mathcal{H} be a collection of subgraphs. The *cone-off* of X with respect to \mathcal{H} is the space $\hat{X}_{\mathcal{H}}$ obtained from X by adding an edge between each pair of distinct x, y , so that $\{x, y\} \subset Y^{(0)}$ for $Y \in \mathcal{H}$. These new edges are called *electric edges*. If there is no ambiguity about \mathcal{H} , we may use \hat{X} to denote the cone-off.

Convention 6.1. In this section we will restrict to continuous quasi-geodesics which are tame; see the discussion after Definition 2.4.

Convention 6.2. In this section, we consider two hyperbolic spaces, X and $\hat{X} = \hat{X}_{\mathcal{H}}$. Fix a basepoint $x_0 \in X$; by definition, $x_0 \in \hat{X}$, as well. We will frequently calculate Gromov products in each space, and so to avoid confusion, we will use $(\cdot | \cdot)_{x_0}$ to denote the Gromov product in X and $\langle \cdot | \cdot \rangle_{x_0}$ to denote the Gromov product in \hat{X} .

There are various other possibilities for a definition of a cone-off of X . It is clear, for example, that if \hat{X} is the cone-off of X with respect to \mathcal{H} , then it is quasi-isometric to the space obtained by adjoining, for each $Y \in \mathcal{H}$, a new vertex v_Y connected by an edge to every vertex of Y . When X is the Cayley graph of a group, this construction is due to Farb [Far98].

In case the collection \mathcal{H} is *quasi-dense*, in the sense that every point is within bounded distance of some element of \mathcal{H} , there is another possibility, explored by Bowditch in [Bow12]. Namely, one can let $\Gamma_{\mathcal{H}, R}$ be a graph with vertex set \mathcal{H} , and connect two vertices if the corresponding sets have distance at most R . For suitably chosen R , the space $\Gamma_{\mathcal{H}, R}$ will be quasi-isometric to \hat{X} .

We will use work of Kapovich–Rafi, Bowditch, and Dowdall–Taylor to understand $\partial\hat{X}$. We start with a theorem of Bowditch (in the case \mathcal{H} is quasi-dense) and Kapovich–Rafi.

thm:conehyp

Theorem 6.3 ([KR14, Proposition 2.6], [Bow12, 7.12]). *Let X be a Gromov hyperbolic graph, and let \mathcal{H} be a collection of uniformly quasi-convex subgraphs. Then the cone-off of X with respect to \mathcal{H} is Gromov hyperbolic. Moreover, geodesics in X can be reparametrized to be uniform quasi-geodesics in \hat{X} .*

Combining Theorem 6.3 with quasi-geodesic stability (Theorem 2.5) yields the following.

cor:nospiraling

Corollary 6.4. *Let X , \mathcal{H} , \hat{X} be as above, and suppose that γ is a quasi-geodesic ray in X . Then the image of γ in \hat{X} is either bounded or has a unique limit point in $\partial\hat{X}$.*

Since the cone-off is hyperbolic, it is natural to ask about its Gromov boundary.

Question 6.5. *If \hat{X} is a cone-off of X , how is the Gromov boundary of \hat{X} related to that of X ?*

Note that neither space quasi-isometrically embeds in the other, so it is not immediately clear that there should be any relation. Dowdall and Taylor provide the following definition and theorem [DT17].

Definition 6.6. Let $\partial_u X \subseteq \partial X$ consist of the points represented by quasi-geodesic rays whose projection to \hat{X} is unbounded.¹

Theorem 6.3 implies that the projection of a quasi-geodesic ray is unbounded if and only if it is a quasi-geodesic ray (after possibly reparametrizing).

thm:DTthm

Theorem 6.7 ([DT17, Theorem 3.2]). *The map $X \rightarrow \hat{X}$ extends continuously to $\partial_u X$, and the restriction $\psi: \partial_u X \rightarrow \partial\hat{X}$ is a homeomorphism.*

Remark 6.8. Dowdall–Taylor’s result is in the slightly more general setting of coarsely surjective *alignment preserving* maps. It follows from the “Moreover” statement in Kapovich–Rafi’s Theorem 6.3 that the cone-off map $X \rightarrow \hat{X}$ is alignment preserving.

Let $\text{Isom}(X, \mathcal{H})$ be the group of isometries of X which preserve the collection \mathcal{H} . The homeomorphism ψ from Theorem 6.7 is $\text{Isom}(X, \mathcal{H})$ -equivariant.

Recall that given $Y \in \mathcal{H}$, we denote by $\Lambda(Y) \subset \partial X$ the limit set of Y in ∂X . It is clear that $\Lambda(Y)$ is in the complement of $\partial_u X$, so

eq:include

$$(\ddagger) \quad \partial_u X \subseteq \partial X \setminus \left(\bigcup_{Y \in \mathcal{H}} \Lambda(Y) \right).$$

In general this can be a proper inclusion, as the following example shows.

ex:bs12

Example 6.9. Let X be the Bass-Serre tree for $BS(1, 2) = \langle a, t \mid tat^{-1} = a^2 \rangle$, so that ∂X is a Cantor set. Then t acts loxodromically on X with axis γ . ($BS(1, 2)$ does not act acylindrically along this axis, since the infinite cyclic group generated by a stabilizes a sub-ray.) Let \mathcal{H} be the collection of translates of γ . The union $\bigcup_{Y \in \mathcal{H}} \Lambda(Y)$ is a countable subset of ∂X , so $\partial X \setminus \bigcup_{Y \in \mathcal{H}} \Lambda(Y)$ is nonempty. However the cone-off \hat{X} of X with respect to \mathcal{H} is bounded, so $\partial\hat{X} = \emptyset$.

We will show that if (G, X, H) is an A/QI triple, then the inclusion (\ddagger) is an equality.

6.1. De-electrification. We need to be able to “lift” paths from \hat{X} to X in a consistent way. There are several notions of this in the literature, usually called “de-electrification.” The following definition is from [Spr17], but it is related to definitions in Bowditch [Bow12] and Dahmani–Mj [DM17]. What we call *de-electrification* here is what Spriano calls *embedded de-electrification*.

¹Dowdall and Taylor use the notation $\partial_{\hat{X}} X$ in [DT17].

def:deelectrification

Definition 6.10. Let \hat{X} be a cone-off of a graph X with respect to a family of uniformly quasi-isometrically embedded subgraphs \mathcal{H} . An \mathcal{H} -geodesic is a path in X which is completely contained in some $Y \in \mathcal{H}$ and is geodesic in Y . Let $\gamma = u_1 \cdot e_1 \cdot \dots \cdot e_n \cdot u_{n+1}$ be a concatenation of geodesics in \hat{X} , where each e_i is an electric edge and the u_i are (possibly trivial) segments of X . A *de-electrification* of γ is a concatenation $u_1 \cdot \eta_1 \cdot \dots \cdot \eta_n \cdot u_{n+1}$ where each η_i is an \mathcal{H} -geodesic joining the endpoints of e_i .

If the subgraphs in \mathcal{H} are not geometrically separated, then a de-electrification of a quasi-geodesic in \hat{X} can be arbitrarily far from being geodesic in X . In particular we cannot expect every de-electrification to be quasi-geodesic. However, Spriano shows that points of \hat{X} are always connected by quasi-geodesics with nice de-electrifications. Say that K is a *constant of quasi-isometric embeddedness* for \mathcal{H} if every element of \mathcal{H} is (K, K) -quasi-isometrically embedded, with K -quasi-convex image.

lem:embedded

Lemma 6.11 ([Spr17, Corollary 2.29]). *For any $\delta \geq 0$, $K \geq 1$, there is a constant $\omega \geq 1$ so that the following holds. Let X be a δ -hyperbolic graph and \mathcal{H} a family of uniformly quasi-isometrically embedded subgraphs with constant of quasi-isometric embeddedness K . Let \hat{X} be the cone-off of X with respect to \mathcal{H} . Then for each pair of vertices $x, y \in X$ there exists an ω -quasi-geodesic γ of \hat{X} from x to y so that every de-electrification of γ is an ω -quasi-geodesic.*

We fix the following data for the rest of the section: (G, X, H) is an A/QI triple where X is a δ -hyperbolic graph containing a basepoint x_0 and Y_0 is an H -cocompact graph in X containing x_0 with constant of quasi-isometric embeddedness K . The space \hat{X} is the cone-off of X with respect to the collection \mathcal{H} of G -translates of Y_0 . The constant ω is obtained by applying Lemma 6.11 to δ and K .

6.2. Characterizing rays with bounded image and proof of Theorem 1.6.

lem:bddimpliesY

Lemma 6.12. *Suppose γ is a quasi-geodesic ray in X . If γ is bounded in \hat{X} then $[\gamma] \in \bigcup_{g \in G} \Lambda(gHg^{-1})$.*

Proof. Fix τ_0 so that γ is a τ_0 -quasi-geodesic ray. We can suppose that γ begins at x_0 . Choose a sequence $\{x_i\}$ on γ tending to infinity in X . We will pass to subsequences several times but use the same notation for the subsequences.

For each i , let σ_i be the ω -quasi-geodesic from x_0 to x_i in \hat{X} provided by Lemma 6.11, and let $\tilde{\sigma}_i$ be any de-electrification. Then $\tilde{\sigma}_i$ is contained in an M -neighborhood of γ , where M is the Morse constant for parameters $\max\{\tau_0, \omega\}$, δ .

Since γ is bounded in \hat{X} , there is some $C \geq 0$ so that $\ell_{\hat{X}}(\sigma_i) \leq C$ for all i . The quasi-geodesics $\tilde{\sigma}_i$ are piecewise quasi-geodesics of the form $\tilde{\sigma}_i = u_{i,1} \cdot \eta_{i,1} \cdot \dots \cdot \eta_{i,k_i} \cdot u_{i,k_i+1}$, where the $u_{i,j}$ are (possibly trivial) paths in X , each $\eta_{i,j}$ is a geodesic in some $Y \in \mathcal{H}$, and $k_i \leq C$ for all i . Since $\ell_X(\tilde{\sigma}_i) \rightarrow \infty$, some of the $\eta_{i,j}$ must be unbounded.

Passing to a subsequence, we may assume that for a fixed j , the lengths $\ell_X(\eta_{i,j})$ tend to infinity, but the lengths of the prefixes $u_{i,1} \cdot \eta_{i,1} \cdot \dots \cdot u_{i,j}$ are uniformly bounded by a constant L . Set $\varepsilon = 3M + 2L$, and let $D = D(\varepsilon, Y_0)$, $N = N(\varepsilon, Y_0)$ be the constants of acylindricity of the action. Discarding the first few terms of the sequence, we may assume that the X -distance between the endpoints of $\eta_{i,j}$

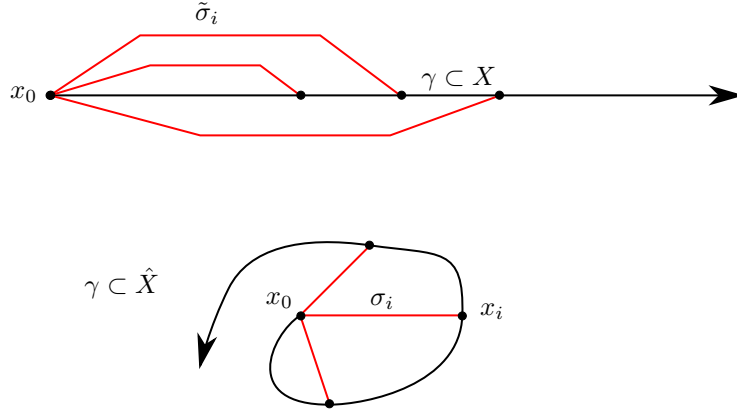


fig:spiral

FIGURE 3. De-electricifications of the paths σ_i must have some subsegments whose lengths go to infinity.

exceeds $D + 2M$ for all i . Let $\gamma_{i,j}$ be the maximal subpath of γ which is contained in the M -neighborhood of $\eta_{i,j}$. Then the length of $\gamma_{i,j}$ is at least D . Moreover, for all i , the initial points of $\gamma_{i,j}$ are at distance at most $2L + 2M$. Let y be a point on γ closest to x_0 that is an endpoint of $\gamma_{i,j}$ for some i , and let γ' be the subpath of γ beginning at y having length D .

Let $\mathcal{Y} \subseteq \mathcal{H}$ be the subset of \mathcal{H} such that each $\eta_{i,j} \subset Y$ for some $Y \in \mathcal{Y}$. Then $\gamma' \subset \mathcal{N}_\varepsilon(Y)$ for each $Y \in \mathcal{Y}$.

Therefore we have

$$\text{diam} \left(\bigcap_{Y \in \mathcal{Y}} \mathcal{N}_\varepsilon(Y) \right) \geq D.$$

As \mathcal{Y} is a collection of translates of the H -cocompact subset Y_0 , Theorem 3.2 implies that \mathcal{Y} can contain at most N distinct elements. In particular the paths $\eta_{i,j}$ lie in a finite union $\bigcup_{k=1}^N Y_k$ of elements of \mathcal{H} . The paths $\eta_{i,j}$ follow travel γ for longer and longer intervals, so the ray γ must limit to a point in $\Lambda(\bigcup_{k=1}^N Y_k)$. The limit set of a finite collection of quasi-convex sets is the union of their limit sets, so $[\gamma] \in \Lambda(Y_k)$ for some fixed $Y_k \in \mathcal{H}$. \square

For the readers convenience we restate the main theorem of this section.

Theorem 1.6. *Suppose (G, X, H) is an A/QI triple and fix $x \in X$. Let $\mathcal{H} = \{gHx \mid gH \in G/H\}$. Let \hat{X} be the cone-off of X with respect to \mathcal{H} . Then $\partial\hat{X}$ is homeomorphic to the subspace of ∂X obtained by deleting the limit sets of all the conjugates of H .*

Proof. Dowdall–Taylor’s Theorem 6.7 tells us that $\partial\hat{X}$ is homeomorphic to the subspace $\partial_u X$ of ∂X consisting of points represented by quasi-geodesic rays whose projection to \hat{X} is unbounded. But Lemma 6.12 tells us that the only way for the image of a quasi-geodesic ray to be bounded is for it to limit to a point in $\Lambda(gHg^{-1})$ for some $g \in G$. \square

6.3. Corollaries.

cor:qiembeddedpersists

Corollary 6.13. *Let (G, X, H) be an A/QI triple, let $K \leq G$ be quasi-isometrically embedded by the action of G on X , and suppose $\Lambda(K) \subseteq \partial X - \bigcup_{g \in G} \Lambda(gHg^{-1})$. Then K is quasi-isometrically embedded by the action of G on \hat{X} .*

The proof of this corollary will make use of a version of the Milnor–Schwartz Lemma which applies to quasi-geodesic spaces. A metric space is *quasi-geodesic* if there exists some $\tau \geq 1$ so that any two points are joined by a τ -quasi-geodesic.

Lemma 6.14 ([Löh17, Proposition 5.4.1]). *If a group G acts coboundedly and metrically properly on a quasi-geodesic metric space Z , then G is finitely generated and for any $z \in Z$, the orbit map $g \mapsto gz$ is a quasi-isometry.*

Proof of Corollary 6.13. We will apply the Milnor–Schwartz Lemma with $Z = Kx_0 \subseteq \hat{X}$, from which the result is immediate. It is clear that K acts cobounded on Kx_0 , so it remains to show that $(Kx_0, d_{\hat{X}})$ is quasi-geodesic and that K acts metrically properly.

By Theorem 6.3, geodesics in X can be reparametrized to be τ -quasi-geodesics in \hat{X} for some fixed $\tau \geq 1$. Let M be the Morse constant for τ -quasi-geodesics in \hat{X} , and let λ be the quasi-convexity constant of Kx_0 in X .

We first show that $(Kx_0, d_{\hat{X}})$ is a quasi-geodesic metric space. As the reader may verify, it suffices to show that Kx_0 is quasi-convex in \hat{X} . Let x, y be points of Kx_0 , and let β be a geodesic of \hat{X} joining them. Let $z \in \beta$, and let α be an X -geodesic joining x to y . Since α is (after reparametrizing) a τ -quasi-geodesic in \hat{X} , there is a point z' of α so that $d_{\hat{X}}(z, z') \leq M$. Since α is a geodesic in X , there is a point $z'' \in Kx_0$ so that $d_X(z', z'') \leq \lambda$. So

$$d_{\hat{X}}(z, Kx_0) \leq d_{\hat{X}}(z, z') + d_{\hat{X}}(z', z'') \leq d_{\hat{X}}(z, z') + d_X(z', z'') \leq M + \lambda,$$

and so Kx_0 is $(M + \lambda)$ -quasi-convex in \hat{X} .

To show that K acts metrically properly on $Kx_0 \subseteq \hat{X}$, let $B \subset Kx_0$ be a finite diameter subset, and suppose toward a contradiction that there exists an infinite collection of elements $k_i \in K$ such that $k_i B \cap B \neq \emptyset$. Since K is quasi-isometrically embedded by the action of G on X , the points $k_i x_0$ do not stay in any bounded subset of X . After passing to a subsequence, we may assume the points $k_i x_0 \in X$ converge to a point $\xi \in \Lambda(K) \subseteq \partial X$. The set $\{k_i x_0\}$ is bounded in \hat{X} , so the point ξ cannot persist in $\partial \hat{X}$. However, this contradicts Theorem 1.6, as $\Lambda(K) \subseteq \partial X - \bigcup_{g \in G} \Lambda(gH)$. \square

We now use the description of the boundary of \hat{X} given in Theorem 1.6 to understand the action of G on \hat{X} .

If a group G acts by isometries on a Gromov hyperbolic metric space Y with basepoint y_0 , then $g \in G$ is *elliptic* if some (equivalently any) orbit in Y is bounded, *loxodromic* if the map $\mathbb{Z} \rightarrow Y$ defined by $n \mapsto g^n y_0$ is a quasi-isometric embedding, and *parabolic* if it is neither elliptic nor loxodromic. One may also use limit sets in ∂Y to distinguish between these types of isometries: an element $g \in G$ is elliptic, parabolic, or loxodromic if $\#\Lambda(\langle g \rangle) = 0, 1$, or 2 , respectively. If $\#\Lambda(\langle g \rangle) = 2$, then we let $\Lambda(\langle g \rangle) = \{g^{\pm\infty}\}$. In this case, there is a quasi-geodesic axis in Y limiting to $g^{\pm\infty}$, and g acts as translation along this axis.

As X is a hyperbolic space, every element of G is either elliptic, loxodromic, or parabolic with respect to $G \curvearrowright X$. Since the canonical map from X to \hat{X} is 1-Lipschitz and G -equivariant, if $g \in G$ is elliptic with respect to $G \curvearrowright X$, then g

is also elliptic with respect to $G \curvearrowright \hat{X}$. It remains to consider elements of G that are parabolic and loxodromic with respect to $G \curvearrowright X$.

cor:loxell

Corollary 6.15. *Suppose (G, X, H) is an A/QI triple, and let \mathcal{H} , \hat{X} be as in Theorem 1.6.*

item:loxodromic

(1) *If $g \in G$ acts loxodromically on X , then either it acts loxodromically on \hat{X} or a power of g stabilizes some $Y \in \mathcal{H}$, and so g acts elliptically on \hat{X} .*

item:parabolic

(2) *An element $g \in G$ acts parabolically on X if and only if it acts parabolically on \hat{X} .*

Proof. Suppose that g acts loxodromically on X . Then $\langle g \rangle$ is quasi-isometrically embedded by the action of G on X . If each of two fixed points of g in ∂X avoids $\Lambda(Y)$ for all $Y \in \mathcal{H}$, then $\langle g \rangle$ is quasi-isometrically embedded by the action of G on \hat{X} by Corollary 6.13, hence g acts loxodromically on \hat{X} .

Thus we may assume that at least one of the fixed points of g lies in the limit set of an element $Y \in \mathcal{H}$. Up to replacing g by g^{-1} , we can assume that $\Lambda(Y)$ contains the repelling fixed point $g^{-\infty} \in \partial X$. The following claim implies statement (1) of the corollary.

claim:lox to ell

Claim 3. *Some power of g stabilizes Y .*

Proof. Let $\alpha_g: \mathbb{R} \rightarrow X$ be a quasi-geodesic axis for g . Let $y_0 \in Y$ be a closest vertex to $\alpha_g(0)$, and let γ_0 be a quasi-geodesic ray lying in Y and limiting to $g^{-\infty}$. By quasi-geodesic stability, there is some $B > 0$ so that $d_X(\alpha_g(0), y_0) \leq B$ and the quasi-geodesic rays γ_0 and $\alpha_g|_{(-\infty, 0]}$ are Hausdorff distance at most B from one another.

Fix $\varepsilon = B$, and let $D = D(\varepsilon, Y)$ and $N = N(\varepsilon, Y)$ be the constants of acylindricity of the action.

For any integer $i \geq 0$, there is a $T_i \geq 0$ so that $\alpha_g(T_i) = g^i \alpha_g(0)$, and we may assume the T_i tend monotonically to infinity. By equivariance, $\alpha_g|_{(-\infty, T_i]}$ is Hausdorff distance at most B from $\gamma_i = g^i \gamma_0 \subseteq g^i Y$. All these rays tend to $g^{-\infty}$ and lie in the B -neighborhood of α_g .

In particular, $\alpha_g|_{(-\infty, T_0]} \subset \mathcal{N}_B(\gamma_0) \subseteq \mathcal{N}_B(g^i Y)$ for each $i \geq 0$. Therefore,

$$\text{diam} \left(\bigcap_{i=1}^{\infty} \mathcal{N}_B(g^i Y) \right) > D.$$

By Theorem 3.2, the collection $\{g^i Y\}$ contains at most N distinct elements.

This implies that there is some $j' \neq j$ such that $g^{j'} Y = g^j Y$. Therefore, $g^{j'-j}$ stabilizes the subgraph Y . \square

We next deal with parabolic elements. If $g \in G$ acts parabolically on \hat{X} , then it has unbounded orbits in X , so it cannot act elliptically on X . But it also cannot act loxodromically, by the first statement. The only remaining possibility is that it acts parabolically.

Conversely, suppose that g acts parabolically on X . Then g fixes a single point $p \in \partial X$.

claim:fp in imphi

Claim 4. *The fixed point p of g lies in $\partial_u X$.*

Proof. We argue by contradiction, supposing that p is not in $\partial_u X$. By Theorem 1.6, this means that $p \in \Lambda(Y)$ for some $Y \in \mathcal{H}$. The proof proceeds similarly to the

proof of Claim 3 with the following changes: we no longer have a quasi-axis α_g , but we instead simply choose some $y_0 \in Y$ and consider a K -quasi-geodesic ray γ_0 in Y from y_0 to p . For some infinite subset $J \subseteq \mathbb{Z}_{\geq 0}$ and some fixed $B > 0$ as before, there are numbers T_j tending monotonically to infinity so that for all $t \geq T_j$, $g^j \gamma_0(t) = \gamma_j(t)$ is contained in the B -neighborhood of γ_0 .

Arguing as in the proof of Claim 3, we conclude that there is some $j' \neq j$ such that $g^{j'-j}$ stabilizes the subgraph Y . However, since Y is a uniformly locally finite graph, it cannot be stabilized by a parabolic isometry. Thus we reach a contradiction, and we conclude that p lies in $\partial_u X$. \square

Let $q = h(p)$ for some loxodromic element $h \in H$, and note that q is in $\partial_u X$. The points $\{g^i q \mid i \in \mathbb{Z}\}$ accumulate on p in $\partial \hat{X}$. If g had a bounded orbit in \hat{X} , this would be impossible. (To see this, consider the image of a τ_0 -quasi-geodesic joining p to q .) Thus g acts parabolically on \hat{X} . \square

There are several similar results in the literature for cone-offs, with varying conditions on X and the subgroup H , for example [Osi06, Theorem 1.14] and [ABO19, Proposition 6.5], which both conclude that if g is loxodromic on X then it is either loxodromic on the cone-off or conjugate into H . In both of these results, there is some hypothesis of properness, which allows the slightly stronger conclusion.

We now consider actions $G \curvearrowright X$ which contain loxodromic WPD elements (see Definition 4.6). We show that if g is WPD with respect to X , and if it is still loxodromic for the action on the cone-off \hat{X} , then it is WPD with respect to \hat{X} . The proof will use the notion of a WWPD element, introduced by Bestvina–Bromberg–Fujiwara in [BBF16]. The definition we use here is due to Handel–Mosher in [HM21]: a loxodromic element $g \in G$ is *WWPD* if the G -orbit of $(g^\infty, g^{-\infty})$ is a discrete subset of $\partial X \times \partial X \setminus \Delta$, where Δ is the diagonal subset. A loxodromic element g is WPD if and only if it is WWPD and the stabilizer of $\{g^\infty, g^{-\infty}\}$ in G is virtually cyclic [HM21, Corollary 2.4].

The following proposition is related to [MMS20, Theorem 2.4], but has slightly different hypotheses.

prop:WPDpersists

Proposition 6.16. *Let (G, X, H) be an A/QI triple. If $g \in G$ is a loxodromic WPD element with respect to the action of G on X and no power of g stabilizes any $Y \in \mathcal{H}$, then g is a loxodromic WPD element with respect to the action on \hat{X} .*

Proof. By Corollary 6.15, the element g acts loxodromically on \hat{X} and the fixed points of g in ∂X lie in $\partial_u X$. Since g is WWPD, the G -orbit of $(g^\infty, g^{-\infty})$ is a discrete subset of $\partial X \times \partial X \setminus \Delta$. This orbit is therefore discrete in $\partial \hat{X} \times \partial \hat{X} \setminus \Delta \cong \partial_u X \times \partial_u X \setminus \Delta$. Therefore g is a WWPD element with respect to the action of G on \hat{X} . Since g is WPD on X , the stabilizer of $\{g^\infty, g^{-\infty}\}$ with respect to the action of G on ∂X is virtually cyclic. Moreover, the stabilizer of the pair $\{g^\infty, g^{-\infty}\}$ does not change when we consider the action of G on $\partial \hat{X} \cong \partial X - \bigcup_{g \in G} \Lambda(gH)$. Therefore g is WPD with respect to the action on \hat{X} , as desired. \square

cor:wpd

Corollary 6.17. *Let G, X, \mathcal{H} , and \hat{X} be as in Theorem 1.6, and suppose additionally that $G \curvearrowright X$ is a WPD action. If there is a loxodromic isometry in the action of G on \hat{X} , then $G \curvearrowright \hat{X}$ is a WPD action.*

sec:examples

7. EXAMPLES

In this section, we use the combination theorem presented in Section 4 to give examples of A/QI triples in $\text{Out}(F_n)$, the Cremona group, and FC-type Artin groups (see Examples 7.1, 7.2, and 7.3). All of these groups are acylindrically hyperbolic, but while we have natural hyperbolic spaces on which each group acts with loxodromic WPD elements, these actions are not known to be acylindrical (and in several cases are known not to be acylindrical). Constructing A/QI triples in these situations provides subspaces along which the action of the group is acylindrical.

We first consider the group $\text{Out}(F_n)$, which acts by isometries on several hyperbolic complexes, including the free factor complex, the cyclic splitting complex, and the free splitting complex. We refer the reader to [HM19, KR14, Man14] for details on these complexes and the actions. Fully irreducible elements are loxodromic WPD elements with respect to the action on all three complexes. These are the only loxodromic isometries of the action on the free factor complex, and so this is a WPD action. However, it is not known if this action is acylindrical. It is also not known if the action on the cyclic splitting complex is WPD or acylindrical. On the other hand, Handel and Mosher [HM19] show that there are loxodromic isometries of the free splitting complex which are not WPD elements, and so this action is neither WPD nor acylindrical. However, in all cases, we are able to construct A/QI triples using Corollary 4.8. As noted in Remark 4.9, the subgroups constructed will typically *not* be hyperbolically embedded.

ex:outfn

Example 7.1 ($\text{Out}(F_n)$). Let $G = \text{Out}(F_n)$, let $h_1, \dots, h_k \in G$ be independent fully irreducible elements, and let X be the free factor complex, the cyclic splitting complex, or the free splitting complex. Then h_1, \dots, h_k are independent loxodromic WPD elements with respect to the action on X . By Corollary 4.8, there is a constant D such that for any $d_1, \dots, d_m \geq D$, if $H = \langle h_1^{d_1}, \dots, h_k^{d_k} \rangle$, then (G, X, H) is an A/QI triple.

A subgroup of $\text{Out}(F_n)$ is called *convex cocompact* if it is quasi-isometrically embedded in the free factor complex via the orbit map. Thus all the subgroups in Example 7.1 are convex cocompact. These subgroups are known to be stable by [ADT17].

We now turn our attention to the Cremona group.

ex:cremona

Example 7.2 (Cremona group). The 2-dimensional Cremona group $\text{Bir}(\mathbb{P}_k^2)$ over an algebraically closed field k is the group of birational transformations of the projective space \mathbb{P}_k^2 . We note that $\text{Bir}(\mathbb{P}_k^2)$ is not a finitely generated group; in fact, it is uncountable. In [CL13], Cantat and Lamy construct an action of $\text{Bir}(\mathbb{P}_k^2)$ on an infinite-dimensional hyperbolic space $\mathbb{H}_{\overline{\mathbb{Z}}}$, which is inspired by work of Manin [Man86] and Zariski [Zar58]. Cantat and Lamy also introduce the notion of a *tight* loxodromic isometry of $\mathbb{H}_{\overline{\mathbb{Z}}}$; see [CL13, Section 2.3.3]. An element $g \in \text{Bir}(\mathbb{P}_k^2)$ is WPD if and only if some positive iterate g^m of g is a tight loxodromic isometry [Can18, Remark 7.5]. (Lonjou also constructs WPD elements in the case that k is not algebraically closed [Lon16].)

Let $G = \text{Bir}(\mathbb{P}_k^2)$, and let $h_1, \dots, h_n \in G$ be independent tight elements. Then h_1, \dots, h_n are independent loxodromic WPD isometries for the action on $\mathbb{H}_{\overline{\mathbb{Z}}}$. By Corollary 4.8, there is a constant D such that for any $d_1, \dots, d_n \geq D$ and $H = \langle h_1^{d_1}, \dots, h_n^{d_n} \rangle$, we have that $(G, \mathbb{H}_{\overline{\mathbb{Z}}}, H)$ is an A/QI triple. Since G is not finitely

generated, there is not a notion of a stable subgroup, but Corollary 4.11 implies that H has finite height.

For our final example, which is more involved than the previous examples, we consider FC-type Artin groups.

ex:artin

Example 7.3 (FC-type Artin groups). Let Γ be a finite graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma) \subset V(\Gamma) \times V(\Gamma)$. Suppose that each edge $(s, t) \in E(\Gamma)$ is labeled with an integer $m(s, t) \geq 2$. The *Artin group* $A(\Gamma)$ associated to the labeled graph Γ is defined by

$$A_\Gamma = \left\langle V(\Gamma) \mid \underbrace{st \cdots}_{m(s,t)} = \underbrace{ts \cdots}_{m(s,t)} \text{ for all } (s, t) \in E(\Gamma) \right\rangle.$$

Let $S = V(\Gamma)$. Given a collection of vertices $T \subset S$, the subgroup A_T of A_Γ generated by T is isomorphic to the Artin group associated to the subgraph of Γ spanned by T . An Artin group is *spherical* if the associated Coxeter group (which is formed by adding the relation $s^2 = 1$ for all $s \in S$) is finite. An important larger class of Artin group are the FC-type Artin groups, which were first introduced by Charney–Davis [CD95]. An Artin group is *FC-type* if A_T is spherical if and only if T is a complete subgraph.

An Artin group A_Γ acts by isometries on its *Deligne complex* $\mathcal{D}(A_\Gamma)$. The Deligne complex (called the *modified Deligne complex* by Charney–Davis) is a cubical complex whose vertices are cosets aA_T where A_T is spherical. Whenever $T \subset T'$ and $\#(T' - T) = 1$, an edge is added between the cosets aA_T and $aA_{T'}$; higher-dimensional cubes are filled in as their 1-skeleta appear. Charney–Davis show in [CD95] that the Deligne complex $\mathcal{D}(A_\Gamma)$ is CAT(0) exactly when A_Γ is FC-type.

Chatterji–Martin show that FC-type Artin groups are acylindrically hyperbolic by applying the following general result about groups acting on CAT(0) cube complexes, whose proof relies on [Mar21, Theorem 1.2]. We refer the reader to [CM19] for the definitions of the terms appearing in theorem. The action of $A_\Gamma \curvearrowright \mathcal{D}(A_\Gamma)$ satisfies the conditions in the first sentence by [CM19, Proposition 5.1].

thm:CM1.1

Theorem 7.4 ([CM19, Theorem 1.1]). *Let G be a group acting essentially and non-elementarily on an irreducible finite-dimensional CAT(0) cube complex. If there exist two hyperplanes whose stabilizers intersect along a finite subgroup, then G admits an acylindrical action on a hyperbolic space.*

The proof of this theorem has two steps. The first is to show that there exists a loxodromic WPD isometry $g \in G$ with respect to the action on the CAT(0) cube complex whose axis in G is strongly contracting. Since the CAT(0) cube complex is not required to be hyperbolic, this is not sufficient to prove that G is acylindrically hyperbolic. The second step is to apply the projection complex machinery of Bestvina–Bromberg–Fujiwara from [BBF15] to conclude that g is a loxodromic WPD isometry in an action of G on a hyperbolic space. Osin shows that this is equivalent to G acting acylindrically on a (possibly different) hyperbolic space [Osi16, Theorem 1.4]. The projection complex is not the only hyperbolic space that could be used as the input for Osin’s theorem. Genevois constructs a family of hyperbolic spaces on which G acts by isometries, and the element g will be a loxodromic WPD isometry of infinitely many of these spaces [Gen19]. (Genevois’s construction generalizes Hagen’s construction of the contact graph [Hag14].)

A graph *has no empty squares* if every circuit of length four has at least one diagonal pair of vertices spanning an edge. Charney–Crisp [CC07, Theorem 4.2] show that if A_Γ is an FC-type Artin group and Γ has no empty squares, then $\mathcal{D}(A_\Gamma)$ is actually CAT(−1); in particular, it is hyperbolic. Thus to prove Theorem 7.4 in the special case of an FC-type Artin group whose defining graph has no empty squares, the Deligne complex can be used directly as the input to Osin’s theorem.

In this setting, the following statement can be extracted from the proof of [CM19, Theorem 1.1]. Given disjoint hyperplanes \hat{H}_1, \hat{H}_2 in an irreducible CAT(0) cube complex X , an automorphism g of X *double-skewers* \hat{H}_1 and \hat{H}_2 if there is a choice of corresponding nested half spaces $H_1 \subset H_2$ such that $H_1 \subset H_2 \subset gH_1$.

Proposition 7.5 ([CM19, Theorem 1.1]). *Let A_Γ be an Artin group of FC type whose defining graph has no empty squares. If there exist two disjoint hyperplanes in the Deligne complex $\mathcal{D}(A_\Gamma)$ whose stabilizers intersect along a finite subgroup, then any element which double-skewers these hyperplanes is a loxodromic WPD element with respect to the action of A_Γ on $\mathcal{D}(A_\Gamma)$.*

Chatterji–Martin use the Sector Lemma [CS11, Lemma 5.2] and the Double-Skewering Lemma [CS11, Double-Skewering Lemma] to show that such elements must always exist. However, the proof does not explicitly construct such elements. We now explicitly construct a family of examples.

The construction involves understanding the intersection pattern of hyperplanes in the Deligne complex. As described in [MW21, Remark 2.2], every hyperplane in $\mathcal{D}(A_\Gamma)$ is a translate of a hyperplane that crosses an edge $(A_\emptyset, A_{\{v\}})$ for some vertex v in Γ . Such a hyperplane is called a *hyperplane of type v* . The action of A_Γ on the Deligne complex preserves hyperplane types, and a hyperplane of type v intersects some hyperplane of type w exactly when w is in the link of v in Γ .

Lemma 7.6. *Let A_Γ be an FC-type Artin group whose defining graph Γ has diameter at least 3 and no empty squares. For any two vertices $s, t \in V(\Gamma)$ at distance at least 3 in Γ and any integers $m, n \in \mathbb{Z} \setminus \{0\}$, the product $s^m t^n$ is a loxodromic WPD element with respect to the action of A_Γ on the Deligne complex. Moreover, if $m', n' \in \mathbb{Z} \setminus \{0\}$ and $(m', n') \neq (m, n)$, then the elements $s^{m'} t^{n'}$ and $s^m t^n$ are independent.*

Proof. If Γ is not connected, then A_Γ splits as a free product and the Deligne complex $\mathcal{D}(A_\Gamma)$ has the structure of a tree of spaces. If s, t are in different connected components of Γ , the result is clear. Thus we may assume that Γ is connected.

Let \hat{H}_s be the hyperplane of $\mathcal{D}(A_\Gamma)$ which crosses the edge $(A_\emptyset, A_{\{s\}})$, and let \hat{H}_t be the hyperplane which crosses the edge $(A_\emptyset, A_{\{t\}})$. As noted in the proof of [CM19, Theorem 1.2], since s and t are at distance at least 3 in Γ , we have $\text{Stab}(\hat{H}_s) \cap \text{Stab}(\hat{H}_t) = \emptyset$. Moreover, $s^m t^n$ is a loxodromic isometry of $\mathcal{D}(A_\Gamma)$ which has a geodesic axis formed by the union of the images of the segment shown in Figure 4 under $\langle s^m t^n \rangle$ by [CM19, Lemma 5.5]. (Strictly speaking, [CM19, Lemma 5.5] only applies to the element st , but the same proof will give the result for $s^m t^n$; this is explicitly stated for the element $s^2 t$ in the proof of [CM19, Lemma 5.4].)

Fix $m, n \in \mathbb{Z} \setminus \{0\}$. Since t is not contained in the link of s , we have $\hat{H}_s \cap \hat{H}_t = \emptyset$. Thus there exist half spaces H_s and H_t satisfying $H_t \subset H_s$ (see Figure 4). We will show that $s^m t^n$ double skewers H_s and H_t , that is, we will show that $H_t \subset H_s \subset s^m t^n H_t$. First, notice that since s is not contained in the link of t , the

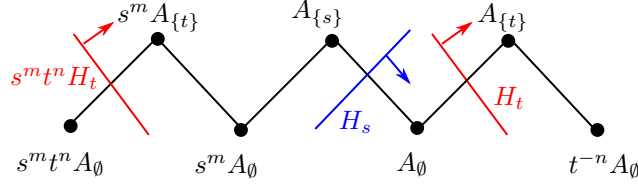
FIGURE 4. A geodesic axis of $s^m t^n$ in $\mathcal{D}(A_\Gamma)$, with the relevant half spaces.

fig:Deligneaxis

hyperplanes \hat{H}_s and $s^m t^n \hat{H}_t$ are disjoint. Thus $H_s \subset s^m t^n H_t$ or $s^m t^n H_t \subset H_s$. If $s^m t^n H_t \subset H_s$, then \hat{H}_s must intersect the path $A_{\{s\}}, s^m A_0, s^m A_{\{t\}}$ exactly once. However, since \hat{H}_s is of type s , it does not cross the edge $(s^m A_0, s^m A_{\{t\}})$. Moreover, since hyperplanes in a CAT(0) cube complex do not self-oscillate and \hat{H}_s crosses the edge $(A_0, A_{\{s\}})$, we see that \hat{H}_s cannot cross $(A_{\{s\}}, s^m A_0)$. Thus $H_s \subset s^m t^n H_t$, and we conclude that $s^m t^n$ double-skewers H_s and H_t .

Therefore $s^m t^n$ is a loxodromic WPD element with respect to the action of A_Γ on the Deligne complex by [CM19, Theorem 1.1].

The proof of [CM19, Lemma 5.4] shows that st and $s^2 t$ are independent, but the proof goes through as written for elements $s^{m'} t^{n'}$ and $s^m t^n$ whenever $(m', n') \neq (m, n)$. \square

Let A_Γ be an FC-type Artin group whose defining graph Γ has diameter at least 3 and no empty squares. Let $s, t \in V(\Gamma)$ be at distance at least 3, and fix distinct elements $h_1, \dots, h_k \in \{s^m t^n \mid m, n \in \mathbb{Z} \setminus \{0\}\}$. By Corollary 4.8 there is a constant D such that for any $d_1, \dots, d_k \geq D$ and $H = \langle h_1^{d_1}, \dots, h_k^{d_k} \rangle$, we have that $(A_\Gamma, \mathcal{D}(A_\Gamma), H)$ is an A/QI triple. In particular, the subgroup H is stable.

sec:questions

8. QUESTIONS

The first question we pose asks for a generalization of Proposition 4.1.

Question 8.1. *Suppose (G, X, H) and (G, Y, K) are A/QI triples but X and Y are not equivariantly quasi-isometric. Does there exist an action $G \curvearrowright Z$ such that $(G, Z, H \cap K)$ is an A/QI triple?*

The next question asks if there is a more general combination theorem than the one presented in Theorem 1.4.

Question 8.2. *Suppose (G, X, H) and (G, X, K) are A/QI triples and $H_0 \leq H$ and $K_0 \leq K$ are quasi-convex subgroups such that every element of $H_0 \setminus K$ and every element of $K_0 \setminus H$ translates the basepoint of X a large distance. Let $P = \langle H_0, K_0 \rangle$. Is (G, X, P) an A/QI triple?*

Corollary 6.17 shows that if (G, X, H) is an A/QI triple and $G \curvearrowright X$ is WPD, then $G \curvearrowright \hat{X}$ is also WPD. Since an acylindrical action is a WPD action, we can conclude that if $G \curvearrowright X$ is acylindrical, then $G \curvearrowright \hat{X}$ is WPD. This motivates the following question.

Question 8.3. *Suppose that (G, X, H) is an A/QI triple where $G \curvearrowright X$ is actually acylindrical. Form the cone-off \hat{X} as in Section 6. Is the action $G \curvearrowright \hat{X}$ also acylindrical?*

Example 7.1 shows that many known convex cocompact subgroups of $\text{Out}(F_n)$ give A/QI triples. Moreover, every convex cocompact subgroup of $\text{Out}(F_n)$ quasi-isometrically embeds in the free factor complex.

Question 8.4. *Does $\text{Out}(F_n)$ always act acylindrically along the image of a convex cocompact subgroup in the free factor complex?*

As well as finite height, Antolín–Mj–Sisto–Taylor show finite width and bounded packing for stable subgroups [AMST19]. We would like to know if these properties hold for H in an A/QI triple (G, X, H) even when G is not finitely generated.

Question 8.5. *When G is not finitely generated, but there is an A/QI triple (G, X, H) , does H have finite width? If G is countable, does H have bounded packing?*

Part of the usefulness of finite height in context of quasi-convex subgroups of hyperbolic groups is that one can pass to an *almost malnormal core* of a finite collection of quasi-convex groups. This core is a collection of quasi-convex subgroups of height 1, so the ambient group is hyperbolic relative to this collection, and one can attempt to run various inductive arguments on height using Dehn filling (as for example in [AGM09] and the appendix to [Ago13]). Rather than aiming to produce a relatively hyperbolic structure on G , one might hope to produce a hyperbolically embedded collection of subgroups. Such a collection would necessarily have height one. As we have mentioned, the H in an A/QI triple (G, X, H) is not necessarily hyperbolically embedded as its height may be strictly bigger than one. In order to pass to a well-behaved almost malnormal core, one would like to know the answer to the following question.

Question 8.6. *Consider the collection of infinite subgroups of H of the form $\bigcap_{i=1}^k g_k H g_k^{-1}$. Do these subgroups fall in finitely many H -conjugacy classes? Do the minimal such subgroups give rise to a hyperbolically embedded collection in G (after possibly replacing each by a finite index supergroup and choosing one per G -conjugacy class)?*

Using the machinery of hierarchically hyperbolic groups, Spriano in [Spr17] proves Theorem 1.6 in the special case where \mathcal{H} is a collection of infinite quasi-convex subspaces which can be extended to form a *weak factor system*. Indeed Spriano gives finer information about the boundary in this case, decomposing it into the boundaries of the various elements of the factor system, one of which is $\partial\tilde{X}$. When X is the Cayley graph of a hyperbolic group and \mathcal{H} is the set of cosets of a finite collection of infinite quasi-convex subgroups, Spriano shows that \mathcal{H} extends to a weak factor system [Spr17, Section 5.2]. Spriano’s argument seems to strongly use properness. In general, X may not be a proper metric space, and thus we ask the following question:

Question 8.7. *Under the assumptions of Theorem 1.6, can the set of translates gHx can be extended to form a weak factor system?*

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