THE BOWDITCH BOUNDARY OF $(G,\mathcal{H})$ WHEN G IS HYPERBOLIC

JASON FOX MANNING

Abstract. In this note we use Yaman's dynamical characterization of relative hyperbolicity to prove a theorem of Bowditch about relatively hyperbolic pairs $(G,\mathcal{H})$ with G hyperbolic. Our proof additionally gives a description of the Bowditch boundary of such a pair. This description of the boundary was previously obtained by Tran [Tra13].

1. Introduction

Let G be a group. A collection $\mathcal{H} = \{H_1, \ldots, H_n\}$ of subgroups of G is said to be *almost malnormal* if every infinite intersection of the form $H_i \cap g^{-1}H_j g$ satisfies both $i = j$ and $g \in H_i$.

In an extremely influential paper from 1999, published in 2012 in *IJAC* [Bow12], Bowditch proves the following useful theorem:

Theorem 1.1. [Bow12, Theorem 7.11] *Let G be a nonelementary hyperbolic group, and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be an almost malnormal collection of proper, quasiconvex subgroups of G. Then G is hyperbolic relative to $\mathcal{H}$.*

Remark 1.2. The converse to this theorem also holds and is implicit in Bowditch's work. If $(G,\mathcal{H})$ is any relatively hyperbolic pair, then the collection $\mathcal{H}$ is almost malnormal by [Osi06, Proposition 2.36] (cf. [Bow12, p. 4]). Moreover the elements of $\mathcal{H}$ are undistorted in G [Osi06, Lemma 5.4] (cf. [Bow12, Lemma 3.5]). Undistorted subgroups of a hyperbolic group are quasiconvex.

In this note, we give a proof of Theorem 1.1 which differs from Bowditch's. The strategy we follow is to exploit the dynamical characterization of relative hyperbolicity given by Yaman in [Yam04]. By doing so, we are able to obtain some more information about the pair $(G,\mathcal{H})$. In particular, we obtain an explicit description of its Bowditch boundary $\partial(G,\mathcal{H})$. (This same strategy was applied by Dahmani to describe the boundary of certain amalgams of relatively hyperbolic groups in [Dah03].) Let ∂G be the Gromov boundary of the group G. If H is quasiconvex in a hyperbolic group G, its limit set $\Lambda(H) \subset \partial G$ is homeomorphic to the Gromov boundary ∂H of H. Our proof of Theorem 1.1 also yields the following result (previously obtained by Tran [Tra13]), which says that $\partial(G,\mathcal{H})$ is obtained by smashing the limit sets of gHg^{-1} to points, for $H \in \mathcal{H}$ and $g \in G$.

Theorem 1.3. *Let G be hyperbolic, and let $\mathcal{H}$ be an almost malnormal collection of infinite quasi-convex proper subgroups of G. Let $\mathcal{L}$ be the set of G–translates of limit sets of elements of $\mathcal{H}$. The Bowditch boundary $\partial(G,\mathcal{H})$ is obtained from the Gromov boundary ∂G as a decomposition space $\partial G/\mathcal{L}$.*

The support of the Simons Foundation (#524176 to J. Manning) and the National Science Foundation (DMS-0804369) is gratefully acknowledged.

Remark 1.4. After I posted a version of this paper on the arXiv, I learned that Theorem 1.3 was already well-known. See in particular the main result of Tran's paper [Tra13] which additionally gives a similar description of the Bowditch boundary in terms of a CAT(0) boundary when G is CAT(0) and relatively hyperbolic. Tran also points out previous results of Gerasimov and Gerasimov–Potyagailo [Ger12, GP13], or alternatively Matsuda–Oguni–Yamagata [MOY12] which can be used to give other proofs of Theorem 1.3. More recently, an "HHS" proof can be found in [Spr17, Section 6].

If there is an advantage to the current approach, it is that we obtain a proof of both Theorems 1.1 and 1.3 at the same time.

One consequence of the explicit description is a bound on the dimension of such a Bowditch boundary.

Corollary 1.5. *Let G be a hyperbolic group and $\mathcal{H}$ an almost malnormal collection of infinite quasi-convex proper subgroups. Then* $\dim \partial(G,\mathcal{H}) \leq \dim \partial G + 1$.

Proof. This follows from the Subspace and Addition Theorems of dimension theory. By Theorem 1.3, $\partial(G,\mathcal{H})$ can be written as a union of a countable set A (coming from the limit sets of the conjugates of the elements of $\mathcal{H}$) with a subspace B of ∂G. The Subspace Theorem implies $\dim(B) \leq \dim \partial G$, and the Addition Theorem implies $\dim(A \cup B) \leq \dim A + \dim B + 1$. □

We can see this as some weak evidence for the following conjecture. (Here $\mathrm{cd}(G,\mathcal{H})$ is the maximum n so $H^n(G,\mathcal{H};M) \neq 0$ for some $\mathbb{Z}G$–module M.)

Conjecture 1.6. [MW] *Let $(G,\mathcal{H})$ be relatively hyperbolic and type F. Then*

$$\dim \partial(G,\mathcal{H}) = \mathrm{cd}(G,\mathcal{H}) - 1.$$

In the absolute setting ($\mathcal{H} = \emptyset$) Conjecture 1.6 is a theorem of Bestvina and Mess [BM91]. It is shown in [MW] that it also holds in case $\mathrm{cd}(G) < \mathrm{cd}(G,\mathcal{H})$.

Here is the connection between Corollary 1.5 and the conjecture. In [MW] it is shown that if $(G,\mathcal{H})$ is relatively hyperbolic and type F_∞, then for all $k \geq 0$ there is an isomorphism

$$(*) \qquad \check{H}^k(\partial(G,\mathcal{H});\mathbb{Z}) \cong H^{k+1}(G,\mathcal{H};\mathbb{Z}G),$$

where the left-hand side is reduced Čech cohomology and the right-hand side is relative group cohomology as defined for example in [BE78].[1] It follows that the inequality $\dim \partial(G,\mathcal{H}) \geq \mathrm{cd}(G,\mathcal{H}) - 1$ always holds for a type F pair, since for any space X we have the inequality

$$\dim(X) \geq \max\{k \mid \check{H}^k(X;\mathbb{Z}) \neq 0\} =: \mathrm{ai\check{C}d}(X).$$

(The notation 'aiČd' stands for *absolute integral Čech dimension.*) The statements [BM91, Corollaries 1.3(b) and 1.4(b)] combined show that, for a hyperbolic group, $\mathrm{ai\check{C}d}(\partial G) = \dim(\partial G)$. The isomorphisms $(*)$ and the long exact sequence of a group pair then give

$$(\dagger) \qquad \mathrm{ai\check{C}d}\,(\partial(G,\mathcal{H})) \leq \mathrm{ai\check{C}d}(\partial G) + 1 = \dim(\partial G) + 1.$$

Corollary 1.5 strengthens (†) in exactly the way that Conjecture 1.6 would predict.

We next recall the definition of a convergence group.

[1] In the absolute case, this is another theorem of Bestvina–Mess [BM91]. In the case of geometrically finite groups of isometries of $\mathbb{H}^n$ it is due to Kapovich [Kap09, Proposition 9.6].

Definition 1.7. Suppose that M is a compact metrizable space with at least 3 points, and let G act on M by homeomorphisms. The action is a *convergence group action* if the induced action on the space $\Theta^3(M)$ of unordered triples of distinct points in M is properly discontinuous.

An element $g \in G$ is *loxodromic* if it has infinite order and fixes exactly two points of M.

A point $p \in M$ is a *bounded parabolic point* if $\mathrm{Stab}_G(p)$ contains no loxodromics, and acts cocompactly on $M \setminus \{p\}$.

A point $p \in M$ is a *conical limit point* if there is a sequence $\{g_i\}$ in G and a pair of points $a \neq b$ in M so that:

(1) $\lim_{i\to\infty} g_i(p) = a$, and
(2) $\lim_{i\to\infty} g_i(x) = b$ for all $x \in M \setminus \{p\}$.

A convergence group action of G on M is *geometrically finite* if every point in M is either a bounded parabolic point or a conical limit point.

Bowditch proved in [Bow98] that if G acts on M as a convergence group and every point of M is a conical limit point, then G is hyperbolic. Conversely, if G is hyperbolic, then G acts as a convergence group on ∂G, and every point in ∂G is a conical limit point. For general geometrically finite actions, we have the following result of Yaman:

Theorem 1.8. [Yam04, Theorem 0.1] *Suppose that M is a non-empty perfect metrizable compact space, and suppose that G acts on M as a geometrically finite convergence group. Let $B \subset M$ be the set of bounded parabolic points. Let $\{p_1, \ldots, p_n\}$ be a set of orbit representatives for the action of G on B. For each i let P_i be the stabilizer in G of p_i, and let $\mathcal{P} = \{P_1, \ldots, P_n\}$.*

Then $(G, \mathcal{P})$ is relatively hyperbolic and M is equivariantly homeomorphic to $\partial(G, \mathcal{P})$.

Outline of proof of Theorems 1.1 and 1.3. We prove Theorem 1.1 by constructing a space M on which G acts as a geometrically finite convergence group, so that the parabolic point stabilizers are all conjugate to elements of $\mathcal{H}$. The space M is a quotient of ∂G, constructed as follows. The hypotheses on $\mathcal{H}$ imply that the boundaries ∂H_i embed in ∂G for each i, and that $g\partial H_i \cap h\partial H_j$ is empty unless $i = j$ and $g^{-1}h \in H_i$. Let

$$\mathcal{A} = \{g\partial H_i \mid g \in G, \text{ and } H_i \in \mathcal{H}\},$$

and let

$$\mathcal{B} = \{\{x\} \mid x \in \partial G \setminus \bigcup A\}.$$

The union $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ is therefore a decomposition of ∂G into closed sets. We let M be the quotient topological space $\partial G/\mathcal{C}$ and write A, B for the images of $\mathcal{A}, \mathcal{B}$, respectively. There is clearly an action of G on M by homeomorphisms.

We now have a sequence of four claims, which we prove in Section 2.

Claim 1. *$M = A \cup B$ is a perfect metrizable space.*

Claim 2. *G acts as a convergence group on M.*

Claim 3. *For $x \in A$, x is a bounded parabolic point, with stabilizer conjugate to an element of $\mathcal{H}$.*

Claim 4. *For $x \in B$, x is a conical limit point.*

Given the claims, we may apply Yaman's theorem 1.8 to conclude that the pair $(G, \mathcal{H})$ is relatively hyperbolic (Theorem 1.1), and that the Bowditch boundary is equivariantly homeomorphic to $\partial G/\mathcal{C}$ (Theorem 1.3). $\square$

2. Proofs of claims

In what follows we fix some δ-hyperbolic Cayley graph Γ of G. We'll use the notation $a \mapsto \bar{a}$ for the map from ∂G to the decomposition space M.

2.1. **Claim 1.**

In this subsection we show the decomposition space M is perfect and metrizable. We first recall the terminology of upper semicontinuous decomposition spaces. Let X be a topological space, and let $\mathcal{D} \subseteq 2^X$ be a decomposition of X into compact subsets. The quotient $X/\mathcal{D}$ is the *decomposition space*. As a set $X/\mathcal{D} = \mathcal{D}$, topologized so that $A \subset \mathcal{D}$ is open in $X/\mathcal{D}$ if and only if $\bigcup A$ is open in X.

A subset of X is *$\mathcal{D}$–saturated* if it is a union of elements of $\mathcal{D}$. The decomposition is said to be *upper semicontinuous* if for every $D \in \mathcal{D}$ and every open set U of X containing D, there is a $\mathcal{D}$–saturated open V so that $D \subseteq V \subseteq U$.

We have the following well-known characterization.

Proposition 2.1. *Let X be compact metrizable, and let $\mathcal{D}$ be a decomposition of X into closed subsets. The following are equivalent:*

(1) *$\mathcal{D}$ is upper semicontinuous.*
(2) *$X/\mathcal{D}$ is compact metrizable.*

Proof. For (1) $\implies$ (2), see [Dav86, p. 13, Proposition 2]. For (2) $\implies$ (1), see [HY88, p. 132, Theorem 3-31]. $\square$

A countable collection of subsets $\mathcal{N}$ of a metric space X is said to be a *null sequence* if, for all $\epsilon > 0$, there are only finitely many $N \in \mathcal{N}$ of diameter greater than ϵ. We need the following useful fact.

Proposition 2.2. [Dav86, p. 14, Proposition 3] *Suppose X is a metric space, and $\mathcal{D}$ is a decomposition so that the collection of nondegenerate elements of $\mathcal{D}$ is a null sequence. Then $\mathcal{D}$ is upper semicontinuous.*

Proof of Claim 1. We first note that the collection $\mathcal{A} \subset \mathcal{C}$ of limit sets of cosets is a null sequence (see [GMRS98, Corollary 2.5]), and that $\mathcal{A}$ consists precisely of the nondegenerate elements of $\mathcal{C}$. Applying Proposition 2.2 we see that $\mathcal{C}$ is an upper semicontinuous decomposition of ∂G. Proposition 2.1 then implies that $M = \partial G/\mathcal{C}$ is compact metric.

We now show M is perfect. Let $p \in M$.

Suppose first that $p \in B$, i.e., that the preimage in ∂G is a single point $\tilde{p}$. Because G is nonelementary, ∂G is perfect. Thus there is a sequence of points $x_i \in \partial G \setminus \{\tilde{p}\}$ limiting on p. The image of this sequence limits on p.

Now suppose that $p \in A$, i.e., the preimage of p in ∂G is equal to $g\partial H$ for some $g \in G$ and some $H \in \mathcal{H}$. Choose any point $x \in \partial G \setminus \partial H$, and any infinite order element h of gHg^{-1}. The points $h^i x$ project to distinct points in $M \setminus \{p\}$, limiting on p. $\square$

2.2. **Claim 2.**

Next we show the action of G on M is a convergence action. In [Bow99], Bowditch gives a characterization of convergence group actions in terms of *collapsing sets*. We rephrase Bowditch slightly in what follows.

Definition 2.3. Let G act by homeomorphisms on M. Suppose that $\{g_i\}$ is a sequence of distinct elements of G. Suppose that there exist points a and b (called the *repelling* and *attracting* points, respectively) so that whenever $K \subseteq M \setminus \{a\}$ and $L \subseteq M \setminus \{b\}$ are compact, the set $\{i \mid g_i K \cap L \neq \emptyset\}$ is finite. Then $\{g_i\}$ is a *collapsing sequence.*

Proposition 2.4. [Bow99, Proposition 1.1] *Let G, a countable group, act on M, a compact Hausdorff space with at least 3 points. Then G acts as a convergence group if and only if every infinite sequence in G contains a subsequence which is collapsing.*

Proof of Claim 2. We use the characterization of 2.4. Let $\{\gamma_i\}$ be an infinite sequence in G. Since the action of G on ∂G is convergence, there is a collapsing subsequence $\{g_i\}$ of $\{\gamma_i\}$; i.e., there are points a and b in ∂G which are repelling and attracting in the sense of Definition 2.3. We will show that $\{g_i\}$ is also a collapsing sequence for the action of G on M, and that the images $\bar{a}$ and $\bar{b}$ in M are the repelling and attracting points for this sequence.

Let $K \subseteq M \setminus \{\bar{a}\}$ and $L \subseteq M \setminus \{\bar{b}\}$ be compact sets, and let $\tilde{K}$ and $\tilde{L}$ be the preimages of K and L in ∂G. We have $\tilde{K} \subseteq \partial G \setminus \{a\}$ and $\tilde{L} \subseteq \partial G \setminus \{b\}$, so $\{i \mid g_i \tilde{K} \cap \tilde{L} \neq \emptyset\}$ is finite. But for each i, $g_i K \cap L = \pi(g_i \tilde{K} \cap \tilde{L})$, so $\{i \mid g_i K \cap L\}$ is also finite. □

Remark 2.5. In the preceding proof it is possible for a and b to be distinct, but $\bar{a} = \bar{b}$.

2.3. **Claim 3.** We next show the nondegenerate sets in the decomposition give rise to bounded parabolic points.

Proof of Claim 3. Let $p \in A \subseteq M$ be the image of $g\partial H$ for $g \in G$ and $H \in \mathcal{H}$. Let $P = gHg^{-1}$. Since H is equal to its own commensurator, so is P, and $P = \mathrm{Stab}_G(p)$. We must show that P acts cocompactly on $M \setminus \{p\}$. The subgroup P is λ-quasiconvex in Γ (the Cayley graph of G) for some $\lambda > 0$. Let N be a closed R-neighborhood of P in Γ for some large integer R, with $R > 2\lambda + 10\delta$. Note that any geodesic from 1 to a point in ∂H stays inside N, and any geodesic from 1 to a point in $\partial G \setminus \partial P$ eventually leaves N. Write $\mathrm{Front}(N)$ for the frontier of N.

Let $K = \{g \in \mathrm{Front}(N) \mid d(g, 1) \leq 2R + 100\delta\}$. Let E be the set of points $e \in \partial X$ so that there is a geodesic from 1 to e passing through K. The set E is compact, and lies entirely in $\partial G \setminus \partial P$. We will show that $PE = \partial G \setminus \partial P$. Let $e \in \partial G \setminus \partial P$, and let $h \in P$ be "coarsely closest" to e in the following sense: If $\{x_i\}$ is a sequence of points in X tending to e, then for large enough i, we have, for any $h' \in P$, $d(h, e_i) \leq d(h', e_i) + 4\delta$. Let γ be a geodesic ray from h to e, and let d be the unique point in $\gamma \cap \mathrm{Front}(N)$. Since $d \in \mathrm{Front}(N)$, there is some h' so that $d(h', d) = R$. Let e' be a point on γ so that $10R < d(h, e') \leq d(h', e') + 4\delta$, and consider a geodesic triangle made up of that part of γ between h and e', some geodesic between h' and h, and some geodesic between h' and e'. This triangle has a corresponding comparison tripod, as in Figure 1. Since any geodesic from h' to h must stay $R - \lambda > \delta$ away from $\mathrm{Front}(N)$, the point $\bar{d}$ must lie on the leg of the tripod corresponding to e'. Let d' be the point on the geodesic from h' to e' which projects to $\bar{d}$ in the comparison tripod. Since $d(h', d) = R$, $d(h', d') \leq R + \delta$. Now

FIGURE 1. Bounding the distance from h to d.

notice that

$$\begin{aligned} d(h,d) &\leq d(h',d') + (e',h')_h - (e',h)_{h'} \\ &\leq d(h',d') + 4\delta \\ &\leq R + 5\delta. \end{aligned}$$

But this implies that the geodesic from 1 to $h^{-1}e$ passes through K, and so $h^{-1}e \in E$ and $e \in hE$. Since e was arbitrary in $\partial G \setminus \partial P$, we have $PE = \partial G \setminus \partial P$, and so the action of P on $\partial G \setminus \partial P$ is cocompact. If $\bar{E}$ is the (compact) image of E in M, then $PE = M \setminus \{p\}$, and so p is a bounded parabolic point. □

2.4. **Claim 4.** Lastly, we must show that the remaining points of M are conical limit points. The proofs of the first two lemmas are left to the reader.

Lemma 2.6. *For all $R > 0$ there is some D, depending only on R, G, $\mathcal{H}$, and S, so that for any g, $g' \in G$, and H, $H' \in \mathcal{H}$,*

$$\operatorname{diam}(N_R(gH) \cap N_R(g'H')) < D.$$

($N_R(Z)$ denotes the R-neighborhood of Z in the Cayley graph $\Gamma = \Gamma(G,S)$.)

Lemma 2.7. *There is some λ depending only on G, $\mathcal{H}$, and S, so that if x, $y \in gH \cup g\partial H$, then any geodesic from x to y lies in a λ-neighborhood of gH in Γ.*

Lemma 2.8. *Let $\gamma\colon \mathbb{R}_+ \to \Gamma$ be a (unit speed) geodesic ray, so that $x = \lim_{t\to\infty} \gamma(t)$ is not in the limit set of gH for any $g \in G$, $H \in \mathcal{H}$, and so that $\gamma(0) \in G$. Let $C > 0$. There is a sequence of numbers $\{n_i\}$ tending to infinity, and a constant χ, so that the following holds, for all $i \in \mathbb{N}$: If $x_i = \gamma(n_i) \in N_C(gH)$ for $g \in G$ and $H \in \mathcal{H}$, then*

$$\operatorname{diam}\left(N_C(gH) \cap \gamma([n_i,\infty))\right) < \chi$$

Proof. Let λ be the quasi-convexity constant from Lemma 2.7. Let D be the constant obtained from Lemma 2.6, setting $R = C + \lambda + 2\delta$, and let $\chi = 2D$.

We define n_i inductively. Let $i \in \mathbb{N}$. If $i = 1$, set $t_1 = 0$; otherwise set $t_i = n_{i-1} + 1$. We will find $n_i \geq t_i$ satisfying the condition in the statement.

If setting $n_i = t_i$ does not work, then there must be some gH with $g \in G$ and $H \in \mathcal{H}$ satisfying $\gamma(t_i) \in N_C(gH)$ and

$$\operatorname{diam}\left(N_C(gH) \cap \gamma([t_i,\infty))\right) \geq \chi.$$

Let $s = \sup\{t \mid \gamma(t) \in N_C(gH)\}$. We claim that we can choose

$$n_i = s - \frac{\chi}{2} = s - D.$$

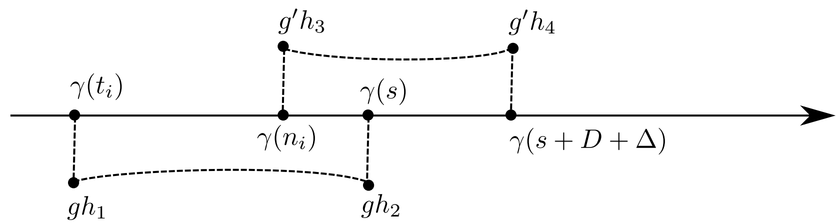

FIGURE 2. The segment $\gamma|_{[n_i,s]}$ is close to both gH and $g'H'$.

Clearly we have

$$\operatorname{diam}\left(N_C(gH) \cap \gamma([n_i, \infty))\right) < \chi.$$

Now suppose that some other $g'H'$ satisfies $x_i = \gamma(n_i) \in N_C(g'H')$ and

$$\operatorname{diam}\left(N_C(g'H') \cap \gamma([n_i, \infty))\right) \geq \chi.$$

There is then some $\Delta \geq 0$ so that $\gamma(s + D + \Delta)$ is within C of $g'H'$. It is straightforward to show (see Figure 2) that $\gamma(n_i)$ and $\gamma(s)$ lie both in the $C + \lambda + 2\delta$ neighborhood of gH and in the $C + \lambda + 2\delta$ neighborhood of $g'H'$. Since $d(\gamma(n_i), \gamma(s)) = s - n_i = D$, this contradicts Lemma 2.6. □

Proof of Claim 4. Let $x \in B = \partial G \setminus \cup\mathcal{A}$. We must show that $\bar{x} \in M$ is a conical limit point for the action of G on M. Fix some $y \in M \setminus \{x\}$, and let γ be a geodesic from y to x in Γ. Let $C = \lambda + 6\delta$, where λ is the constant from Lemma 2.7. Using Lemma 2.8, we can choose a sequence of (inverses of) group elements $\{{x_i}^{-1}\}$ in the image of γ so that whenever $x_i \in N_C(gH)$ for some $g \in G$, $H \in \mathcal{H}$, and $i \in \mathbb{N}$, we have

$$\operatorname{diam}\left(N_C(gH) \cap \gamma([n_i, \infty))\right) < \chi, \tag{1}$$

for some constant χ independent of g, H, and i.

Now consider the geodesics $x_i\gamma$. They all pass through 1, so we may pick a subsequence $\{x_i'\}$ so that the geodesics $x_i'\gamma$ converge setwise to a geodesic σ running from b to a for some b, $a \in \partial G$. In fact this sequence $\{x_i'\}$ will satisfy $\lim_{i\to\infty} x_i'x = a$ and $\lim_{i\to\infty} x_i'y' = b$ for all $y' \in \partial G \setminus \{x\}$. We will be able to use this sequence to see that $\bar{x}$ is a conical limit point for the action of G on M, *unless* we have $\bar{a} = \bar{b}$ in M.

By way of contradiction, we therefore assume that a and b both lie in $g\partial H$ for some $g \in G$, and $H \in \mathcal{H}$. The geodesic σ lies in a λ-neighborhood of gH, by Lemma 2.7. Let $R > \chi$, and let $B_R(1)$ be the R–ball around the identity in the Cayley graph Γ. The set $x_i\gamma \cap B_R(1)$ must eventually be constant, equal to $\sigma_R := \sigma \cap B_R(1)$. Now σ_R a geodesic segment of length $2R$ lying entirely inside $N_C(gH)$. It follows that, for sufficiently large i, ${x_i'}^{-1}\sigma_R \subseteq \gamma$ lies inside $N_C({x_i'}^{-1}gH)$. In particular, if ${x_i'}^{-1} = \gamma(t_i)$, then we have $\gamma([t_i, t_i + R)) \subseteq N_C({x_i'}^{-1}gH)$. $R > \chi$, this contradicts (1). □

3. ACKNOWLEDGMENTS

Thanks to Saul Schleimer and the referee for helpful corrections.

References

[BE78] R. Bieri and B. Eckmann. Relative homology and Poincaré duality for group pairs. *J. Pure Appl. Algebra*, 13(3):277–319, 1978.

[BM91] M. Bestvina and G. Mess. The boundary of negatively curved groups. *J. Amer. Math. Soc.*, 4(3):469–481, 1991.

[Bow98] B. H. Bowditch. A topological characterisation of hyperbolic groups. *J. Amer. Math. Soc.*, 11(3):643–667, 1998.

[Bow99] B. H. Bowditch. Convergence groups and configuration spaces. In *Geometric group theory down under (Canberra, 1996)*, pages 23–54. de Gruyter, Berlin, 1999.

[Bow12] B. H. Bowditch. Relatively hyperbolic groups. *Internat. J. Algebra Comput.*, 22(3):1250016, 66, 2012. Based on the 1999 preprint.

[Dah03] F. Dahmani. Combination of convergence groups. *Geom. Topol.*, 7:933–963 (electronic), 2003.

[Dav86] R. J. Daverman. *Decompositions of manifolds*, volume 124 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1986.

[Ger12] V. Gerasimov. Floyd maps for relatively hyperbolic groups. *Geom. Funct. Anal.*, 22(5):1361–1399, 2012.

[GMRS98] R. Gitik, M. Mitra, E. Rips, and M. Sageev. Widths of subgroups. *Trans. Amer. Math. Soc.*, 350(1):321–329, 1998.

[GP13] V. Gerasimov and L. Potyagailo. Quasi-isometric maps and Floyd boundaries of relatively hyperbolic groups. *J. Eur. Math. Soc. (JEMS)*, 15(6):2115–2137, 2013.

[HY88] J. G. Hocking and G. S. Young. *Topology*. Dover Publications Inc., New York, second edition, 1988.

[Kap09] M. Kapovich. Homological dimension and critical exponent of Kleinian groups. *Geom. Funct. Anal.*, 18(6):2017–2054, 2009.

[MOY12] Y. Matsuda, S. Oguni, and S. Yamagata. Blowing up and down compacta with geometrically finite convergence actions of a group, 2012. Preprint, arXiv:1201.6104.

[MW] J. F. Manning and O. H. Wang. Cohomology and the Bowditch boundary. *Michigan Mathematical Journal*. To appear, preprint available at arXiv:1806.07074.

[Osi06] D. V. Osin. Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems. *Mem. Amer. Math. Soc.*, 179(843):vi+100, 2006.

[Spr17] D. Spriano. Hyperbolic HHS I: Factor systems and quasi-convex subgroups, 2017. Preprint, arXiv:1711.10931.

[Tra13] H. C. Tran. Relations between various boundaries of relatively hyperbolic groups. *Internat. J. Algebra Comput.*, 23(7):1551–1572, 2013.

[Yam04] A. Yaman. A topological characterisation of relatively hyperbolic groups. *J. Reine Angew. Math.*, 566:41–89, 2004.