Math 6620, Problem Set 1
Due Thursday, February 7.

1. [L], Exercise 2.7. (Prove Lemma 2.4.)

2. Suppose that (M, g) and (N, h) are Riemannian manifolds, and that $f \in C^\infty(M)$ is positive everywhere. The *warped product* of M and N with warping function f is $(M \times N, g \times_f h)$ where $g \times_f h \in \mathcal{T}^2(M \times N)$ is defined by

$$(g \times_f h)_{(p,q)}\left((X, Y)(X', Y')\right) = g_p(X, Y) + (f(p))^2 g_q(X', Y').$$

   If the metrics on M and N are understood, we can also write $M \times_f N$.

   (a) Show that $g \times_f h$ is a Riemannian metric on $M \times N$.

   (b) We talked about surfaces of revolution in class. Express a surface of revolution (with the induced metric from $\mathbb{E}^3$) as a warped product.

   (c) Let $n \geq 1$, let $\mathbb{S}^{n-1}$ be the unit $(n-1)$–sphere in $\mathbb{E}^n$, with the standard metric (incuded by the Euclidean one). Let $f(x) = x$ where $x \in (0, \infty)$. Show that if $(0, \infty)$ is given the standard Euclidean metric, then the warped product $(0, \infty) \times_f \mathbb{S}^{n-1}$ is isometric to $\mathbb{E}^n \setminus \{0\}$.

3. [L], 3-10.

4. [L], 3-8.

5. [L], 3-9.