BAUMGARTNER'S ISOMORPHISM PROBLEM FOR $\aleph_2$-DENSE SUBORDERS OF $\mathbb{R}$

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Dedicated to the memory of James Baumgartner

Abstract. In this paper we will analyze Baumgartner's problem asking whether it is consistent that $2^{\aleph_0} \geq \aleph_2$ and every pair of $\aleph_2$-dense subsets of $\mathbb{R}$ are isomorphic as linear orders. The main result is the isolation of a combinatorial principle $(**)$ which is immune to c.c.c. forcing and which in the presence of $2^{\aleph_0} \leq \aleph_2$ implies that two $\aleph_2$-dense sets of reals can be forced to be isomorphic via a c.c.c. poset. Also, it will be shown that it is relatively consistent with ZFC that there exists an $\aleph_2$ dense suborder X of $\mathbb{R}$ which cannot be embedded into $-X$ in any outer model with the same $\aleph_2$.

1. Introduction

In one of the first results concerning the structure of abstract linear orders, Cantor showed that any two countable dense linear orders are isomorphic. Here a linear order is *dense* if it has no first or last elements and between any two elements there is a third. For uncountable linear orders, the situation is more complicated. First, there is the trivial observation that two uncountable linear orders of the same cardinality might have the property that one has a countable interval while the other does not. For this reason, one usually focuses attention on κ-*dense* linear orders, for some cardinal κ — linear orders with the property that every nonempty interval contains exactly κ elements. Even in the class of $\aleph_1$-dense linear orders, however, there are many

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have qualitative differences, the most notable of which is separability. For the purpose of this paper, we will focus our attention on separable linear orders which are κ-dense (see, e.g., [14] for a discussion of the more general setting).

Dushnik and Miller showed that the class of $2^{\aleph_0}$-dense linear orderings is already extremely complicated.

Theorem 1.1. [5] *If $X \subseteq \mathbb{R}$ has cardinality $2^{\aleph_0}$, then there is a $Y \subseteq X$ of cardinality $2^{\aleph_0}$ such that if f is a partial monotone function from Y to Y, then f differs from the identity on a set of cardinality less than the continuum. In particular such a Y satisfies that any two distinct suborders of Y of cardinality $2^{\aleph_0}$ are not isomorphic.*

Hence if $2^{\aleph_0} = \aleph_1$, then there are $2^{\aleph_1}$ pairwise nonisomorphic $\aleph_1$-dense suborders of $\mathbb{R}$. Baumgartner showed, however, that it is consistent that every two $\aleph_1$-dense suborders of $\mathbb{R}$ be isomorphic.

Theorem 1.2. [4] *Assume $2^{\aleph_0} = \aleph_1 < 2^{\aleph_1} = \aleph_2$. Then there is a poset which satisfies the countable chain condition and which forces the statement "every two $\aleph_1$-dense suborders of $\mathbb{R}$ are isomorphic."*

In particular, the Proper Forcing Axiom implies that every two $\aleph_1$-dense suborders of $\mathbb{R}$ are isomorphic. In the same paper, Baumgartner asked whether it was consistent that $2^{\aleph_0} \geq \aleph_2$ and every two $\aleph_2$-dense suborders of $\mathbb{R}$ are isomorphic. For ease of writing, we will let BA_κ denote the assertion that $2^{\aleph_0} \geq \kappa$ and every two κ-dense suborders of $\mathbb{R}$ are isomorphic.

While this problem has been popularized by Shelah and others (see e.g. [10, 2.20]), little progress has been made on the problem until now. Notice that, by Dushnik and Miller's result, any model of $\mathrm{BA}_{\aleph_2}$ would necessarily satisfy $2^{\aleph_0} > \aleph_2$. Moreover, the second author has proved the following result.

Theorem 1.3. [12] *If there is an unbounded chain in $(\omega^\omega, <^*)$ of cofinality κ, then BA_κ is false.*

Here $<^*$ is the order of eventual domination on integer sequences.

Since the methods for building models of $2^{\aleph_0} > \aleph_2$ are relatively limited, the difficulty of this problem has often been attributed to the more general difficulty of obtaining models of set theory with a "large continuum" while exerting more control than is allowed with c.c.c. forcings. The purpose of this article is to relate Baumgartner's problem to a seemingly different problem of a more combinatorial nature. Consider the following assertions:

($*$): If $\mathcal{F}$ is a collection of one-to-one functions from ω_2 to ω_2 and $|\mathcal{F}| \leq \aleph_2$, then there is a $g : \omega_2 \to \omega_2$ which is one-to-one such that for every f in $\mathcal{F}$, $\{\alpha \in \omega_2 : f(\alpha) = g(\alpha)\}$ is countable.

($**$): If $\mathcal{F}$ is a collection of one-to-one functions from ω_2 to ω_2 and $|\mathcal{F}| \leq \aleph_2$, then there is a $g : \omega_2 \to \omega_2$ which is one-to-one such that:

- for every f in $\mathcal{F}$, $\{\alpha \in \omega_2 : f(\alpha) = g(\alpha)\}$ is countable and
- for every f in $\mathcal{F}$, there is a countable set $D \subseteq \omega_2$ such that if $\alpha \neq \beta \in \omega_2 \setminus D$, then $f(g(\alpha)) \neq g(\beta)$.

Notice that both of these principles are preserved in c.c.c. forcing extensions.

The following are the main results of the present paper.

Theorem 1.4. *If* $\mathrm{MA}_{\aleph_2}$*(*σ*-linked) is true and* ($*$) *is false, then there are two* $\aleph_2$*-dense suborders of* $\mathbb{R}$ *which are not isomorphic.*

Theorem 1.5. *Assume* ($**$) *in conjunction with* $2^{\aleph_0} \leq \aleph_2$ *and* $2^{\aleph_2} = \aleph_3$*. There is a poset which satisfies the countable chain condition and which forces* $\mathrm{BA}_{\aleph_1}$*,* $\mathrm{BA}_{\aleph_2}$*, and* $\mathrm{MA}_{\aleph_2}$*.*

Notice the analogy between Theorems 1.2 and 1.5: the only difference is that in the case of $\aleph_2$-dense suborders of $\mathbb{R}$, we must adequately prepare the ground model first, beyond what is provided by cardinal arithmetic alone. It should be noted that the hypothesis $2^{\aleph_2} = \aleph_3$ is only needed in Theorem 1.5 in order to obtain a c.c.c. forcing; we can always collapse $2^{\aleph_2}$ to $\aleph_3$ while not adding subsets of $\aleph_2$ (and thus preserving both ($**$) and $2^{\aleph_0} \leq \aleph_2$).

It should be noted that it would be completely unexpected if $\mathrm{BA}_{\aleph_2}$ is consistent but implies the failure of $\mathrm{MA}_{\aleph_2}$(σ-linked). Also, the existing techniques which seem most relevant for obtaining the consistency of ($**$) involve iterating σ-closed posets, necessarily resulting in a model of $2^{\aleph_0} = \aleph_1$. Thus we have shown that obtaining the consistency of the conjunction of $\mathrm{BA}_{\aleph_2}$ and $\mathrm{MA}_{\aleph_2}$(σ-linked) lies somewhere between obtaining the consistency of ($*$) and the consistency of ($**$) with $2^{\aleph_0} \leq \aleph_2$. While ($**$) is formally stronger than ($*$), the exact relationship between these principles is at present unclear; see the discussion in the concluding remarks section concerning some recent developments.

The significance of Theorem 1.4 is amplified by a result of Abraham and Shelah [2] which we will see implies that there are models of the failure of ($*$) in which the witness to the failure of ($*$) persists in any outer model with the same $\aleph_2$. In fact the proof of Theorem 1.4 will show that there are models of set theory in which there are $\aleph_1$-dense and $\aleph_2$-dense sets of reals which are not isomorphic to their reverse and which retain this property in any outer model with the same $\aleph_2$.

This paper is organized as follows. Section 2 will recall some of the work of Abraham and Shelah on closed unbounded subsets of ω_1 and relate it to $(*)$. Section 3 will give a proof of Theorem 1.4. Finally, Section 4 will give a proof of Theorem 1.5.

2. Fast clubs, $(*)$, and collapsing $\aleph_2$

Suppose that $\mathcal{E}$ is a collection of closed unbounded subsets of ω_1. A club $C \subseteq \omega_1$ is *fast* with respect to $\mathcal{E}$ if $C \setminus E$ is countable whenever E is in $\mathcal{E}$. If there is a fast club with respect to $\mathcal{E}$, then we will say that $\mathcal{E}$ can be *diagonalized*. It is a well known result of Jensen that if $2^{\aleph_0} = \aleph_1$, then there is a σ-closed forcing which is $\aleph_2$-c.c. — and in particular preserves $\aleph_2$ — and which adds a club which is fast with respect to the ground model club filter.

Notice that if $\mathcal{E}$ is a collection of clubs of cardinality $\aleph_1$, then there is a club C which is fast with respect to $\mathcal{E}$. In particular, if $\mathcal{E}$ is any family of clubs, we can force to add a club which is fast with respect to $\mathcal{E}$ by collapsing the cardinality of $\mathcal{E}$ to be $\aleph_1$ while preserving $\aleph_1$. Abraham and Shelah have proved the following result which shows that this is necessary in some cases.

Theorem 2.1. [2] *Assume CH. There is a proper cardinal preserving forcing extension in which there is a family $\mathcal{E}$ of closed unbounded subsets of ω_1 such that $\mathcal{E}$ has cardinality $\aleph_2$ and in any outer model with the same $\aleph_1$, the intersection of every uncountable subset of $\mathcal{E}$ is finite. In particular in any outer model of the generic extension with the same $\aleph_2$, $\mathcal{E}$ is not diagonalized.*

Next we will relate $(*)$ to the assertion that every collection of clubs of cardinality at most $\aleph_2$ can be diagonalized. Notice that this latter assertion is equivalent to the assertion that every subset of $\omega_1^{\omega_1}$ of cardinality $\aleph_2$ is bounded by a single function in the order of eventual dominance.

Fix a sequence $\langle e_\beta : \beta \in \omega_2 \rangle$ such that for each $\beta \in \omega_2 \setminus \omega_1$, $e_\beta : \beta \to \omega_1$ is a bijection.

Proposition 2.2. $(*)$ *is equivalent to the following statement: whenever $\mathcal{F}$ is a collection of at most $\aleph_2$ many one-to-one functions from ω_1 to ω_1, there is a countable-to-one $g : \omega_2 \to \omega_2$ such that whenever β is closed under g and f is in $\mathcal{F}$, there is a countable $D \subseteq \beta$ such that if $xi \in \beta \setminus D$,*

$$f(\min(e_\beta(\xi), e_\beta(g(\xi))) < \max(e_\beta(\xi), e_\beta(g(\xi))).$$

In particular if there is a family of $\aleph_2$-many closed unbounded subsets of ω_1 which cannot be diagonalized, then $(*)$ is false.

Proof. First assume $(*)$ and let $\mathcal{F} \subseteq \omega_1^{\omega_1}$ have cardinality $\aleph_2$. For each f in $\mathcal{F}$, let $\mathcal{F}_f$ be a countable collection of one-to-one functions which cover

$$\{(\xi, \eta) \in \omega_1^2 : \max(\xi, \eta) \leq f(\min(\xi, \eta))\}.$$

Define $\mathcal{H}$ to be the set of all functions h such that for some f in $\mathcal{F}$, $f' \in \mathcal{F}_f$ and $\beta \in \omega_2$,

$$h(\xi) = \begin{cases} e_\beta^{-1}(f'(e_\beta(\xi))) & \text{if } \xi \in \beta \\ \xi & \text{otherwise.} \end{cases}$$

Now suppose that $g : \omega_2 \to \omega_2$ is such that, for every h in $\mathcal{H}$, $\{\xi \in \omega_2 : g(\xi) = h(\xi)\}$ is countable. Then for each $\beta \in \omega_2$ which is closed under g and each f in $\mathcal{F}$, there are at most countably many $\xi \in \beta$ such that

$$\max(e_\beta(\xi), e_\beta(g(\xi))) \leq f(\min(e_\beta(\xi), e_\beta(g(\xi))).$$

Next suppose that the hypothesis stated in the lemma holds and that $\mathcal{H} \subseteq \omega_2^{\omega_2}$ is as in the statement of $(*)$. Define $\mathcal{F}$ to be the set of all functions of the form $\xi \mapsto e_\beta(h(e_\beta^{-1}(\xi)))$ such that h is in $\mathcal{H}$ and β is uncountable and closed under h. It is easily checked that if $g : \omega_2 \to \omega_2$ satisfies the conclusion of the hypothesis in the proposition with respect to $\mathcal{F}$, then g satisfies the conclusion of $(*)$ with respect to $\mathcal{H}$. □

The following proposition has a similar proof which is left to the reader.

Proposition 2.3. $(**)$ *is equivalent to the following statement: whenever $\mathcal{F}$ is a collection of at most $\aleph_2$ many one-to-one functions from ω_1 to ω_1, there is a one-to-one $g : \omega_2 \to \omega_2$ such that whenever β is closed under g and f is in $\mathcal{F}$, there is a countable $D \subseteq \beta$ such that:*

- *if $\xi \in \beta \setminus D$, then*

$$f(\min(e_\beta(\xi), e_\beta(g(\xi)))) < \max(e_\beta(\xi), e_\beta(g(\xi))),$$

- *if $\xi \neq \eta \in \beta \setminus D$, then*

$$f(\min(e_\beta(g(\xi)), e_\beta(g(\eta)))) < \max(e_\beta(g(\xi)), e_\beta(g(\eta))), .$$

3. Robust counterexamples to BA_κ

In this section, we will show how the results discussed in the previous section can be used to build robust counterexamples to $\mathrm{BA}_{\aleph_1}$ and $\mathrm{BA}_{\aleph_2}$. This will demonstrate limitations as to what methods could be used to establish the consistency of $\mathrm{BA}_{\aleph_2}$. It will also give a new proof of an old theorem of Abraham and Shelah that $\mathrm{BA}_{\aleph_1}$ is not a consequence of $\mathrm{MA}_{\aleph_1}$.

In this section we will examine the following combinatorial statement for cardinals $\mu \leq \kappa \leq \lambda$:

$\mathrm{ED}(\kappa, \lambda, \mu)$: If $\mathcal{F} \subseteq \kappa^\kappa$ consists of countable-to-one functions and $|\mathcal{F}| \leq \lambda$, then there exists a countable-to-one $g \in \kappa^\kappa$ such that for all f in $\mathcal{F}$, $\{\alpha \in \kappa : f(\alpha) = g(\alpha)\}$ has cardinality less than μ.

Thus $(*)$ is just the assertion $\mathrm{ED}(\omega_2, \omega_2, \omega_1)$. Also, it is not difficult to show that $\mathrm{ED}(\omega_1, \lambda, \omega_1)$ is equivalent to the assertion that every subset of $\omega_1^{\omega_1}$ of cardinality at most λ is bounded in the order of eventual dominance.

Proposition 3.1. *Assume* $\mathrm{MA}_\lambda(\sigma$*-linked*$)$. *If* $\kappa \leq \lambda$ *are cardinals,* $\mathcal{F} \subseteq \kappa^\kappa$ *consists of partial one-to-one functions, and* $|\mathcal{F}| \leq \lambda$, *then there is an enumeration* $r : \kappa \to 2^\omega$ *of a* κ*-dense subset of* 2^ω *such that, for each* f *in* $\mathcal{F}$, *the function* $r(\alpha) \mapsto r(f(\alpha))$ *can be covered by countably many increasing functions. In particular if* $\mathrm{ED}(\kappa, \lambda, \omega_1)$ *is false and* $\mathrm{MA}_\lambda(\sigma$*-linked*$)$ *is true, then there is a* κ*-dense* $X \subseteq \mathbb{R}$ *such that* X *is not isomorphic to a suborder of* $-X$.

Proof. Let $\mathcal{F} = \{f_\xi : \xi \in \lambda\}$ be given. Define a poset Q as follows. The underlying set consists of all pairs $q = (r_q, X_q)$ such that:

(1) for some finite subset D_q of κ and $l_q \in \omega$, r_q is a one-to-one function from D_q into 2^{l_q};
(2) X_q is a function from $\lambda \times \omega$ into the powerset of D_q and
$$\{(\xi, k) \in \lambda \times \omega : X_q(\xi, k) \neq \emptyset\}$$
is finite;
(3) if $(\xi, k) \in \lambda \times \omega$ and $\alpha \in X_q(\xi, k)$ is in the domain of f_ξ, then $f_\xi(\alpha)$ is in D_q. Similarly, if $\alpha \in X_q(\xi, k)$ is in the range of f_ξ, then $f_\xi^{-1}(\alpha)$ is in D_q;
(4) if $(\xi, k) \in \lambda \times \omega$ and $\alpha \neq \alpha' \in X_q(\xi, k)$, then $r_q(\alpha) <_{\text{lex}} r_q(\alpha')$ if and only if $r_q(f_\xi(\alpha)) <_{\text{lex}} r_q(f_\xi(\alpha'))$.

Define $q \leq p$ for $p, q \in Q$ to mean:

(5) $D_p \subseteq D_q$ and $l_p \leq l_q$;

(6) if α is in D_p, then $r_p(\alpha)$ is an initial part of $r_q(\alpha)$;
(7) if $(\xi, k) \in \lambda \times \omega$, then $X_p(\xi, k) \subseteq X_q(\xi, k)$.

If q is in Q, let $D_q(i)$ $(i < m_q)$ be an increasing enumeration of D_q and let $F_q(j)$ $(j < n_q)$ be an increasing enumeration of

$$F_q = \{\xi \in \lambda : \exists k \in \omega(X_q(\xi, k) \neq \emptyset)\}$$

If $i < m_q$, define $\bar{r}_q(i) = r_q(D_q(i))$. If $j < n_q$ and $k < \omega$, define

$$\bar{X}_q(j, k) = \{i < m_q : D_q(i) \in X_q(F_q(j), k)\}.$$

If $i, i' < m_q$ and $j < n_q$, set $\bar{f}_q(i, j) = i'$ if $f_{F_q(j)}(D_q(i)) = D_q(i')$; $\bar{f}_q(i, j)$ is undefined if no such i' exists. Notice that by our hypothesis, $\lambda \leq 2^{\aleph_0}$ and hence there exists a sequence s_ξ $(\xi \in \lambda)$ of distinct elements of 2^ω. Let $\bar{s}_q$ denote the pair

$$(\langle s_{D_q(i)} \restriction k : i < m_q\rangle, \langle s_{F_q(i)} \restriction k : i < n_q\rangle)$$

where k is minimal such that no element of 2^k occurs more than once in either coordinate. The tuple $(m_q, n_q, \bar{r}_q, \bar{X}_q, \bar{f}_q, \bar{s}_q)$ will be referred to as the *type* of q.

Claim 3.2. *Q is σ-linked.*

Remark 3.3. It is unclear whether Q is in fact σ-centered or whether MA_λ(σ-centered) is sufficient as a hypothesis is Theorem 1.4.

Proof. It suffices to show that if $p, p' \in Q$ have the same type, then p and p' are compatible. Given such p and p', define q as follows:

$$D_q = D_p \cup D_{p'} \qquad X_q(\xi, j) = X_p(\xi, j) \cup X_{p'}(\xi, j).$$

If α is in D_p, define $r_q(\alpha) = r_p(\alpha)^\frown 0$; if α is in $D_{p'} \setminus D_p$, define $r_q(\alpha) = r_{p'}(\alpha)^\frown 1$. First observe that if α is in $D_p \cap D_q$, then since the types of p and p' are in the same, there is an $i < m$ such that $D_p(i) = \alpha = D_{p'}(i)$. In particular, $r_p(\alpha) = \bar{r}(i) = r_{p'}(\alpha)$. Thus it is sufficient to verify that q is in Q as it will then follow immediately that $q \leq p, p'$.

Since both p and p' are in Q, the only condition which is nontrivial to check is (4). Toward this end, suppose that $(\xi, k) \in \lambda \times \omega$, $\alpha \neq \alpha' \in X_q(\xi, k)$. Without loss of generality, we may assume that $\alpha \in D_p \setminus D_{p'}$ and $\alpha' \in D_{p'} \setminus D_p$. Notice that since p and p' satisfy condition (3), we have that $f_\xi(\alpha)$ is in $D_p \setminus D_{p'}$ and $f_\xi(\alpha')$ is in $D_{p'} \setminus D_p$. Let $i, i' < m$ be such that $D_p(i) = \alpha$ and $D_{p'}(i') = \alpha'$. Notice that since the types of p and p' coincide, there is a $j < n$ such that $F_p(j) = F_{\bar{p}}(j) = \xi$.

First suppose that $i = i'$. In this case $r_q(\alpha) = \bar{r}(i)^\frown 0$ and $r_q(\alpha') = \bar{r}(i)^\frown 1$. Furthermore,

$$r_q(f_\xi(\alpha)) = \bar{r}(\bar{f}(i, j))^\frown 0 \qquad r_q(f_\xi(\alpha')) = \bar{r}(\bar{f}(i', j))^\frown 1.$$

It follows that $r_q(\alpha) <_{\text{lex}} r_q(\alpha')$ and $r_q(f_\xi(\alpha)) <_{\text{lex}} r_q(f_\xi(\alpha'))$.

Next, suppose that $i \neq i'$ and let $\beta = D_p(i')$. Since α is in $X_p(\xi, k)$ and α' is in $X_{p'}(\xi, k)$, both i and i' are in $\bar{X}(j, k)$. It follows that β is in $X_p(\xi, k)$. Then

$$r_p(\beta) = r_{p'}(\alpha') = \bar{r}(i') \neq \bar{r}(i) = r_p(\alpha)$$

$$r_p(f_\xi(\beta)) = r_{p'}(f_\xi(\alpha')) = \bar{r}(\bar{f}(i', j)) \neq \bar{r}(\bar{f}(i, j)) = r_p(f_\xi(\alpha))$$

and hence $r_q(\alpha) <_{\text{lex}} r_q(\alpha')$ is equivalent to $r_p(\alpha) <_{\text{lex}} r_p(\beta)$. Similarly

$$r_q(f_\xi(\alpha)) <_{\text{lex}} r_q(f_\xi(\alpha'))$$

is equivalent to

$$r_p(f_\xi(\alpha)) <_{\text{lex}} r_p(f_\xi(\beta)).$$

Since p is in Q, this establishes that $r_q(\alpha) <_{\text{lex}} r_q(\alpha')$ is equivalent to $r_q(f_\xi(\alpha)) <_{\text{lex}} r_q(f_\xi(\alpha'))$. □

Notice that, for each $\alpha \in \kappa$ and $l \in \omega$, the set of all q in Q such that α is in D_q and $l \leq l_q$ is dense. Furthermore, for each $\alpha \in \kappa$ and $\xi \in \lambda$, the set of q in Q for which there is a j such that α is in $X_q(\xi, j)$ is dense in Q. Finally, if s is a finite binary sequence and $\alpha \in \kappa$, then the set of q such that there is a $\beta \in D_q$ with $r_q(\beta)$ extending s and $\alpha \in \beta$ is dense in Q. Let $G \subseteq Q$ be a filter which meets each of these dense sets and define, for each $\alpha \in \kappa$, $\xi \in \lambda$, and $j \in \omega$,

$$r(\alpha) = \bigcup_{q \in G} r_q(\alpha) \qquad\qquad X(\xi, j) = \bigcup_{q \in G} X_q(\xi, j)$$

It follows that r enumerates a κ-dense subset of 2^ω and that for all $\xi \in \lambda$, $\bigcup_{j=0}^{\infty} X(\xi, j) = \kappa$. Moreover, if $\alpha \neq \beta$ are in $X(\xi, j)$ for some $\xi \in \lambda$ and $j \in \omega$, then $r(\alpha) <_{\text{lex}} r(\beta)$ if and only if $r(f(\alpha)) <_{\text{lex}} r(f(\beta))$. □

Notice that from this we can readily prove that $\text{MA}_{\aleph_1}$ does not imply $\text{BA}_{\aleph_1}$: observe that in any c.c.c. forcing extension of a model of $2^{\aleph_1} = \aleph_2$, $\text{ED}(\omega_1, \omega_2, \omega_1)$ fails. Thus if we force $\text{MA}_{\aleph_2}$ by a c.c.c. forcing over any model of $2^{\aleph_1} = \aleph_2$ (which is possibly by [11]), we obtain a model of $\text{MA}_{\aleph_1}$ in which $\text{BA}_{\aleph_1}$ fails. This should be compared with the more involved iterated forcing construction of [3].

4. A strategy for obtaining the consistency $\text{BA}_{\aleph_2}$

In this section we will prove Theorem 1.5. The bulk of the work is in showing that if X and Y are subsets of $\mathbb{R}$ of cardinality $\aleph_2$, $(*)$ is true, and $2^{\aleph_0} \leq \aleph_2$, then there is a c.c.c. forcing of cardinality $\aleph_2$ which forces that there is a function $f : X \to Y$ which is a countable union of increasing subfunctions. The main ingredient in the proof is the following lemma which is essentially a consequence of the proof of [13, 4.2].

Lemma 4.1. *Suppose that x_α ($\alpha \in \omega_1$) and y_α ($\alpha \in \omega_1$) are two given ω_1-sequences of elements of $\mathbb{R}$. There exists a family $\mathcal{F} \subseteq \omega_1^{\omega_1}$ consisting of countable-to-one functions such that $|\mathcal{F}| = 2^{\aleph_0}$ and if $g : \omega_1 \to \omega_1$ is a countable-to-one function such that for every f in $\mathcal{F}$ there is a countable $D \subseteq \omega_1$ such that*

- *for all $\alpha \in \omega_1 \setminus D$, $f(\min(\alpha, g(\alpha))) < \max(\alpha, g(\alpha))$ and*
- *for all $\alpha \neq \beta \in \omega_1 \setminus D$, $f(\min(g(\alpha), g(\beta))) < \max(g(\alpha), g(\beta))$,*

then the poset Q consisting of all finite $q \subseteq \omega_1$ such that $\{(x_\alpha, y_{g(\alpha)}) : \alpha \in q\}$ is increasing is powerfully c.c.c.

Proof. For each continuous function F from a Borel subset $\mathbb{R}^{2n-1}$ into $\mathbb{R}$ for some n, define $f_F : \omega_1 \to \omega_1$ by setting $f_F(\alpha)$ equal to the least upper bound of the set of all β such that there exist ξ_i ($i < n$) and η_i ($i < n$) less than α such that one of the following holds:

$$F(x_{\xi_0}, y_{\eta_0}, \ldots, y_{\eta_{n-2}}, x_{\xi_{n-1}}) = y_\beta$$

$$F(x_{\xi_0}, y_{\eta_0}, \ldots, y_{\eta_{n-2}}, y_{\eta_{n-1}}) = x_\beta$$

Let $\mathcal{F}$ denote the collection of all such f_F. The proof of [13, 4.2] shows that this $\mathcal{F}$ satisfies the conclusion of the lemma. □

Lemma 4.2. *Assume (**) and $2^{\aleph_0} \leq \aleph_2$. If X and Y are two subsets of $\mathbb{R}$ of cardinality $\aleph_2$, then there is a c.c.c. forcing Q of cardinality $\aleph_2$ such that, after forcing with Q, there is a one-to-one function from X to Y which is a countable union of increasing subfunctions.*

Proof. Fix, for each uncountable $\beta < \omega_2$, a bijection $e_\beta : \beta \to \omega_1$. Let x_α ($\alpha \in \omega_2$) and y_α ($\alpha \in \omega_2$) be enumerations of X and Y, respectively, without repetition. For each uncountable $\beta < \omega_2$, let $\mathcal{F}_\beta \subseteq \omega_1^{\omega_1}$ satisfy the conclusion of Lemma 4.1 for the sequences $\langle x_{e_\beta^{-1}(\xi)} : \xi \in \omega_1 \rangle$ and $\langle y_{e_\beta^{-1}(\xi)} : \xi \in \omega_1 \rangle$. Set $\mathcal{F} = \bigcup_{\beta \in \omega_2} \mathcal{F}_\beta$, noting that by our assumption, $|\mathcal{F}| \leq \aleph_2$. Applying (**), there is a countable-to-one function $g : \omega_2 \to \omega_2$ such that which satisfies the conclusion of Lemma 2.3 with respect to $\mathcal{F}$.

Define Q to consist of all finite subsets q of ω_2 such that $\{(x_\alpha, y_{g(\alpha)}) : \alpha \in q\}$ is strictly increasing as a partial function from $\mathbb{R}$ to $\mathbb{R}$; we view Q as a poset with the order of reverse inclusion. By Lemma 4.1, every suborder of Q of cardinality $\aleph_1$ is powerfully c.c.c. and hence Q is powerfully c.c.c.. It is easily verified that the countable (finite support) power of Q forces that $x_\alpha \mapsto y_{g(\alpha)}$ is a union of countably many increasing subfunctions. □

We also recall the following technique of the second author (see the proof of Corollary 8.3 of [13]) for proving the results in Baumgartner's [4].

Lemma 4.3. [13] *Suppose that* MA_κ(σ*-centered*) *holds and whenever* $X, Y \subseteq \mathbb{R}$ *with* $|X| = |Y| = \kappa$*, there is a* σ*-increasing function from* X *into* Y*. Then every two* κ*-dense subsets of* $\mathbb{R}$ *are order isomorphic.*

Theorem 4.4. *Assume* $(*)$*,* $2^{\aleph_0} \leq \aleph_2$*, and* $2^{\aleph_1} = 2^{\aleph_2} = \aleph_3$*. There is a c.c.c. poset which forces:*

- $\mathrm{MA}_{\aleph_2}$*;*
- *if* $\kappa \leq \aleph_2$*, then every two* κ*-dense sets of reals are isomorphic.*

Proof. First observe that $(*)$ is preserved by c.c.c. forcing, since every countable-to-one function from ω_2 to ω_2 in a generic extension can be covered by countably many such ground model functions. Using Lemma 4.2 and standard bookkeeping arguments (see [11] or [7]), build a finite support iteration of c.c.c. forcings $\langle P_\alpha; \dot{Q}_\alpha : \alpha \in \omega_3 \rangle$ such that:

- if $\dot{Q}$ is a P_{ω_3}-name for a c.c.c. forcing of cardinality at most $\aleph_2$, then there is an $\alpha \in \omega_3$ such that $\dot{Q}_\alpha = \dot{Q}$.
- if $\kappa \leq \aleph_2$ and $\dot{X}$ and $\dot{Y}$ are P_{ω_2}-names for subsets of $\mathbb{R}$ of cardinality κ, then there is an $\alpha \in \omega_3$ such that every condition of $\dot{Q}_\alpha$ forces that there is an at most countable-to-one σ-increasing function from $\dot{X}$ into $\dot{Y}$.

It follows from Lemma 4.3 that the resulting model satisfies $\mathrm{MA}_{\aleph_2}$ and both $\mathrm{BA}_{\aleph_1}$ and $\mathrm{BA}_{\aleph_2}$. □

In order to see how to obtain the relative consistency of $\mathrm{BA}_{\aleph_1}$ with $2^{\aleph_0}$ large, start with a model of GCH and let $\kappa \geq \aleph_2$ be a cardinal. First iterate Jensen's fast club forcing with countable support to obtain a forcing extension with the same cardinals such that every subset of $\omega_1^{\omega_1}$ of cardinality at most κ is bounded in the order of eventual dominance — i.e. that $\mathrm{ED}(\omega_1, \kappa, \omega_1)$ holds. Notice that $\mathrm{ED}(\omega_1, \kappa, \omega_1)$ is preserved by c.c.c. forcing and if $2^{\aleph_0} \leq \kappa$, then by Lemma 4.1, it implies that if X and Y are two $\aleph_1$-dense suborders of $\mathbb{R}$, then there is a c.c.c. forcing of cardinality $\aleph_1$ which makes X and Y isomorphic as linear orders. Now construct a finite support iteration of c.c.c. forcings of length κ^+ such that every intermediate stage of the construction forces $2^{\aleph_0} \leq \kappa$ and such that, by the end of the iteration, $\mathrm{BA}_{\aleph_1}$ holds (of course we can arrange MA_κ as well). (This is essentially the same proof as given in [1], although it is cast in a different language.)

5. Concluding remarks and recent developments

The original submission of this paper was made in March 2014 and contained only the combinatorial statement $(*)$ and a flawed proof of the version of Theorem 1.5 in which $(**)$ was replaced with $(*)$. In late September 2014, Itay Neeman pointed out the error (which was in an analog of Lemma 4.1). At a workshop at CIRM Luminy the following week he announced that under a suitable large cardinal assumption, there is a forcing extension by a σ-closed forcing in which $(*)$ and CH both hold. The combinatorial statement $(**)$ was formulated later by the authors following discussions with Neeman at the time of the workshop; it is natural strengthening of $(*)$ which is needed to make the original arguments valid. Neeman has since adapted his methods to yield the consistency of $(**)$ relative to a large cardinal hypothesis.

The following problems are, at least at present, left open by Neeman's work.

Problem 5.1. *Does there exist an n such that either $\mathfrak{t} \leq \aleph_n$ or else there is a $\kappa \leq \aleph_n$ such that BA_κ fails?*

Problem 5.2. *Does the conjunction of $\mathrm{BA}_{\aleph_2}$ and $\mathrm{MA}_{\aleph_2}$ imply that there is a cardinal which is inaccessible in the constructable universe?*

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