# COMPACT SPACES WITH HEREDITARILY NORMAL SQUARES

JUSTIN TATCH MOORE

## 1. Introduction

In 1948, Katětov proved the following metrization theorem.

**Theorem 1.1.** [3] *If $X$ is a compact space*[1] *and every subspace of $X^3$ is normal, then $X$ is metrizable.*

This is an immediate consequence of the following two results which are of independent interest.

**Theorem 1.2.** [3] *If $X \times Y$ is hereditarily normal, then either $X$ is perfectly normal or else every countable subspace of $Y$ is closed and discrete.*

**Theorem 1.3.** [7] *If $X$ is a compact space and the diagonal is a $G_\delta$ subset of $X^2$, then $X$ is metrizable.*

Katětov then asked whether the dimension in his theorem could be lowered to 2. In [1] Gruenhage and Nyikos present two examples which show that consistently this is not possible.

**Theorem 1.4.** [1] *If there is a $Q$-set then there is a separable compact space $X$ such that $X^2$ contains an uncountable discrete subspace and yet has every subspace normal.*

**Theorem 1.5.** [1] *If the Continuum Hypothesis is true, then there is a non-metrizable compact space $X$ such that every subspace of $X^2$ is separable and normal.*

The first construction is due to Nyikos and is optimal in the sense that the existence of such a space implies the existence of a $Q$-set [1]. The second construction is due to Gruenhage and does not obviously require the full strength of the Continuum Hypothesis.

In [4], Larson and Todorcevic proved that it is consistent that Katětov's problem has a positive answer.

---

The research presented in this paper was supported by NSF grant DMS–0401893.

[1] In this article, all spaces are assumed to be regular.

**Theorem 1.6.** [4] *It is relatively consistent with* ZFC *that if $X$ is a compact space and $X^2$ is hereditarily normal, then $X$ is metrizable.*

The solution they give represents a set theoretic breakthrough. The purpose of this section is to suggest how one might obtain a positive solution to Katětov's problem via an analysis which is almost purely topological. The broader goal is to obtain a better understanding of hereditary and perfect normality in compact topological spaces.

I will begin by giving a list of questions which have so far have not received much attention. I was made aware of most if not all of them by Todorcevic.

**Question 1.7.** *If $X$ is compact and $X^2$ is hereditarily normal, must $X$ be separable?*

Recall that a space $X$ is *premetric of degree* $\leq 2$ iff there is a continuous map $f$ from $X$ into a metric space such that the preimage of any point contains at most two elements. Both Gruenhage's and Nyikos's examples in [1] are premetric of degree $\leq 2$.

**Question 1.8.** *(see* [8]*) If $X$ is compact and $X^2$ is hereditarily normal, must $X$ be premetric of degree $\leq 2$?*

**Question 1.9.** *If there is a compact non-metrizable $X$ which is premetric of degree $\leq 2$ such that $X^2$ is hereditarily normal, must there exist either a Q-set or a Luzin set?*

In each case, a positive answer to the question is a consequence of a positive answer to Katětov's problem and hence is consistent by [4]. The hope is that it is possible to prove positive answers to these questions in ZFC.

Notice that a counterexample to Question 1.7 is necessarily a compact L space. While a Suslin line comes to mind as a candidate for an example, M. E. Rudin has shown that this is not possible — if $L$ is a compact Suslin line, then $L^2$ is not hereditarily normal [6]. Interestingly, however, $2^{\aleph_0} < 2^{\aleph_1}$ implies that a counterexample to Question 1.7 must have a square which does not satisfy the countable chain condition. This is a consequence of the following results of Shapirovskii and Todorcevic.

**Theorem 1.10.** *(see* [11]*) The regular open algebra of any hereditarily normal ccc space has size at most continuum.*

**Theorem 1.11.** [10] *If $X$ is compact and $X^2$ does not contain an uncountable discrete subspace, then $X$ is separable.*

Observe that a positive answer to Question 1.8 would give a positive answer to Question 1.7 since every premetric compactum of degree $\leq 2$ is separable.

Question 1.9 is motivated Theorem 1.14 below which shows that Gruenhage's construction requires the existence of a Luzin set. Observe that it is relatively easy to obtain a model of set theory in which there are no $Q$-sets or Luzin sets — this is true after adding $\aleph_2$ random reals to any model, for instance. Hence a positive solution of the above questions would yield a different solution to Katětov's problem.

We will now revisit Gruenhage's example mentioned above. The construction is closely based around a well known construction of Kunen.

**Theorem 1.12.** [2] *If the Continuum Hypothesis is true, then there is a strengthening of the topology on $\mathbb{R}$ to a topology which is locally countable, locally compact, and such that the difference between the closure of a set in this and the usual topology is countable. In particular such a space is hereditarily separable but not Lindelöf.*

M. Wage observed that the construction could be carried out on an arbitrary uncountable set of reals instead of just $\mathbb{R}$ assuming the Continuum Hypothesis. Such spaces have come to be known as *Kunen lines*. Gruenhage's construction is connected in the sense that his $X^2$ contains a subspace $Z$ which maps 2–1 onto a Kunen line where the underlying set of reals is a Luzin set.

In order to state Theorem 1.14 concisely, I will first introduce some notation.

**Definition 1.13.** Suppose that $X$ and $Y$ are topological spaces and $f : X \to Y$ is continuous. Define $\Delta_f$ to be all pairs $(x_0, x_1)$ in $X^2$ such that $f(x_0) = f(x_1)$. The function $f_* : \Delta_f \to Y$ is defined by $f_*(x, y) = f(x) = f(y)$.

If $f$ is the identity function, then $\Delta_f$ is the diagonal and the subscript is suppressed, giving the standard notation.

**Theorem 1.14.** *Suppose that $X$ is a compact non-metrizable space such that*

*(1) $X^2$ is hereditarily normal,*
*(2) $X$ is premetric of degree $\leq 2$, and*
*(3) the quotient of $\Delta_f \setminus \Delta$ by $f_*$ is a Kunen line.*

*Then there is a Luzin set.*

*Remark.* It is not clear whether Gruenhage's construction can be carried out from the existence of a Luzin set. Todorcevic has shown that an analogue of Wage's construction can be carried out if $\mathfrak{b} = \aleph_1$, an assumption which follows from the existence of a Luzin set.

**Theorem 1.15.** [9] *($\mathfrak{b} = \aleph_1$) If $X$ is a set of reals of size $\aleph_1$, then there is a refinement of the metric topology which is locally compact, locally countable, perfectly normal and hereditarily separable in all of its finite powers.*

Carrying out Gruenhage's construction assuming only the existence of a Luzin set seems to be a considerably more subtle matter — see my note [5] for some limited progress. I conjecture that this is possible.

*Proof.* Let $X$ be given as in the statement of the theorem and $f : X \to K$ witness that $X$ is premetric of degree $\leq 2$. If $U$ is an open subset of $X$ and $\{x_0, x_1\}$ is a pair of points in $X$ then we say that $U$ *splits* $\{x_0, x_1\}$ if both $U$ and $X \setminus \overline{U}$ contain an element of $\{x_0, x_1\}$. Since $X$ is non-metric and compact, it is possible to recursively select points $z_\xi$ in $K$ and open sets $U_\xi$ in $X$ such that $U_\xi$ splits $f^{-1}(z_\xi)$ but does not split $f^{-1}(z_\eta)$ if $\xi < \eta < \omega_1$. Let $Z = \{z_\xi : \xi < \omega_1\}$ and let $V_n$ $(n < \omega)$ enumerate a base for the topology on $K$. By removing points from $Z$ if necessary, we may assume that it has no countable neighborhoods.

Observe that if $f(x) = z_\xi$ then one of the collections

$$\{f^{-1}(V_n) \cap U_\xi : x \in f^{-1}(V_n)\}$$
$$\{f^{-1}(V_n) \setminus \overline{U_\xi} : x \in f^{-1}(V_n)\}$$

intersects to the singleton $\{x\}$ and hence forms a local base for $x$. Also observe that since $f^{-1}(V_n)$ does not split any pair of the form $f^{-1}(z)$ for $z \in K$, sets of the form $f^{-1}(V_n) \cap U_\xi$ and $f^{-1}(V_n) \setminus \overline{U_\xi}$ can split $f^{-1}(z_\eta)$ only when $\eta \leq \xi$.

Suppose that $Z$ is not a Luzin set in $\mathrm{cl}_K(Z)$. It suffices to show that $X^2$ is not hereditarily normal. To this end, let $E \subseteq K$ be a closed set such that $E \cap Z$ is relatively nowhere dense and uncountable. Define the following sets

$$G = \{(x_0, x_1) \in \Delta_f : x_0 \neq x_1 \text{ and } f_*(x_0, x_1) \in E \cap Z\}$$
$$H = \{(x, x) \in X^2 : f(x) \notin E\}.$$

Clearly $\overline{G} \cap H = G \cap \overline{H} = \emptyset$. It is sufficient to show that if $W \subseteq X^2$ is open and contains $H$ then $\overline{W} \cap G$ is nonempty.

By shrinking $W$ if necessary, we may assume that is a union of sets of the form

$$\big((f^{-1}(V_n) \cap U_\xi) \times (f^{-1}(V_n) \cap U_\xi)\big) \cup \big((f^{-1}(V_n) \setminus \overline{U_\xi}) \times (f^{-1}(V_n) \setminus \overline{U_\xi})\big)$$

for $n < \omega$ and $\xi < \omega_1$ such that $V_n \cap E = \emptyset$. Since $X^2$ is hereditarily normal, it follows from [3] that $X$ is perfect and therefore that $W$ is a countable union of such sets. Let $\delta$ be an upper bound for all $\xi < \omega_1$ required in this union. If $\delta < \xi < \omega_1$ and $(x_0, x_1)$ is in $\Delta_f \setminus \Delta$ with $f_*(x_0, x_1) = z_\xi$, then $(x_0, x_1)$ is in $W$ provided that $z_\xi$ is not in $E$. Put $D = \{z_\xi : \xi \leq \delta\}$.

By our assumption on $\Delta_f \setminus \Delta$, the closure of $Z \setminus (E \cup D)$ in the metric topology and in the quotient topology induced by $f_*$ differ by a countable set $D'$. Since $E$ is nowhere dense, $D$ is countable, and $Z$ has no countable neighborhoods, $Z$ is contained in the metric closure of $Z \setminus (E \cup D)$.

I will now show that if $(x_0, x_1)$ is in $\Delta_f \setminus \Delta$ with $f_*(x_0, x_1)$ in $Z \setminus (D \cup D')$, then either $(x_0, x_1)$ or $(x_0, x_1)$ is in the closure of $W$. This finishes the proof since there is a $(x_0, x_1)$ such that $f_*(x_0, x_1)$ is in $Z \setminus (D \cup D')$ and both $(x_0, x_1)$ and $(x_1, x_0)$ are in $G$. To this end, suppose that $(x_0, x_1)$ are given as above and let $z = f_*(x_0, x_1)$. Since $z$ is not in $D'$, $z$ is a limit point of $Z \setminus (E \cup D)$ in the quotient topology since it is in the metric topology. This means that there is an element of $f_*^{-1}(z)$ which is in the closure of the $f_*$-preimage of $Z \setminus (E \cup D)$. Since this preimage is contained in $W$, either $(x_0, x_1)$ or $(x_1, x_0)$ is in the closure of $W$ as desired. □

## References


[1] Gary Gruenhage and Peter Nyikos. Normality in $X^2$ for compact $X$. *Trans. Amer. Math. Soc.*, 340(2):563–586, 1993.

[2] Istvan Juhász, Kenneth Kunen, and Mary Ellen Rudin. Two more hereditarily separable non-Lindelöf spaces. *Canad. J. Math.*, 28(5):998–1005, 1976.

[3] Mirosłav Katětov. Complete normality in Cartesian products. *Fund. Math.*, 36:271–274, 1948.

[4] Paul Larson and Stevo Todorcevic. Katětov's problem. *Trans. Amer. Math. Soc.*, 354:1783–1792, 2002.

[5] Justin Tatch Moore. Splitting Todorčević's set mapping from $\mathfrak{b} = \omega_1$. Unpublished note 8/04.

[6] Mary Ellen Rudin. Hereditary normality and Souslin lines. *General Topology Appl.*, 10(1):103–105, 1979.

[7] V. E. Šneĭder. Continuous images of Suslin and Borel sets. Metrization theorems. *Doklady Akad. Nauk SSSR (N.S.)*, 50:77–79, 1945.

[8] Stevo Todorcevic. Chain-condition methods in topology, ii.

[9] Stevo Todorcevic. *Partition Problems In Topology.* Amer. Math. Soc., 1989.

[10] Stevo Todorcevic. Free sequences. *Topology Appl.*, 35(2-3):235–238, 1990.

[11] Stevo Todorcevic. Chain-condition methods in topology. *Topology Appl.*, 101(1):45–82, 2000.



Department of Mathematics, Boise State University, Boise, Idaho 83725–1555

*E-mail address*: justin@math.boisestate.edu