

# SET MAPPING REFLECTION

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ABSTRACT. In this note we will discuss a new reflection principle which follows from the Proper Forcing Axiom. The immediate purpose will be to prove that the bounded form of the Proper Forcing Axiom implies both that  $2^\omega = \omega_2$  and that  $L(\mathcal{P}(\omega_1))$  satisfies the Axiom of Choice. It will also be demonstrated that this reflection principle implies that  $\square(\kappa)$  fails for all regular  $\kappa > \omega_1$ .

## 1. INTRODUCTION

The notion of properness was introduced by Shelah in [14] and is a weakening of both the countable chain condition and the property of being countably closed. Its purpose was to provide a property of forcing notions which implies that they preserve  $\omega_1$  and which is preserved under countable support iterations. With the help of a supercompact cardinal, one can prove the consistency of the following statement (see [6]).

**PFA:** If  $\mathcal{P}$  is a proper forcing notion and  $\mathcal{D}$  is a family of dense subsets of  $\mathcal{P}$  of size  $\omega_1$ , then there is a filter  $G \subseteq \mathcal{P}$  which meets every element of  $\mathcal{D}$ .

The Proper Forcing Axiom (PFA) is therefore a strengthening of the better known and less technical  $\text{MA}_{\omega_1}$  [16]. It has been extremely useful, together with the stronger Martin's Maximum (MM) [7], in resolving questions left unresolved by Martin's Axiom.

Early on it was known that it was not possible to replace  $\omega_1$  by  $\omega_2$  and get a consistent statement (see [3]). The stronger forcing axiom MM was known to already imply that the continuum is  $\omega_2$  [7]. Later Todorćević and Velicković showed that PFA also implies that the continuum is  $\omega_2$  (see [23] and [26]). This proof used a deep analysis of the gap structure of  $\omega^\omega/\text{fin}$  [22] and of the behavior of the oscillation map [21]. Their proof, however, was less generous than some of the proofs

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that the continuum was  $\omega_2$  from Martin's Maximum. In particular, while MM implies that  $L(\mathcal{P}(\omega_1))$  satisfies AC [27], the same was not known for PFA (compare to the final remark section 3 of [26]).

MM was also shown to have a variety of large cardinal consequences. Many of these are laid out in [7]. Much of this was proved via stationary reflection principles which seemed to typify the consequences of MM which do not follow from PFA. Todorcevic showed that PFA implies that the combinatorial principle  $\square(\kappa)$  fails for all  $\kappa > \omega_1$  [19]. This, combined with modern techniques in inner model theory [12], gives a considerable lower bound on the consistency strength of PFA.<sup>1</sup> Both the impact on the continuum and the large cardinal strength of these forcing axioms have figured prominently in their development.

The purpose of this note is to introduce a new reflection principle, MRP, which follows from PFA. The reasons are threefold. First, this axiom arose as a somewhat natural abstraction of one its consequences which in turn implies that there is a well ordering of  $\mathbb{R}$  which is  $\Sigma_1$ -definable over  $(H(\omega_2), \epsilon)$ . A corollary of the proof will be that the Bounded Proper Forcing Axiom implies that there is such a well ordering of  $\mathbb{R}$ , thus answering a question from [8].

Second, this principle seems quite relevant in studying consequences of PFA which do not follow from the  $\omega$ -Proper Forcing Axiom. The notion of  $\omega$ -properness was introduced by Shelah in the course of studying preservation theorems for not adding reals in countable support iterations (see [15]). For our purpose it is sufficient to know that both c.c.c. and countably closed forcings are  $\omega$ -proper and that  $\omega$ -proper forcings are preserved under countable support iterations. While I am not aware of the  $\omega$ -PFA having been studied in the literature, nearly all of the studied consequences of PFA are actually consequences of the weaker  $\omega$ -PFA.<sup>2</sup> It is my hope and optimism that the Mapping Reflection Principle will be useful tool in studying the consequences of PFA which do not follow from the  $\omega$ -PFA in much the same way that the Strong Reflection Principle has succeeded in implying the typical consequences of MM which do not follow from PFA [17] (see also [5]).

Finally, like the Open Coloring Axiom, the Ramsey theoretic formulation of Martin's Axiom, and the Strong Reflection Principle, this principle can be taken as a black box and used without knowledge of forcing. The arguments using it tend to be rather elementary in nature

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<sup>1</sup>It is not known if " $\square(\kappa)$  fails for all regular  $\kappa > \omega_1$ " is equiconsistent with the existence of a supercompact cardinal.

<sup>2</sup>For example:  $MA_{\omega_1}$ , the non-existence of S-spaces [18], all  $\omega_1$ -dense sets of reals are isomorphic [4], the Open Coloring Axiom [22], the failure of  $\square(\kappa)$  for all regular  $\kappa > \omega_1$  [19], the non-existence of Kurepa trees [3].

and require only some knowledge of the combinatorics of the club filter on  $[X]^\omega$  and Löwenheim-Skolem arguments.

The main results of this note are summarized as follows.

**Theorem 1.1.** *The Proper Forcing Axiom implies the Mapping Reflection Principle.*

**Theorem 1.2.** *The Mapping Reflection Principle implies that  $2^\omega = 2^{\omega_1} = \omega_2$  and that  $L(\mathcal{P}(\omega_1))$  satisfies the Axiom of Choice.*

**Theorem 1.3.** *The Bounded Proper Forcing Axiom implies that  $2^\omega = \omega_2$  and that  $L(\mathcal{P}(\omega_1))$  satisfies the Axiom of Choice.*

**Theorem 1.4.** *The Mapping Reflection Principle implies that  $\square(\kappa)$  fails for every regular  $\kappa > \omega_1$ .*

The notation used in this paper is more or less standard. If  $\theta$  is a regular cardinal, then  $H(\theta)$  is the collection of all sets of hereditary cardinality less than  $\theta$ . As is common, when I refer to  $H(\theta)$  as a structure I will actually mean  $(H(\theta), \in, \triangleleft)$  where  $\triangleleft$  is some well order of  $H(\theta)$  which can be used to compute Skolem functions and hence generate the club  $E \subseteq [H(\theta)]^\omega$  of countable elementary submodels of  $H(\theta)$ . If  $X$  is a set of ordinals, then  $\text{otp}(X)$  represents the ordertype of  $(X, \in)$  and  $\pi_X$  is the unique collapsing isomorphism from  $X$  to  $\text{otp}(X)$ . While an attempt has been made to keep parts of this paper self contained, a knowledge of proper forcing is assumed in Section 3. The reader is referred to [3], [15], and [22] for more reading on proper forcing and PFA. Throughout the paper the reader is assumed to have a familiarity with set theory ([9] and [11] are standard references).

## 2. THE MAPPING REFLECTION PRINCIPLE

The following definition will be central to our discussion. Recall that for an uncountable set  $X$ ,  $[X]^\omega$  is the collection of all countable subsets of  $X$ .

**Definition 2.1.** Let  $X$  be an uncountable set,  $M$  be a countable elementary submodel of  $H(\theta)$  for some regular  $\theta$  such that  $[X]^\omega \in M$ . A subset  $\Sigma$  of  $[X]^\omega$  is  *$M$ -stationary* if whenever  $E \subseteq [X]^\omega$  is a club in  $M$  there is an  $N$  in  $E \cap \Sigma \cap M$ .

**Example 2.2.** If  $M$  is a countable elementary submodel of  $H(\omega_2)$  and  $A \subseteq M \cap \omega_1$  has order type less than  $\delta = M \cap \omega_1$ , then  $\delta \setminus A$  is  *$M$ -stationary*.

The set  $[X]^\omega$  is equipped with the Ellentuck topology obtained by declaring the sets

$$[x, N] = \{Y \in [X]^\omega : x \subseteq Y \subseteq N\}$$

to be open for all  $N$  in  $[X]^\omega$  and finite  $x \subseteq N$ . In this paper “open” will always refer to this topology. It should be noted that the sets which are closed in the Ellentuck topology and cofinal in the order structure generate the closed unbounded filter on  $[X]^\omega$ .

For ease of reading I will make the following definition.

**Definition 2.3.** A set mapping  $\Sigma$  is said to be *open stationary* if, for some uncountable set  $X$  and regular cardinal  $\theta$  with  $X$  in  $H(\theta)$ , it is the case that elements of the domain of  $\Sigma$  are elementary submodels of  $H(\theta)$  which contain  $M$  and  $\Sigma(M) \subseteq [X]^\omega$  is open and  $M$ -stationary for all  $M$  in the domain of  $\Sigma$ . If necessary, the parameters  $X$  and  $\theta$  will be referred to as  $X_\Sigma$  and  $\theta_\Sigma$ .

The following is among the simplest example of an open stationary set mapping.

**Example 2.4.** Let  $r : \omega_1 \rightarrow \omega_1$  be regressive on the limit ordinals. If  $\Sigma$  is defined by putting  $\Sigma(N) = [\{r(\delta)\}, \delta]$  for  $N$  a countable elementary submodel of  $H(\omega_2)$ , then  $\Sigma$  is open and stationary.

This motivates the following reflection principle which asserts that this example is present inside any open stationary set mapping.

**MRP:** If  $\Sigma$  is an open stationary set mapping whose domain is a club, then there is a continuous  $\in$ -chain  $\langle N_\nu : \nu < \omega_1 \rangle$  in the domain of  $\Sigma$  such that for all limit  $0 < \nu < \omega_1$  there is a  $\nu_0 < \nu$  such that  $N_\xi \cap X_\Sigma \in \Sigma(N_\nu)$  whenever  $\nu_0$  is in  $N_\xi$  and  $\xi < \nu$ .

If  $\langle N_\nu : \nu < \omega_1 \rangle$  satisfies the conclusion of MRP for  $\Sigma$ , then it is said to be a *reflecting sequence for  $\Sigma$* . We now continue with the first example.

**Example 2.5.** An immediate consequence of MRP is that if  $C_\delta$  is a cofinal  $\omega$ -sequence in  $\delta$  for each countable limit ordinal  $\delta$ , then there is a club  $E \subseteq \omega_1$  such that  $E \cap C_\delta$  is finite for all  $\delta$ . To see this, let  $C_\delta$  ( $\delta < \omega_1$ ) be given, and if  $N$  is a countable elementary submodel of  $H(\omega_2)$ , let  $\Sigma(N)$  be the complement of  $C_\delta \cup \{\delta\}$ . It is easily checked that  $\Sigma$  is an open-stationary set mapping and that if  $\langle N_\nu : \nu < \omega_1 \rangle$  is a reflecting sequence for  $\Sigma$ , then  $E = \{\nu < \omega_1 : N_\nu \cap \omega_1 = \nu\}$  is closed unbounded set with the desired properties.<sup>3</sup>

<sup>3</sup>It is easy to verify, however, that this consequence of MRP can not be forced with an  $\omega$ -proper forcing over a model of  $\clubsuit$ . That this statement follows from PFA appears in [15]. It is essentially the only example in the literature that I am aware of which is a combinatorial consequence of PFA but not of  $\omega$ -PFA.

### 3. PFA IMPLIES MRP

The purpose of this section is to prove the following theorem. Recall that a forcing notion  $\mathcal{P}$  is *proper* if whenever  $M$  is a countable elementary submodel of  $H(|2^{\mathcal{P}}|^+)$  containing  $\mathcal{P}$  and  $p$  is in  $\mathcal{P} \cap M$ , there is a  $\bar{p} \leq p$  which is  $(M, \mathcal{P})$ -generic.<sup>4</sup>

**Theorem 3.1.** *PFA implies MRP.*

*Proof.* Let  $\Sigma$  be a given open stationary set mapping defined on a club of models and abbreviate  $X = X_\Sigma$  and  $\theta = \theta_\Sigma$ . Let  $\mathcal{P}_\Sigma$  denote the collection of all continuous  $\in$ -increasing maps  $p : \alpha + 1 \rightarrow \text{dom}(\Sigma)$  such that for all  $0 < \nu \leq \alpha$  there is a  $\nu_0 < \nu$  with  $p(\xi) \cap X \in \Sigma(p(\nu))$  whenever  $\nu_0$  is in  $p(\xi)$  and  $\xi < \nu$ .  $\mathcal{P}_\Sigma$  is ordered by extension. I will now prove that  $\mathcal{P}_\Sigma$  is proper. Notice that if this is the case, then the sets  $\mathcal{D}_\alpha = \{p \in \mathcal{P}_\Sigma : \alpha \in \text{dom}(p)\}$  must be dense. This is because  $\mathcal{D}_x^* = \{p \in \mathcal{P}_\Sigma : \exists \nu \in \text{dom}(p)(x \in p(\nu))\}$  is clearly dense for all  $x$  in  $X$  and therefore, after forcing with  $\mathcal{P}_\Sigma$ , there is always a surjection from  $\{\alpha : \exists p \in G(\alpha \in \text{dom}(p))\}$  onto the uncountable set  $X$ .

To see that  $\mathcal{P}_\Sigma$  is proper, let  $p$  be in  $\mathcal{P}_\Sigma$  and  $M$  be an elementary submodel of  $H(\lambda)$  for regular  $\lambda$  sufficiently large such that  $\Sigma$ ,  $\mathcal{P}_\Sigma$ ,  $p$ , and  $H(|\mathcal{P}_\Sigma|^+)$  are all in  $M$ . Let  $\{D_i : i < \omega\}$  enumerate the dense subsets of  $\mathcal{P}_\Sigma$  which are in  $M$ . We will now build a sequence of conditions  $p_0 \geq p_1 \geq \dots$  by recursion. Set  $p_0 = p$  and let  $p_i$  be given. Let  $E_i$  be the collection of all intersections of the form  $N = N^* \cap X$  where  $N^*$  is a countable elementary submodel of  $H(|\mathcal{P}_\Sigma|^+)$  containing  $H(\theta)$ ,  $D_i$ ,  $\mathcal{P}_\Sigma$ , and  $p_i$  as elements. Then  $E_i \subseteq [X]^\omega$  is a club in  $M$ . Since  $\Sigma(M \cap H(\theta))$  is open and  $M \cap H(\theta)$ -stationary, there is an  $N_i$  in  $E_i \cap \Sigma(M \cap H(\theta)) \cap M$  and an  $x_i$  in  $[N_i]^{<\omega}$  such that  $[x_i, N_i] \subseteq \Sigma(M \cap H(\theta))$ . Extend  $p_i$  to

$$q_i = p_i \cup \{(\zeta_i + 1, \text{hull}(\{p_i\} \cup x_i))\}$$

where  $\zeta_i$  is the last element of the domain of  $p_i$  and  $\text{hull}(\{p_i\} \cup x_i)$  is the Skolem hull taken in  $H(\theta)$ . The function  $q_i : \zeta_i + 1 \rightarrow \text{dom}(\Sigma)$  is  $\in$ -increasing since it extends  $p_i$  by a single entry which, by arrangement, contains the range of  $p_i$  as a subset. Since  $p_i$  has successor length and the domain of  $q_i$  contains no new limit ordinals,  $q_i$  is continuous and is in  $\mathcal{P}_\Sigma$ .

Notice that  $q_i$  is in  $N_i^*$  since  $N_i^*$  contains  $p_i$  and  $H(\theta)$ . Working in  $N_i^*$ , find an extension  $p_{i+1}$  of  $q_i$  which is in  $N_i^* \cap D_i$ . The key observation here is that everything in the range of  $p_{i+1}$  which is not

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<sup>4</sup>Here condition  $\bar{p}$  is  $(M, \mathcal{P})$ -generic if whenever  $r$  is an extension of  $\bar{p}$  and  $D \subseteq \mathcal{P}$  is a predense set in  $M$ , there is a  $s$  in  $D \cap M$  such that  $s$  is compatible with  $r$ .

in the range of  $p_i$  has an intersection with  $X$  which is in the interval  $[x_i, N_i]$  and therefore in  $\Sigma(M \cap H(\theta))$  by the virtue  $x_i, N_i$  witnessing that  $\Sigma(M \cap H(\theta))$  is a neighborhood of  $N_i$ . Define  $p_\infty(\xi) = p_i(\xi)$  if  $\xi \leq \zeta_i$  and  $p_\infty(\sup_i \zeta_i) = M \cap H(\theta)$ . It is easily checked that  $p_\infty$  is well defined and that  $p_\infty$  is a condition which is moreover  $(M, \mathcal{P}_\Sigma)$ -generic.  $\square$

#### 4. MRP AND THE CONTINUUM.

In this section we will see that MRP can be used to code reals in a way which is somewhat reminiscent of the SRP style coding methods (e.g.  $\phi_{AC}, \psi_{AC}$  of [27]). Before we begin, we first need to introduce some notation.

Fix a sequence  $\langle C_\xi : \xi \in \lim(\omega_1) \rangle$  such that  $C_\xi$  is a cofinal subset of  $\xi$  of ordertype  $\omega$  for each limit  $\xi < \omega_1$ . Let  $N, M$  be countable sets of ordinals such that  $N \subseteq M$ ,  $\text{otp}(M)$  is a limit, and  $\text{sup}(N) < \text{sup}(M)$ . Define

$$w(N, M) = |\text{sup}(N) \cap \pi_M^{-1}[C_\alpha]|$$

where  $\alpha$  is the ordertype of  $M$ . A trivial but important observation is that  $w$  is *left monotonic* in the sense that  $w(N_1, M) \leq w(N_2, M)$  whenever  $N_1 \subseteq N_2 \subseteq M$  and  $\text{sup}(N_2) < \text{sup}(M)$ . Also, if  $N \subseteq M$  are countable sets of ordinals with  $\text{sup}(N) < \text{sup}(M)$  and  $\iota$  is an order preserving map from  $M$  into the ordinals, then  $w(N, M) = w(\iota''N, \iota''M)$ .

We will now consider the following statement about a given subset  $A$  of  $\omega_1$ :

$v_{AC}(A)$ : There is an uncountable  $\delta < \omega_2$  and an increasing sequence  $\langle N_\xi : \xi < \omega_1 \rangle$  which is club in  $[\delta]^\omega$  such that for all limit  $\nu < \omega_1$  there is a  $\nu_0 < \nu$  such that if  $\nu_0 < \xi < \nu$ , then  $N_\nu \cap \omega_1 \in A$  is equivalent to  $w(N_\xi \cap \omega_1, N_\nu \cap \omega_1) < w(N_\xi, N_\nu)$ .

The formula  $v_{AC}$  is the assertion that  $v_{AC}(A)$  holds for all  $A \subseteq \omega_1$ .

**Proposition 4.1.**  $v_{AC}$  implies that in  $L(\mathcal{P}(\omega_1))$  there is a well ordering of  $\mathcal{P}(\omega_1)/\text{NS}$  in length  $\omega_2$ . In particular  $v_{AC}$  implies both  $2^{\omega_1} = \omega_2$  and that  $L(\mathcal{P}(\omega_1))$  satisfies the Axiom of Choice.

*Proof.* For each  $[A]$  in  $\mathcal{P}(\omega_1)/\text{NS}$  let  $\delta_{[A]}$  be the least uncountable  $\delta < \omega_2$  such that there is a club  $N_\xi$  ( $\xi < \omega_1$ ) in  $[\delta]^\omega$  such that for all limit  $\nu < \omega_1$  there is a  $\nu_0 < \nu$  such that for all  $\nu_0 < \xi < \nu$  we have that

$$N_\nu \cap \omega_1 \in A \text{ iff } w(N_\xi \cap \omega_1, N_\nu \cap \omega_1) < w(N_\xi, N_\nu).$$

It is easily checked that this definition is independent of the choice of representative of  $[A]$ . It is now sufficient to show that if  $A$  and  $B$  are subsets of  $\omega_1$  and  $\delta_{[A]} = \delta_{[B]}$ , then  $[A] = [B]$ . To this end suppose that

$\delta < \omega_2$  is an uncountable ordinal such that for clubs  $N_\xi^A$  ( $\xi < \omega_1$ ) and  $N_\xi^B$  ( $\xi < \omega_1$ ) in  $[\delta]^\omega$  we have for every limit  $\nu < \omega_1$  there is a  $\nu_0 < \nu$  such that for all  $\nu_0 < \xi < \nu$  we have both

$$w(N_\xi^A \cap \omega_1, N_\nu^A \cap \omega_1) < w(N_\xi^A, N_\nu^A) \text{ iff } \xi \in A$$

$$w(N_\xi^B \cap \omega_1, N_\nu^B \cap \omega_1) < w(N_\xi^B, N_\nu^B) \text{ iff } \xi \in B.$$

Now there is a closed unbounded set  $C \subseteq \omega_1$  such that if  $\xi$  is in  $C$ , then  $N_\xi^A = N_\xi^B$ . It is easily seen that if  $\nu$  is a limit point of  $C$ , then  $\nu$  is in  $A$  iff  $\nu$  is in  $B$ .  $\square$

**Proposition 4.2.**  $v_{AC}$  implies that  $2^\omega = 2^{\omega_1}$ .

*Proof.* This is virtually identical to the proof that the statement  $\theta_{AC}$  of Todorcevic implies  $2^\omega = 2^{\omega_1}$  as proved in [25].  $\square$

The reason for formulating  $v_{AC}$  is that it is a consequence of MRP.

**Theorem 4.3.** MRP implies  $v_{AC}$ .

This is a consequence of the following fact.

**Lemma 4.4.** If  $M$  is a countable elementary submodel of  $H((2^{\omega_1})^+)$ , then the following sets are open and  $M$ -stationary:

$$\Sigma_{<}(M) = \{N \in M \cap [\omega_2]^\omega : w(N \cap \omega_1, M \cap \omega_1) < w(N, M \cap \omega_2)\}$$

$$\Sigma_{\geq}(M) = \{N \in M \cap [\omega_2]^\omega : w(N \cap \omega_1, M \cap \omega_1) \geq w(N, M \cap \omega_2)\}.$$

To see how to prove Theorem 4.3 from the lemma, define

$$\Sigma_A(M) = \begin{cases} \Sigma_{<}(M) & \text{if } M \cap \omega_1 \in A \\ \Sigma_{\geq}(M) & \text{if } M \cap \omega_1 \notin A. \end{cases}$$

Now let  $N_\xi^*$  ( $\xi < \omega_1$ ) be a reflecting sequence for  $\Sigma_A$ . Let

$$\delta = \bigcup_{\xi < \omega_1} N_\xi \cap \omega_2$$

$$N_\xi = N_\xi^* \cap \omega_2.$$

It is easily verified that  $\delta$  is an ordinal, taken together with  $\langle N_\xi : \xi < \omega_1 \rangle$ , satisfies the conclusion of  $v_{AC}(A)$ . We will now return our attention to the proof of the lemma.

*Proof.* To see that  $\Sigma_{<}(M)$  is  $M$ -stationary, let  $E \subseteq [\omega_2]^\omega$  be a club in  $M$ . By the pigeonhole principle, there is a  $\gamma < \omega_1$  such that

$$\{\sup(N) : N \in E \text{ and } N \cap \omega_1 \subseteq \gamma\}$$

is unbounded in  $\omega_2$ . By elementarity of  $M$  there is a  $\gamma < M \cap \omega_1$  such that

$$\{\sup(N) : N \in E \cap M \text{ and } N \cap \omega_1 \subseteq \gamma\}$$

is unbounded in  $M \cap \omega_2$ . Pick an  $N$  in  $E \cap M$  such that

$$w(N \cap \omega_2, M \cap \omega_2) = |\sup(N) \cap \pi_{M \cap \omega_2}^{-1}[C_{\text{otp}(M \cap \omega_2)}]| > |C_{M \cap \omega_1} \cap \gamma|.$$

Since  $N$  is in  $M$  and  $N$  is countable,  $N \subseteq M$  and  $\sup(N) < \sup(M)$ . It now follows from the definition of  $\Sigma_{<}(M)$  that  $N$  is in  $E \cap \Sigma_{<}(M) \cap M$ .

To see that  $\Sigma_{<}(M)$  is open, let  $N$  be in  $\Sigma_{<}(M)$ . If  $N$  does not have a last element, let  $\xi$  be the least element of  $N$  greater than

$$\max(\sup(N) \cap \pi_{M \cap \omega_2}^{-1}[C_{\text{otp}(M \cap \omega_2)}]).$$

If  $N$  has a greatest element, set  $\xi = \max(N)$ . Define  $x$  to be the finite set  $(N \cap C_{M \cap \omega_1}) \cup \{\xi\}$ . It is easy to see that

$$w(x \cap \omega_1, M \cap \omega_1) = w(N \cap \omega_1, M \cap \omega_1)$$

$$w(x \cap \omega_2, M \cap \omega_2) = w(N \cap \omega_2, M \cap \omega_2)$$

and hence by left monotonicity of  $w$  we have that  $[x, N] \subseteq \Sigma_{<}(M)$ .

In order to see that  $\Sigma_{\geq}(M)$  is  $M$ -stationary, let  $E \subseteq [\omega_2]^\omega$  be a club in  $M$  and let  $\gamma < \omega_2$  be uncountable such that  $E \cap [\gamma]^\omega$  is a club in  $[\gamma]^\omega$ . By elementarity of  $M$ , such a  $\gamma$  can be found in  $M$ . Working in  $M$  it is possible to find an  $N$  in  $E \cap [\gamma]^\omega$  such that

$$|N \cap \omega_1 \cap C_{M \cap \omega_1}| \geq |\gamma \cap \pi_{M \cap \omega_2}^{-1}[C_{\text{otp}(M \cap \omega_2)}]|$$

and  $\gamma \cap \pi_{M \cap \omega_2}^{-1}[C_{\text{otp}(M \cap \omega_2)}] \subseteq N$ . Then  $N$  is in  $E \cap \Sigma_{\geq}(M) \cap M$ . The proof that  $\Sigma_{\geq}(M)$  is open is similar to the corresponding proof for  $\Sigma_{<}(M)$ .  $\square$

## 5. THE BOUNDED PROPER FORCING AXIOM AND THE CONTINUUM

In this section we will see that the Bounded Proper Forcing Axiom implies  $\nu_{\text{AC}}$ . Before arguing this, I will first give a little context to the result. The Bounded Proper Forcing Axiom is equivalent to the assertion that

$$(H(\omega_2), \in) \prec_{\Sigma_1} V^{\mathcal{P}}$$

for every proper forcing  $\mathcal{P}$ . That is if  $\phi$  is a  $\Sigma_1$ -formula with a parameter in  $H(\omega_2)$ , then  $\phi$  is true iff it can be forced to be true by some proper forcing. The original statement of BPFA is due to Goldstern and Shelah [8] and is somewhat different, though equivalent. The above formulation and its equivalence to the original is due to Bagaria [2]. The consistency strength of BPFA is much weaker than that of PFA — it is exactly that of a reflecting cardinal<sup>5</sup> [8] [24]. It should be noted though that many of the consequences of PFA —  $\text{MA}_{\omega_1}$ , the

<sup>5</sup>For instance, if  $\kappa$  is Mahlo, then  $V_\kappa$  satisfies that there is a proper class of reflecting cardinals.



non-existence of S-space and Kurepa trees, the assertion that all  $\omega_1$ -dense sets of reals are order isomorphic — are actually consequences of BPFA.

Recently there has been a considerable amount of work on bounded forcing axioms and well orderings of the continuum. Woodin was the first to give such a proof from the assumption of Bounded Martin's Maximum and "there is a measurable cardinal" [27]. The question of whether Bounded Martin's Maximum alone sufficed remained an intriguing question. In light of Woodin's result, a reasonable approach was to show that BMM has considerable large cardinal strength and use this to synthesize the role of the measurable cardinal. Asperó showed that BMM implies the dominating number  $\mathfrak{d}$  is  $\omega_2$  [1]. Soon after, Todorćević showed that BMM implies that  $\mathfrak{c} = \omega_2$  and that, moreover, there is a well ordering of  $\mathbb{R}$  which is  $\Sigma_1$ -definable from an  $\omega_1$ -sequence of reals [25]. Very recently Schindler showed that BMM does have considerable consistency strength [13]. It should be noted, however, that it is still unclear whether the measurable cardinal can be removed from Woodin's argument.

Now we will see that the Bounded Proper Forcing Axiom is already sufficient to give a definable well ordering of  $\mathbb{R}$  from parameters in  $H(\omega_2)$ . Notice that for a fixed  $A$ ,  $v_{AC}(A)$  is a  $\Sigma_1$ -sentence which takes the additional parameter  $\langle C_\xi : \xi \in \lim(\omega_1) \rangle$  (in order to define  $w$ ). Further examination of the above proof reveals that for each  $A$ , there is a proper forcing which forces  $v_{AC}(A)$ . Hence the Bounded Proper Forcing Axiom implies  $v_{AC}$ . Asperó pointed out that, unlike statements such as  $\psi_{AC}$  and  $\phi_{AC}$ , the statement  $v_{AC}$  can be forced over any model with an inaccessible cardinal.

## 6. MRP AND $\square(\kappa)$

Recall the following combinatorial principle — defined for a regular  $\kappa > \omega_1$  — which was introduced by Todorćević in [20]:<sup>6</sup>

$\square(\kappa)$ : There is a sequence  $\langle C_\alpha : \alpha < \kappa \rangle$  such that:

- (1)  $C_{\alpha+1} = \{\alpha\}$  and  $C_\alpha \subseteq \alpha$  is closed and cofinal if  $\alpha$  is a limit ordinal.
- (2) If  $\alpha$  is a limit point of  $C_\beta$ , then  $C_\alpha = C_\beta \cap \alpha$ .
- (3) There is no club  $C \subseteq \kappa$  such that for all limit points  $\alpha$  in  $C$  the equality  $C_\alpha = C \cap \alpha$  holds.

In this section we will prove Theorem 1.4 — that MRP implies that  $\square(\kappa)$  fails for all regular  $\kappa > \omega_1$ . The reason for proving such a result is

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<sup>6</sup>This seems to be the first place  $\square(\kappa)$  was defined explicitly. A similar principle  $\square_\kappa$  was defined and studied by Jensen in [10].

that it shows MRP has considerable consistency strength. The essence of Theorem 1.4 is contained in the following lemma.

**Lemma 6.1.** *If  $M$  is a countable elementary submodel of  $H(\kappa^+)$  containing a  $\square(\kappa)$ -sequence  $\langle C_\alpha : \alpha < \kappa \rangle$ , then the set  $\Sigma(M)$  of all  $N \subseteq M \cap \kappa$  such that  $\sup(N)$  is not in  $C_{\sup(M \cap \kappa)}$  is open and  $M$ -stationary.*

*Remark 6.2.* Given the lemma, let  $N_\nu$  ( $\nu < \omega_1$ ) be a reflecting sequence for  $\Sigma$  and set  $E = \{\sup(N_\nu \cap \kappa) : \nu < \omega_1\}$ . Then  $E$  is closed and of order type  $\omega_1$ . Let  $\beta$  be the supremum of  $E$ . Now there must be a limit point  $\alpha$  in  $E \cap C_\beta$ . Let  $\nu$  be such that  $\alpha = \sup(N_\nu \cap \kappa)$ . But now there is a  $\nu_0 < \nu$  such that  $\sup(N_{\xi} \cap \kappa)$  is not in  $C_\alpha = C_\beta \cap \alpha$  whenever  $\nu_0 < \xi < \nu$ . This means that  $\alpha$  is not a limit point of  $E$ , a contradiction.

*Proof.* First we will check that  $\Sigma(M)$  is open. To see this, let  $N$  be in  $\Sigma(M)$ . If  $N$  has a last element  $\gamma$ , then  $[\{\gamma\}, N] \subseteq \Sigma(M)$ . If  $N$  does not have a last element, then, since  $C_{\sup(M \cap \kappa)}$  is closed, there is a  $\gamma$  in  $N$  such that if  $\xi < \sup(N)$  is in  $C_{\sup(M \cap \kappa)}$ , then  $\xi < \gamma$ . Again  $[\{\gamma\}, N] \subseteq \Sigma(M)$ .

Now we will verify that  $\Sigma(M)$  is  $M$ -stationary. To this end, let  $E \subseteq [\kappa]^\omega$  be a club in  $M$ . Let  $S$  be the collection of all  $\sup(N)$  such that  $N$  is in  $E$ . Clearly  $S$  has cofinally many limit points in  $\kappa$ . If  $S \cap M$  is contained in  $C_{\sup(M \cap \kappa)}$ , then we have that whenever  $\alpha < \beta$  are limit points in  $S \cap M$ ,

$$\begin{aligned} C_\alpha &= C_{\sup(M \cap \kappa)} \cap \alpha \\ C_\beta &= C_{\sup(M \cap \kappa)} \cap \beta \end{aligned}$$

and hence  $C_\alpha = C_\beta \cap \alpha$ . But, by elementarity of  $M$ , this means that for all limit points  $\alpha < \beta$  in  $S$ ,  $C_\alpha = C_\beta \cap \alpha$ . This would in turn imply that the union  $C$  of  $C_\alpha$  for  $\alpha$  a limit point of  $S$  is a closed unbounded set such that  $C_\alpha = C \cap \alpha$  for all limit points  $\alpha$  of  $C$ , contradicting the definition of  $\langle C_\alpha : \alpha < \kappa \rangle$ . Hence there is an  $N$  in  $E$  such that  $\sup(N)$  is not in  $C_{\sup(M \cap \kappa)}$ .  $\square$

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