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THE METRIZATION PROBLEM FOR FRÉCHET GROUPS

STEVO TODORCEVIC AND JUSTIN TATCH MOORE

1. Introduction

Let us begin this paper by recalling the following classical metrization theorem of Birkhoff and Kakutani.

Theorem 1.1. *Every first countable group is metrizable.*

In this article, we will be interested in the extent to which the assumption of first countability in this theorem can be weakened. Recall that a Hausdorff[1] topological space X is *Fréchet* if whenever x is a limit point of $A \subseteq X$, there is a sequence a_n $(n < \omega)$ of elements of A which converges to x. This is a natural weakening of first countability which has been extensively studied in the literature. It turns out that this property by itself is not sufficient to ensure the metrizability of a topological group.

Example 1.2. The direct sum of ω_1 copies of the circle group $(\mathbb{R}/\mathbb{Z}, +)$ is a σ-compact Fréchet group which is not first countable.

Such an example, however is easily ruled out by requiring that the group be separable. Hence we arrive at the following problem posed by Malykhin in 1978.

Problem 1.3. *Is every separable Fréchet group metrizable?*

The requirement of separability in Malykhin's problem can be replaced by a more restrictive notion without changing the problem as the following proposition shows.

Proposition 1.4. *If every countable Fréchet group is metrizable, then so is every separable Fréchet group.*

We have already noted that some countability requirement is necessary in the formulation of Malykhin's problem. Later Todorcevic provided an example with an additional striking property.

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[1]All spaces in this article are assumed to be Hausdorff.

Example 1.5. [14] There are two σ-compact Fréchet groups whose product is not countably tight.

This highlights an auxiliary consideration — the productivity of the Fréchet property in groups — which will also be part of our focus.

It was known from the beginning that Malykhin's problem is really a consistency question. At the time he posed Problem 1.3, Malykhin was aware of the following consistent counterexample.

Example 1.6. ($\mathfrak{p} > \omega_1$) If G is a separable metrizable group with at least two elements, then G^{ω_1} is a separable Fréchet group which is not metrizable.

On the other hand, Shibakov later showed that CH can also be used to generate counterexamples with additional properties.

Example 1.7. [9] ($\mathfrak{c} = \omega_1$) There are separable Fréchet groups whose product is not Frećhet.

It is also not clear what role the group structure plays in Malykhin's problem. Recall that $[\omega]^{<\omega}$ — the finite subsets of ω — is a group when equipped with the operation $\triangle$ of symmetric difference.

Question 1.8. *Is there a topology which makes* $([\omega]^{<\omega}, \triangle)$ *a non-metrizable Fréchet group?*

Question 1.9. *Suppose that there is a separable non-metrizable Fréchet group. Is there a topology which makes* $([\omega]^{<\omega}, \triangle)$ *a non-metrizable Fréchet group?*

These questions really ask about the existence of certain filters — known as FUF filters in the literature — on $[\omega]^{<\omega}$ (see [7]).

Since there is already a recent survey [7] by Shakhmatov on convergence in settings where there is additional algebraic structure, we will focus on a scenario for proving a positive answer to Malykhin's problem and refer the reader to that article for further information.

2. The role of gaps

Suppose that G is a countable topological group on ω with 0 serving as the identity. A central object in the study of Malykhin's problem is the ideal $\mathscr{I}_G$ of subsets of ω which do not accumulate to 0. It is easily verified that G is Fréchet iff $\mathscr{I}_G^{\perp\perp} = \mathscr{I}_G$.[2]

Recall that two families $\mathscr{A}$ and $\mathscr{B}$ of subsets of ω form a *gap* if they are orthogonal (i.e. $\mathscr{B} \subseteq \mathscr{A}^\perp$) and yet there is no $C \subseteq \omega$ such that

[2]If $\mathscr{A}$ is a family of subsets of ω, we let $\mathscr{A}^\perp$ denote the collection of all subsets of ω which have finite intersection with every element of $\mathscr{A}$.

every element of $\mathscr{A}$ is almost contained in C and every element of $\mathscr{B}$ is almost disjoint from C. If $\mathscr{A}^{\perp\perp}$ is countably generated, then the gap is said to be *Rothberger*. Hence Malykhin's problem is equivalent to asking whether there is a countable Fréchet group G such that $\mathscr{I}_G$ does not form a Rothberger gap with $\mathscr{I}_G^{\perp}$.

Todorcevic's Open Coloring Axiom is an assertion which has a strong influence on the structure of gaps in $\mathscr{P}(\omega)$:

> If X is a separable metric space and $G \subseteq [X]^2$ is an open graph on X, then either G is countably chromatic or else has an uncountable complete subgraph.

This axiom was defined in [13], where its influence on gaps is presented. Moreover, if the underlying set of reals is analytic, then OCA is provable — in ZFC — for all open graphs on X. This can either be deduced[3] from the consistency proof of OCA or proved directly as in [3]. Not surprisingly, it is possible to carry out a parallel analysis of gaps in $\mathscr{P}(\omega)$ in which $\mathscr{A}$ is analytic. This led to the proof of the following effective solution to Malykhin's problem. Recall that a countable topological space X is *analytic* if its topology is an analytic subset of $\mathscr{P}(X)$ when identified with 2^X.

Theorem 2.1. [12] *Suppose that G is a separable group with a countable dense analytic subspace. Then G is metrizable.*

Countable metrizable spaces are always analytic but in general this is a considerably larger class which includes a number of important test spaces. For example the countable sequential fan and Arens space are examples of analytic topologies. In fact a countable space is analytic if and only if it can be embedded into $C_p(X)$ for some Polish space X.

While Malykhin's problem can only have a consistent positive solution and Theorem 2.1 is a ZFC theorem, the analysis of the combinatorial difficulties seems likely to be similar. The reader is referred to [12] and [16] for applications of OCA which are closely related to the subject matter.

3. Other convergence properties

There are two other weakenings of first countability which are important in the present context.

[3]This is a consequence of the following observations: (1) the partial order for forcing an instance of OCA (see [13, §8]) can be modified so that the resulting homogeneous set is moreover relatively closed, (2) analytic sets have the perfect set property yielding a perfect homogeneous set in the extension, and (3) by Shoenfeld's absoluteness theorem [10], the homogeneous set exists in the ground model.

Definition 3.1. A topological space is said to have the *weak diagonal sequence property*[4] if whenever S_i $(i < \omega)$ is a collection of sequences which converges to a given point x, there is a sequence S_∞ which converges to x such that $S_i \cap S_\infty$ is non-empty for infinitely many $i < \omega$. If S_∞ can always be selected so as to intersect every S_i, then the space is said to have the *diagonal sequence property*.[5]

In the general setting of topological spaces, they are unrelated to the Fréchet property. Nyikos demonstrated that this is not the case is the more restrictive setting of topological groups.

Theorem 3.2. [6] *Fréchet groups have the weak diagonal sequence property.*

Whether Fréchet groups have the stronger of these properties, however, is unclear and may be closely related to Malykhin's problem.

Question 3.3. [8] *Is it consistent that every countable Fréchet group has the diagonal sequence property?*

Question 3.4. [8] *Is it consistent that every countable Fréchet group with the diagonal sequence property is metrizable?*

Both of these questions could also be also asked without the assumption of countability. Let us note the following reformulation of a result of Szlenk (see [11] and [15, p.65], or [12, p.516]) that there is a positive solution to the effective version of Question 3.4.

Theorem 3.5. *Every analytic Fréchet space with the diagonal sequence property is first countable.*

An important question becomes whether (and how much) Theorem 3.2 can be strengthened. The ultimate target is to demonstrate that — consistently — separable Fréchet groups are *bi-sequential*: whenever $\mathscr{U}$ is a convergent ultrafilter, there is a sequence U_n $(n < \omega)$ of elements of $\mathscr{U}$ which converges to the same point as $\mathscr{U}$. This property is easily shown to be productive and strengthens both the Fréchet and diagonal sequence properties. Furthermore, in the class of topological groups, this condition is as strong as metrizability, as the following result of Arkhangel'ski and Malykhin [1] demonstrates.

Theorem 3.6. *Bi-sequential groups are first countable and therefore metrizable.*

[4] This property is often referred to as α_4.

[5] This property is often referred to as α_2.

Proof. Clearly we may assume that G has no isolated points and therefore that the nowhere dense subsets of G extend the cofinite filter. Suppose that g is in G and let $\mathscr{U}$ be an ultrafilter converging to g which is disjoint from the collection of nowhere dense subsets of G. Applying the bi-sequentiality of G, let U_n $(n < \omega)$ be a sequence of elements of $\mathscr{U}$ which converge to g. Let V_n be the interior of the closure of U_n and set $W_n = V_n * V_n^{-1}$. It follows then that $\{W_n : n < \omega\}$ forms a countable neighborhood base at 0 and hence G is first countable. By the Birkhoff-Kakutani theorem, every first countable group is metrizable, finishing the proof. □

The point is, however, that it may be more natural to verify that the group at hand is bi-sequential and indeed this is the approach taken in [12].

We will now recall a set theoretic definition. Now suppose that X is a countable set. A collection $\mathscr{H}$ of subsets of X is a *co-ideal* (on X) if

(1) $\mathscr{H}$ is closed under supersets relative to X and
(2) If Z is in $\mathscr{H}$ and $Z = \bigcup_{i<n} Z_i$, then there is an $i < n$ such that Z_i is in Z.

Definition 3.7. A co-ideal $\mathscr{H}$ on X is said to be *selective* if it satisfies the following two additional conditions:

p^+: Whenever Z_n $(n < \omega)$ is a decreasing sequence of elements of $\mathscr{H}$, then there is a Z_∞ in $\mathscr{H}$ such that $Z_\infty \setminus Z_n$ is finite for all $n < \omega$.

q^+: Whenever Z is in $\mathscr{H}$ and $\phi : Z \to \omega$ is finite-to-one, there is a $Z_* \subseteq Z$ in $\mathscr{H}$ such that $\phi \restriction Z_*$ is one-to-one.

If G is a topological group, define

$$\mathscr{H}_G = \{X \subseteq G : X \text{ accumulates at } 0\}.$$

It is easily verified that $\mathscr{H}_G$ is always a co-ideal — even if G is not a group and 0 is replaced by an arbitrary limit point in the space. The following proposition shows that in the context which is of interest to us, this co-ideal is selective.

Proposition 3.8. [12] *If G is a countable Fréchet group, then $\mathscr{H}_G$ is a selective co-ideal.*

4. In search of a test model

Examples 1.2 and 1.6 suggest the need for a model which allows for the failure of certain consequences of PFA close to $\mathrm{MA}_{\aleph_1}(\sigma-\text{centered})$

while maintaining other consequences — and OCA in particular. Models with these general properties were considered by Larson and Todorcevic in [4], [5] and were obtained by forcing with a Souslin tree over a model of a strong fragment of PFA. Note, however, that since $\mathfrak{p} > \omega_1$ in their ground model and since examples solving Malykhin's problem are preserved in forcing extensions which do not add reals, such models can not yield a solution to Malykhin's problem. The problem is essentially that one needs the conjunction of OCA and together with a failure of $\mathfrak{p} > \omega_1$ which is more pertinent to the problem at hand. Michael Hrušák has suggested the following problems in relation to this.

Question 4.1. *Is* OCA *consistent with the assertion that every ω-splitting family in $[\omega]^\omega$ contains an ω-splitting subfamily of size $\aleph_1$?*

Question 4.2. *Assume* OCA *and that every ω-splitting family in $[\omega]^\omega$ contains an ω-splitting subfamily of size $\aleph_1$. Is every separable Fréchet group metrizable?*

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Stevo Todorcevic — Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 2E4

Université Paris 7 - C.N.R.S., UMR 7056,, 2 Place Jussieu, 75251 Paris, Cedex 05, France

Justin Tatch Moore — Department of Mathematics, Boise State University, Boise, Idaho 83725–1555