

Open colorings, the continuum and the second uncountable cardinal *

Justin Tatch Moore

June 15, 2002

Abstract

The purpose of this article is to analyze the cardinality of the continuum using Ramsey theoretic statements about open colorings or “open coloring axioms.” In particular I will show that the conjunction of two well known axioms, $\mathbf{OCA}_{[\text{ARS}]}$ and $\mathbf{OCA}_{[\text{T}]}$, implies that the size of the continuum is \aleph_2 .

Our focus in this paper will be the following two open coloring axioms ¹ and their influence on the size of the continuum.

$\mathbf{OCA}_{[\text{ARS}]}$ If X is a separable metric space of size \aleph_1 and $c : [X]^2 \rightarrow \{1, \dots, n\}$ is a continuous map then there is a decomposition of X into countably many pieces X_i ($i \in \mathbb{N}$) such that c is constant on $[X_i]^2$ for all $i \in \mathbb{N}$.

$\mathbf{OCA}_{[\text{T}]}$ If X is a separable metric space and $G \subseteq [X]^2$ is open then either G is countably chromatic (there is a decomposition of X into countably many pieces X_i such that $[X_i]^2 \cap G$ is empty for all i) or there is an uncountable $H \subseteq X$ such that $[H]^2 \subseteq G$.

*The research for this paper was supported by EPSRC grant GR/M71121 during my stay at the University of East Anglia. I would also like to acknowledge the support I received from the Institut Mittag-Leffler during my visit there.

¹The subscripts [ARS] and [T] refer to [1] and [11] where these axioms originally appeared. In the current literature \mathbf{OCA} has come to mean $\mathbf{OCA}_{[\text{T}]}$.

Here $[X]^2$ is the set of unordered pairs of distinct elements of X . A set $G \subseteq [X]^2$ is *open* if for every $\{x, y\}$ in G there are disjoint neighborhoods $U \ni x$ and $V \ni y$ such that $\{\{x', y'\} : x' \in U, y' \in V\}$ is contained in G . One defines continuity in a similar way for functions of the form $c : [X]^2 \rightarrow \{1, \dots, n\}$.

What I will demonstrate through the course of this note is that the conjunction of these two axioms implies that there are exactly \aleph_2 real numbers. The proof is divided into four parts, each contributing a section to the paper along with some review and concluding remarks. The first section recalls some of the basic properties of the partial order $(\mathbb{N}^{\mathbb{N}}, <^*)$ of eventual dominance. Section 2 gives some of the flavor of the proof from [11] that $\mathbf{OCA}_{[T]}$ implies that the minimum size of an unbounded subset of $(\mathbb{N}^{\mathbb{N}}, <^*)$ is equal to \aleph_2 . Section 3 introduces the alternation map and its properties. This map is then used in Section 4 to code reals inside of certain unbounded subsets of $(\mathbb{N}^{\mathbb{N}}, <^*)$ in the presence of $\mathbf{OCA}_{[T]}$. Section 5 shows that $\mathbf{OCA}_{[ARS]}$ prevents too many reals from being coded inside of any subset of $\mathbb{N}^{\mathbb{N}}$ of size \aleph_1 . The final section closes with some questions and comments.

None of the mathematics in this paper should be beyond the understanding of someone able to make sense of the statement of the main theorem. Those familiar with the order $(\mathbb{N}^{\mathbb{N}}, <^*)$ should feel free to skip the first section. Those familiar with $\mathbf{OCA}_{[T]}$ may wish to skip directly to Section 3. Readers interested in further reading on 2^{\aleph_0} and \aleph_2 are referred to [3],[8],[9],[12], and [13].

I would like to thank Stevo Todorčević for giving me Theorem 2.1 as an exercise back in 1997 during our first evening in a coffee shop. He also offered some useful suggestions on the presentation of the material in Section 2.

1 The partial order $(\mathbb{N}^{\mathbb{N}}, <^*)$

Our discussion in this paper will frequently focus around the topological space $\mathbb{N}^{\mathbb{N}}$ of all sequences of natural numbers. This is a complete separable metric space when equipped with the product topology. Basic open sets in this context take the form

$$[u] = \{x \in \mathbb{N}^{\mathbb{N}} : x \text{ extends } u\}$$

where u is a finite sequence of natural numbers. The complete metric is given by $d(x, y) = 2^{-\Delta(x, y)}$ where

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$

We will also be interested in $\mathbb{N}^{\mathbb{N}}$ as a partially ordered set under the order $<^*$ of eventual dominance:

$$x <^* y \text{ iff } x(n) < y(n) \text{ for all but finitely many } n.$$

The partial order $(\mathbb{N}^{\mathbb{N}}, <^*)$ has a rich structure and frequently shows up in mathematical problems, particularly those close to analysis, set theory, and general topology. The following observations are well known and easily proven.

Fact 1.1. $(\mathbb{N}^{\mathbb{N}}, <^*)$ is countably directed (every countable subset has an upper bound).

Fact 1.2. If X is an unbounded and countably directed subset of $(\mathbb{N}^{\mathbb{N}}, <^*)$ then whenever X is decomposed into countably many sets X_n ($n \in \mathbb{N}$), there is an n such that X_n is unbounded and countably directed.

In light of Fact 1.1 and the trivial observation that $\mathbb{N}^{\mathbb{N}}$ is unbounded in $(\mathbb{N}^{\mathbb{N}}, <^*)$ it makes sense to define the following cardinal.

Definition 1.3. The unbounding number \mathfrak{b} is the minimum size of an unbounded subset of $(\mathbb{N}^{\mathbb{N}}, <^*)$.

This cardinal is therefore somewhere between \aleph_1 and 2^{\aleph_0} . It turns out to have some rather surprising combinatorial properties (see [11, §1]). The following fact is well known and easily verified.

Fact 1.4. There is an unbounded chain in $(\mathbb{N}^{\mathbb{N}}, <^*)$ of order type \mathfrak{b} . In particular there is an unbounded and countably directed subset of $(\mathbb{N}^{\mathbb{N}}, <^*)$ of size \mathfrak{b} .

The next observation is behind much of the combinatorics associated with unboundedness in $(\mathbb{N}^{\mathbb{N}}, <^*)$.

Fact 1.5. Suppose $X \subseteq \mathbb{N}^{\mathbb{N}}$ is nonempty and has the property that every neighborhood of X is unbounded. For every basic open set $[u]$, if $[u] \cap X$ is nonempty, then there is a v extending u such that $[v \hat{\ } i] \cap X$ is nonempty for infinitely many i .

Notice that since the union of countably many bounded sets is bounded (by Fact 1.1), if X is an unbounded subset of $(\mathbb{N}^{\mathbb{N}}, <^*)$ then it contains a nonempty set X' such that every neighborhood of X' is unbounded.

2 \aleph_2 and the unbounding number

This section will review the following result and give a sketch of its proof.

Theorem 2.1. [11] $OCA_{[T]}$ implies that \mathfrak{b} is \aleph_2 .

This result breaks into two parts, one showing \mathfrak{b} is at least \aleph_2 and the later showing that it is at most \aleph_2 . Proving that \mathfrak{b} is at least \aleph_2 under $OCA_{[T]}$ was done originally via the oscillation map (see [11]). It was this proof that led me to the discovery of the alternation map (which is essentially a continuous version of the oscillation map) and to the coding technique which will be presented in the later sections. An interested reader is encouraged to deduce this portion of the Theorem 2.1 from the properties of the alternation map given below.

The half of Theorem 2.1 which deals with the inequality $\mathfrak{b} \leq \aleph_2$ makes crucial use of the following well studied notion.

Definition 2.2. If κ and λ are cardinals then a (κ, λ^*) gap in $(\mathbb{N}^{\mathbb{N}}, <^*)$ is a pair of sequences a_ξ ($\xi < \kappa$) and b_η ($\eta < \lambda$) in $\mathbb{N}^{\mathbb{N}}$ such that

$$a_\xi <^* a_{\xi'} <^* b_{\eta'} <^* b_\eta$$

for all $\xi < \xi' < \kappa$ and $\eta < \eta' < \lambda$ but for which there is no single c in $\mathbb{N}^{\mathbb{N}}$ such that $a_\xi <^* c <^* b_\eta$ for all $\xi < \kappa$ and $\eta < \lambda$.

Long ago Hausdorff made the following important connection between gaps and unbounded chains in $(\mathbb{N}^{\mathbb{N}}, <^*)$.

Theorem 2.3. [4] If κ is a regular cardinal, then there is an unbounded chain in $(\mathbb{N}^{\mathbb{N}}, <^*)$ of order type κ iff there is a (κ, ω^*) gap in $(\mathbb{N}^{\mathbb{N}}, <^*)$.

Hausdorff also discovered another type of gap — the (ω_1, ω_1^*) gap. Rather remarkably, this is the best that one can do without using extra assumptions.

Theorem 2.4. [11] Under $OCA_{[T]}$ all gaps in $(\mathbb{N}^{\mathbb{N}}, <^*)$ are of the form (ω_1, ω_1^*) , (κ, ω^*) , or (ω^*, κ) for some cardinal κ .

Also quite striking is the following fact which now completes the proof of Theorem 2.1. An interested reader is encouraged to supply its proof via Theorem 2.3.

Lemma 2.5. [11] If every subset of $\mathbb{N}^{\mathbb{N}}$ of size at most \aleph_2 is bounded, then there is an (ω_2, κ^*) gap in $(\mathbb{N}^{\mathbb{N}}, <^*)$ for some regular uncountable cardinal κ .

3 The alternation map

For technical reasons we will now restrict our discussion to the suborder $(\mathbb{N}^{\uparrow\mathbb{N}}, <^*)$ of strictly increasing sequences of natural numbers. Since $\mathbb{N}^{\uparrow\mathbb{N}}$ is both cofinal in and order isomorphic to $(\mathbb{N}^{\mathbb{N}}, <^*)$ this is really not a loss of generality. The following notion will be central to our discussion in this section.

Definition 3.1. Suppose that S is a subset of \mathbb{N} . Two sequences x and y in $\mathbb{N}^{\uparrow\mathbb{N}}$ are said to *alternate* on S if $x(n) \neq y(n)$ for all n in S and for every pair of consecutive elements m and n of S , $x(m) < y(m)$ iff $x(n) > y(n)$.

Notice that this notion also makes sense if x and y are only partial functions whose domains include S .

Now suppose that x and y are two distinct elements of $\mathbb{N}^{\uparrow\mathbb{N}}$ and that x is lexicographically less than y . Define a set $S(x, y) \subseteq \mathbb{N}$ by recursion. The first element of $S(x, y)$ is $\Delta(x, y)$. Given the $2n^{\text{th}}$ element k of $S(x, y)$ define the next two elements to be $y(k)$ and $x(y(k))$. So the elements of $S(x, y)$ are

$$\Delta(x, y), y(\Delta(x, y)), x(y(\Delta(x, y))), y(x(y(\Delta(x, y)))) , \dots$$

If x and y are only partial functions, repeat this procedure so long as the elements generated remain in the domains of both x and y . Since x and y are strictly increasing, the elements of S being defined are getting successively larger and larger. If i is an integer and $S(x, y)$ has an i^{th} element (starting with $i = 0$), then we will occasionally find it convenient to denote it as $S(x, y)(i)$.

Notice that the map $S : [\mathbb{N}^{\uparrow\mathbb{N}}]^2 \rightarrow \mathcal{P}(\mathbb{N})$ is continuous since to determine a finite amount of information about $S(x, y)$ one needs only a finite amount of information about x and y . Given $S(x, y)$, $\text{alt}(x, y)$ is defined to be the cardinality of the largest initial segment of $S(x, y)$ on which x and y alternate minus one. Thus $\text{alt} : [\mathbb{N}^{\uparrow\mathbb{N}}]^2 \rightarrow \mathbb{N} \cup \{\infty\}$ is also a continuous map, where $\mathbb{N} \cup \{\infty\}$ is the one point compactification of \mathbb{N} . Notice that the alternation map is finite on all pairs which are comparable in $(\mathbb{N}^{\uparrow\mathbb{N}}, <^*)$.

The following theorem shows that the alternation map exhibits a behavior similar to that of the more well known oscillation map of Todorćević (see, e.g., [11]). It has essentially appeared in [5] and chapter 4 of [6]. A proof has been included for completeness.

Theorem 3.2. *If X is an unbounded and countably directed in $(\mathbb{N}^{\mathbb{N}}, <^*)$ then there is an n such that for every k there is a pair x, y in X with $\Delta(x, y) = n$ and $\text{alt}(x, y) = k$.*

Proof. Let X be given and choose a countable dense subset D of X . Since $(X, <^*)$ is countably directed, there is an a in X which is an upper bound for D . Now consider the set of all $\{y \in X : a <^* y\}$. Since X is directed and unbounded, this set is also unbounded. It is therefore possible by Fact 1.2 to find an n and a subcollection Y of this set such that every neighborhood of Y is unbounded and $a(i) < y(i)$ for every y in Y and $i \geq n$.

Now choose a sequence u_k ($i \in \mathbb{N}$) such that for all k

1. u_k is a finite sequence of natural numbers such that $[u_k \hat{\ } i] \cap Y$ is nonempty for infinitely many i ,
2. u_k is an initial segment of u_{k+2} ,
3. $\Delta(u_k, u_{k+1}) = |u_0| > m$,
4. $|u_k| < S(u_{k+1}, u_{k+2})(k) < |u_{k+1}|$, and
5. $u_{k+2}(n_k) < u_{k+1}(n_k)$ where $n_k = S(u_{k+1}, u_{k+2})(k)$.

The recursion is easily carried out using Fact 1.5. Notice that $u_{2k+1} <_{\text{lex}} u_{2k+2}$ follows from 2, 3, and 5. Also, by 2 and 3, $S(u_{k+1}, u_{k+2})(i)$ is a fixed integer n_i which does not depend on k provided that $i \leq k$. By 5 u_k and u_{k+1} alternate on $S(u_k, u_{k+1}) = \{n_0, \dots, n_{k-1}\}$ ($S(u_0, u_1)$ is empty).

Now let d_k be in $[u_{k+1}] \cap D$ and fix an m_k such that $d_k(i) < a(i)$ for all $i > m_k$. Choose y_k in $[u_k] \cap Y$ such that $y_k(|u_k|) > a(m_k)$. Note that y_k dominates d_k on the interval $[|u_k|, m_k]$ by monotonicity and y_k dominates d_k $[m_k, \infty)$ since $m_k > m$. It follows that $\Delta(d_k, y_k) = n_0$ and that $\{n_0, \dots, n_k\}$ are the first $k+1$ elements of $S(d_k, y_k)$. Note that $y_k(n_k) > d_k(n_k)$ and that the next element of $S(d_k, y_k)$ is greater than $|u_k|$ and hence y_k dominates d_k at this place as well. Hence by our observations above $\{n_0, \dots, n_k\}$ is exactly the largest initial segment of $S(d_k, y_k)$ on which d_k and y_k alternate, making $\text{alt}(d_k, y_k) = k$. \square

4 Continuously coding a real number

In this section we will see how to code countable binary sequences into subsets of $\mathbb{N}^{\mathbb{N}}$. Since the real numbers have the same size as $\{0, 1\}^{\mathbb{N}}$ this can be

considered as a means for coding reals. To facilitate the coding that we are about to define, it is useful if we change the map alt to a map t which takes values which are finite binary sequences rather than natural numbers. To this end let $\varphi(k, n)$ be the unique binary sequence t of length n such that

$$k \equiv \sum_{i=0}^{n-1} t(i)2^i \pmod{2^n}.$$

It is easily seen that $\varphi(\cdot, n)$ maps onto the binary sequences of length n .

If x, y are in $\mathbb{N}^{\uparrow\mathbb{N}}$ and $\text{alt}(x, y)$ is finite, define

$$t(x, y) = \varphi(\text{alt}(x, y), \Delta(x, y)).$$

Notice that, since it is a composition of continuous functions, t is continuous on its domain. Its domain is the preimage of \mathbb{N} under the alternation map and is therefore an open set. Similarly t is defined on all pairs x and y which are comparable in $(\mathbb{N}^{\uparrow\mathbb{N}}, <^*)$. Theorem 3.2 immediately gives the following result.

Theorem 4.1. *If X is unbounded and countably directed in $(\mathbb{N}^{\uparrow\mathbb{N}}, <^*)$ and r is in $\{0, 1\}^{\mathbb{N}}$ then there is a pair x, y in X such that $t(x, y)$ is defined and is an initial segment of r .*

Now we are ready to define what it means for a set $H \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$ to code an element r of $\{0, 1\}^{\mathbb{N}}$.

Definition 4.2. If $H \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$ is uncountable, t is defined on $[H]^2$, and the image of $[H]^2$ under t contains no two incompatible elements then H is said to be a *code*. If r is the unique element of $\{0, 1\}^{\mathbb{N}}$ such that $t(x, y)$ is an initial segment of r for all x, y in H then we will say that r is *coded by H* .

Remark 4.3. Notice that the map Δ is unbounded on the pairs of any uncountable set X . This implies that no set can be a code for two distinct r and r' in $\{0, 1\}^{\mathbb{N}}$.

Theorem 4.4. (*OCA*_[T]) *If $X \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$ is unbounded and countably directed then for every r in $\{0, 1\}^{\mathbb{N}}$ there is an $H \subseteq X$ which codes it.*

Proof. Define $G_r \subseteq [X]^2$ by putting $\{x, y\}$ in G_r iff $t(x, y)$ is defined and is an initial segment of r . Since t is continuous and has an open domain, G_r is open. By Fact 1.2 and Theorem 4.1, G_r isn't countably chromatic. Applying *OCA*_[T], there is an uncountable $H \subseteq X$ such that $[H]^2 \subseteq G_r$ which, by definition, is a code for r . \square

5 Sealing codes

In this section we finish the proof of the main result by showing that under $\mathbf{OCA}_{[\text{ARS}]}$ any chain in $(\mathbb{N}^{\uparrow\mathbb{N}}, <^*)$ of size \aleph_2 can contain codes for at most \aleph_2 reals.

Theorem 5.1. ($\mathbf{OCA}_{[\text{ARS}]}$) *Suppose that $X_0 \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$ has size \aleph_1 and t is defined on $[X_0]^2$. There is a disjoint family A_r ($r \in \{0, 1\}^{\mathbb{N}}$) of (possibly empty) subsets of X_0 such that if $H \subseteq X_0$ is a code for r then $H \setminus A_r$ is at most countable. In particular at most \aleph_1 many different elements of $\{0, 1\}^{\mathbb{N}}$ have codes contained in X_0 .*

Proof. Define $t_n : [X_0]^2 \rightarrow \{0, 1\}^{\leq n}$ for each n by setting $t_n(x, y)$ to be the restriction of $t(x, y)$ to its first n entries (set $t_n(x, y) = t(x, y)$ if $\Delta(x, y) \leq n$). Note that t_n is also continuous for each n and has finite range. Construct a family $\{A_s\}$ indexed by the finite binary sequences with the following properties:

1. For each n the collection $\{A_s : s \in \{0, 1\}^n\}$ is a partition of X_0 into disjoint sets.
2. Each A_s can be decomposed into countably many sets $A_{s,i}$ such that $t_{|s|}$ is constantly s on $[A_{s,i}]^2$ for all i .

Let n be fixed and apply $\mathbf{OCA}_{[\text{ARS}]}$ to $t_n : [X_0]^2 \rightarrow \{0, 1\}^{\leq n}$ to get countably many sets Y_i such that t_n is constant on $[Y_i]^2$ for all i . Notice we may assume that, for each i , if $x \neq y$ are in Y_i then $\Delta(x, y) > n$. It follows that for every i there is an s in $\{0, 1\}^n$ such that t is constantly s on $[Y_i]^2$ (possibly for trivial reasons). The sets A_s ($s \in \{0, 1\}^n$) are now constructed so that they are pairwise disjoint and each A_s is contained in union of all Y_i such that t_n is constantly s on $[Y_i]^2$. It should be clear that these sets satisfy properties 1 and 2 above.

If r is in $\{0, 1\}^{\mathbb{N}}$ set $A_r = \bigcap_{n=0}^{\infty} A_{r \upharpoonright n}$. By property 1 A_r and $A_{r'}$ are disjoint for distinct r and r' in $\{0, 1\}^{\mathbb{N}}$. Now observe that, by property 2, if $s \in \{0, 1\}^n$ is different than $r \upharpoonright n$ then $H \cap A_s$ is countable since $t_n''[H]^2 \subseteq \{\langle \rangle, \dots, r \upharpoonright n\}$. Applying property 1 again, $H \setminus A_{r \upharpoonright n}$ is countable for all n and hence $H \setminus A_r$ is countable. \square

6 Closing remarks

I will now close this paper by mentioning the following two open problems.

Question 6.1. Does $\mathbf{OCA}_{[\text{ARS}]}$ imply 2^{\aleph_0} is \aleph_2 ?

Question 6.2. Does $\mathbf{OCA}_{[\text{T}]}$ imply 2^{\aleph_0} is \aleph_2 ?

Both of these questions are interesting for different reasons and, in spite of their similar appearance, seem to be quite different in nature. Question 6.1 is a special case of the more well known open problem of whether the Bounded Proper Forcing Axiom implies $2^{\aleph_0} = \aleph_2$ (see [10] for more information on bounded forcing axioms). Question 6.2 has been pursued since $\mathbf{OCA}_{[\text{T}]}$ was introduced. Interest in it stems both from the wide variety of applications which this axiom has seen (see, e.g., [2], [11, §8]) and from the fact that it is consistent with an effective form of the Continuum Hypothesis in the absence of the Axiom of Choice (see [7]). I. Farah has shown in an unpublished note that it is consistent that $\mathbf{OCA}_{[\text{T}]}$ holds for graphs of size \aleph_1 and $2^{\aleph_0} > \aleph_2$ but it is known that this restricted form of the axiom is much weaker than $\mathbf{OCA}_{[\text{T}]}$ (see [10]). Both questions are also interesting if they are supplemented with \mathbf{MA}_{\aleph_1} , an axiom often used in conjunction with these axioms.

Any complete resolution to these questions would be of great interest. If either question has a positive answer (particularly in the case of Question 6.1) this would seem to require a new method for proving that 2^{\aleph_0} is \aleph_2 . On the other hand, if both questions have a negative answer, this would give two Ramsey theoretic statements which are individually consistent with $2^{\aleph_0} > \aleph_2$ but which jointly imply $2^{\aleph_0} = \aleph_2$. Situations of this kind (and lack thereof) have recently received a great deal of attention — see [13].

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