PROPER FORCING, CARDINAL ARITHMETIC, AND UNCOUNTABLE LINEAR ORDERS

JUSTIN TATCH MOORE

Abstract. In this paper I will communicate some new consequences of the Proper Forcing Axiom. First, the Bounded Proper Forcing Axiom implies that there is a well ordering of $\mathbb{R}$ which is Σ_1-definable in $(H(\omega_2), \in)$. Second, the Proper Forcing Axiom implies that the class of uncountable linear orders has a five element basis. The elements are X, ω_1, ω_1^*, C, C^* where X is any suborder of the reals of size ω_1 and C is any Countryman line. Third, the Proper Forcing Axiom implies the Singular Cardinals Hypothesis at κ unless stationary subsets of $S^\omega_{\kappa^+}$ reflect. The techniques are expected to be applicable to other open problems concerning the theory of $H(\omega_2)$.

1. Introduction

The purpose of this note is to communicate the following results.

Theorem 1.1. [11] *(*BPFA*) There is a well ordering of* $\mathcal{P}(\omega_1)/\mathrm{NS}$ *(and hence of* $\mathbb{R}$*) which is* Σ_1*-definable in* $(H(\omega_2), \in)$*.*

Theorem 1.2. [12] *(*PFA*) Suppose that* X *is a subset of* $\mathbb{R}$ *of size* $\aleph_1$ *and* C *is a Countryman line. Then any uncountable linear order contains an isomorphic copy of one of the following five orders:* X*,* ω_1*,* ω_1^**,* C*, or* C^**.*[1]

Theorem 1.3. [13] *(*PFA*) If* $\kappa^+ < \kappa^\omega$ *for an uncountable cardinal* κ *then stationary subsets of*

$$S^\omega_{\kappa^+} = \{\alpha < \kappa^+ : \mathrm{cf}(\alpha) = \omega\}$$

reflect.

Another proof of the following result is also obtained.[2]

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[1]If L is a linear order then L^* denotes the converse of L.

[2]The first proof can be obtained by a combination of Magidor's unpublished proof that PFA implies $\square_{\kappa,\omega_1}$ fails for $\kappa > \omega_1$ and Cummings and Schimmerling's result in [6] that $\square_{\kappa,\omega}$ holds in Prikry extensions. I would like to point to thank an anonymous referee for bringing this to my attention.

Theorem 1.4. [13] *The Proper Forcing Axiom fails in any generic extension by Prikry forcing.*

The first of these theorems answers a question from the folklore which is made explicit in [25]. The second confirms a well known conjecture of Shelah [14] (see also [15], [20], [21]). The third and fourth address an approach to proving the consistency of PFA and the negation of the Singular Cardinals Hypothesis. They strongly suggest that PFA implies SCH.

The first of these theorems resulted in the development of a new reflection principle which follows from PFA. An analysis of this new principle then gave rise to the remaining theorems.

This note is organized as follows. In the first section we will discuss the proof of the first theorem. In the second section we will introduce the notion of trace reflection for sets of ordinals and discuss how it relates to the Singular Cardinals Problem. In the third section we will look at a new reflection principle which was isolated from the proof and mention some of its basic consequences and how they relate to Theorems 1.1, 1.3, and 1.4. In the forth section we will briefly discuss the techniques involved in proving the more involved Theorem 1.2 as well as some historical discussion of the theorem. In the final section I will offer some open questions and future directions.

This paper is not intended to provide any detailed proofs of the claims it makes. The details will be in the forthcoming [12], [11], and [13]. The intention is to announce the results and to give the reader an idea of the arguments involved. The reader is referred to [8], [25], [26], and [29] for some further information on results related to forcing axioms and well orderings of the continuum. For further reading on the basis problem for uncountable linear orders see [14], [20], and [21]. See [8] and [28] for more information on the Singular Cardinals Hypothesis and forcing axioms.

2. A Σ_1-definable well ordering of $\mathbb{R}$ from BPFA

Recall the Bounded Proper Forcing Axiom of Goldstern and Shelah [9] (stated here in an equivalent formulation due to Bagaria [3]):

> BPFA**:** For every proper forcing $\mathcal{Q}$ the structure $(H(\omega_2), \in)$ is Σ_1-elementary in $V^{\mathcal{Q}}$.

In this section we will see that BPFA can be used to code reals in a way which is reminiscent of the SRP style coding methods (e.g. ϕ_{AC}, ψ_{AC} of [29]).

Theorem 2.1. [11] *(*BPFA*) There is a well ordering of* $\mathcal{P}(\omega_1)/\mathrm{NS}$ *which is* Σ_1*-definable in* $(H(\omega_2), \in)$.

In order to discuss the idea behind the proof, we first need to introduce some notation. Fix a sequence $\langle C_\xi : \xi \in \lim(\omega_1)\rangle$ such that C_ξ is a cofinal subset of ξ of ordertype ω for each limit $\xi < \omega_1$. Let N, M be countable sets of ordinals such that $N \subseteq M$, $\mathrm{otp}(M)$ is a limit, and $\sup(N) < \sup(M)$. Define

$$w(N, M) = |\sup(N) \cap \pi_M^{-1}[C_\alpha]|$$

where α is the ordertype of M and $\pi_M : M \to \alpha$ is the collapsing map.

Consider the following assertion for a subset A of ω_1:

$\upsilon_{\mathrm{AC}}(A)$**:** There is an uncountable $\delta < \omega_2$ and an increasing sequence $\langle N_\xi : \xi < \omega_1\rangle$ which is club in $[\delta]^\omega$ such that for all limit $\nu < \omega_1$ there is a $\nu_0 < \nu$ such that if $\nu_0 < \xi < \nu$ then $N_\nu \cap \omega_1 \in A$ is equivalent to $w(N_\xi \cap \omega_1, N_\nu \cap \omega_1) < w(N_\xi, N_\nu)$.

The sentence υ_{AC} (which takes $\langle C_\xi : \xi \in \lim(\omega_1)\rangle$ as a parameter) is the assertion that $\upsilon_{\mathrm{AC}}(A)$ holds for all $A \subseteq \omega_1$. It is routine to show that υ_{AC} implies that there is a Σ_1-definable well order of $\mathcal{P}(\omega_1)/\mathrm{NS}$.

What is actually proved in [11] is that for any $A \subseteq \omega_1$, $\upsilon_{\mathrm{AC}}(A)$ can be forced by a proper partial order. It follows immediately that BPFA implies υ_{AC} which in turn generates the requisite well ordering. David Aspero pointed out that these techniques can also be used to force a well order of $\mathcal{P}(\omega_1)/\mathrm{NS}$ which is Σ_1-definable in $(H(\omega_2), \in)$ over any model with an inaccessible cardinal. The sentences ϕ_{AC} and ψ_{AC} are both known to have considerable large cardinal strength. While the strength of θ_{AC} of [26] is unknown, the best upper bound also large.

3. Trace reflection and singular cardinals

In [8], Foreman, Magidor, and Shelah proved the following theorem.

Theorem 3.1. [8] *(*MM*) If* $\kappa \geq \omega_2$ *is a regular cardinal and* $\mathcal{F}$ *is a collection of* $\aleph_1$ *many stationary subsets of* S^ω_κ *then there is a* $\delta < \kappa$ *such that* $\mathrm{cf}(\delta) = \omega_1$ *and* $X \cap \delta$ *is stationary for all* X *in* $\mathcal{F}$.

Since S^ω_κ can be partitioned into κ many disjoint stationary sets, it easily follows that, assuming MM, $[\kappa]^{\omega_1}$ has cofinality κ. By Silver's theorem [18] this in turn shows that Martin's Maximum implies the Singular Cardinals Hypothesis [8].

In this section, I will discuss a new notion of simultaneous reflection which can be used to prove Theorems 1.3 and 1.4. First, we need to recall Todorčević's notion of a minimal walk [22]. A C-sequence on an ordinal θ is a sequence $\vec{C} = \langle C_\alpha : \alpha < \theta\rangle$ such that $C_\alpha \subseteq \alpha$ is closed and unbounded for every limit $\alpha \in \theta$ and $C_{\alpha+1} = \{\alpha\}$. If $\Omega \subseteq \theta$, then we say that $\vec{C}$ *avoids* Ω if C_α is disjoint from Ω for limit α. Observe that if $\vec{C}$ avoids Ω, then $\Omega \cap \alpha$ is non-stationary for all limit α. Conversely if $\Omega \subseteq \theta$ is such that $\Omega \cap \alpha$ is non-stationary for all limit α, then there is a C-sequence which avoids it.

The trace function for $\vec{C}$ is defined recursively as follows:[3]

$$\mathrm{tr}(\alpha, \alpha) = \emptyset,$$

$$\mathrm{tr}(\alpha, \beta) = \mathrm{tr}(\alpha, \min(C_\beta \setminus \alpha)) \cup \{\beta\}.$$

A key property of the trace function is the following.

Proposition 3.2. *If $\delta < \beta < \theta$ and δ is a limit ordinal then there is a $\delta_0 < \delta$ such that*

$$\mathrm{tr}(\delta, \beta) \subseteq \mathrm{tr}(\alpha, \beta)$$

whenever α is in (δ_0, δ). If $\vec{C}$ avoids $\{\delta\}$ then it can be further arranged that δ is in $\mathrm{tr}(\alpha, \beta)$ *as well.*

If $\vec{C}$ is given, let $\mathcal{H}(\vec{C})$ be all $X \subseteq \theta$ such that if $Z \subseteq \theta$ is closed and unbounded then

$$X \cap \bigcup_{\alpha, \beta \in Z} \mathrm{tr}(\alpha, \beta) \neq \emptyset$$

We are now ready to define the new notion of reflection.

Definition 3.3. *An element X of $\mathcal{H}(\vec{C})$ is said to* trace reflect *at $\delta < \theta$ if $X \cap \delta$ is in $\mathcal{H}(\vec{C} \restriction \delta)$.*

The following theorem is then used to deduce Theorems 1.3 and 1.4 as corollaries.

Theorem 3.4. [13] *(*PFA*) Suppose that $\theta \geq \omega_2$ is a regular cardinal and $\langle C_\alpha : \alpha < \theta \rangle$ is a C-sequence which avoids some stationary $\Omega \subseteq S^\omega_\theta$. If $\mathcal{F}$ is a collection of $\aleph_1$ many elements of $\mathcal{H}(\vec{C})$ which are contained in Ω then there is a $\delta < \theta$ such that* $\mathrm{cf}(\delta) = \omega_1$ *and every element of $\mathcal{F}$ trace reflects at δ.*

It should be noted that while Theorem 1.3 is easily deduced from the comments made above, Theorem 1.4 requires some further argument — see [13].

[3]The convention that $\mathrm{tr}(\alpha, \alpha)$ be equal to $\emptyset$ rather than $\{\alpha\}$ is not entirely standard but has little effect on the theory of tr. Our convention is needed for Proposition 3.2.

4. Set Mapping Reflection

In this section I will define a reflection principle which was isolated from the proof in the previous section. First we will need some definitions. Recall that for an uncountable set X, $[X]^\omega$ is the collection of all countable subsets of X.

Definition 4.1. *Let X be an uncountable set and M be a countable elementary submodel of $H(\theta)$ for some regular θ such that $[X]^\omega$ is in M. A subset Σ of $[X]^\omega$ is M-*stationary *if whenever $E \subseteq [X]^\omega$ is a club in M there is an N in $E \cap \Sigma \cap M$.*

The set $[X]^\omega$ is equipped with the Ellentuck topology obtained by declaring the intervals

$$[x, N] = \{Y \in [X]^\omega : x \subseteq Y \subseteq N\}$$

to be open for all N in $[X]^\omega$ and finite $x \subseteq N$. That is, $U \subseteq [X]^\omega$ is open if whenever A is in U, there is an a in $[A]^{<\omega}$ such that $[a, A] \subseteq U$. Also, if $F : [X]^{<\omega} \to X$, then the set of all A in $[X]^\omega$ such that $F''A \subseteq A$ is closed in this topology.

For ease of reading I will also make the following definition.

Definition 4.2. *A set mapping Σ is said to be* open stationary *if, for some uncountable set X_Σ and regular cardinal θ_Σ with X_Σ in $H(\theta_\Sigma)$, it is the case that elements of the domain of Σ are elementary submodels of $H(\theta_\Sigma)$ which contain X and $\Sigma(M) \subseteq [X_\Sigma]^\omega$ is open and M-stationary for all M in the domain of Σ.*

Among the simplest examples of open stationary set mappings are the following.

Example. Let $r : \omega_1 \to \omega_1$ be regressive on the limit ordinals. If $\langle P_\xi : \xi < \omega_1 \rangle$ is a continuous $\in$-chain of elementary submodels of $H(\omega_2)$ then define $\Sigma(P_\xi) = [\{r(\delta_\xi)\}, \delta_\xi]$ where $\delta_\xi = P \cap \omega_1$.

The *Mapping Reflection Principle* asserts that any open stationary set mapping contains a copy of such an example.

> MRP**:** If Σ is an open stationary set mapping whose domain is a club then there is a continuous $\in$-chain $\langle N_\nu : \nu < \omega_1 \rangle$ in the domain of Σ such that for all limit $0 < \nu < \omega_1$ there is a $\nu_0 < \nu$ such that $N_\xi \cap X_\Sigma \in \Sigma(N_\nu)$ whenever $\xi \in (\nu_0, \nu)$.

Now consider the following examples which are motivated by the previous section.

Example. If M is a countable elementary submodel of $H(2^{\omega_1 +})$ then the sets

$$\Sigma_<(M) = \{N \in M \cap [\omega_2]^\omega : w(N \cap \omega_1, M \cap \omega_1) < w(N, M \cap \omega_2)\}$$

$$\Sigma_\geq(M) = \{N \in M \cap [\omega_2]^\omega : w(N \cap \omega_1, M \cap \omega_1) \geq w(N, M \cap \omega_2)\}$$

are open. With a little effort, they are also seen to be M-stationary.

Example. Let $\langle C_\alpha : \alpha < \theta \rangle$ be a C-sequence which avoids a stationary set $\Omega \subseteq S^\omega_\theta$. If M is a countable elementary submodel of $H(2^{\theta^+})$ which contains $\vec{C}$ and some stationary $\Omega_0 \subseteq \Omega$ then put

$$\Sigma_{\Omega_0}(M) = \{N \in M \cap [\theta]^\omega : \Omega_0 \cap \mathrm{tr}(\sup(N), \sup(M \cap \theta)) \neq \emptyset\}.$$

This mapping is open and stationary by Proposition 3.2.

It follows from the comments made in the example above that MRP implies υ_{AC} and Theorem 3.4 on trace reflection.

The connection to PFA is that the following is true.

Theorem 4.3. *The Proper Forcing Axiom implies the Mapping Reflection Principle.*

The forcing which is used is the collection of all countable successor length approximations to a reflecting sequence, ordered by extension. The open stationarity of the set mapping, together with the fact that the domain is a club, seems to be exactly what is needed to make the partial order proper.

It is worth noting that MRP and Todorčević's Strong Reflection Principle (SRP) [5] have no formal relationship to one another. Of course MRP can't imply SRP (or even many of its typical consequences such as Chang's Conjecture and the saturation of the non-stationary ideal) since MRP follows from PFA. On the other hand, it can be shown that counterexamples to MRP are typically preserved by ω-semiproper forcing.[4] Since SRP can be forced by an ω-semiproper forcing, it follows that SRP does not imply MRP.

5. A five element basis for the uncountable linear orders

In this section we will discuss the proof of Shelah's conjecture for uncountable linear orders from PFA. The following statement was conjectured to be consistent by Shelah in [14].

[4] A ♣-sequence which guesses closed unbounded sets, for instance.

Shelah's Conjecture. *If X is any set of reals of size $\aleph_1$ and C is any Countryman order then any uncountable linear order contains an isomorphic copy of one of the following five orders: X, ω_1, ω_1^*, C, or C^*.*

It was inspired in part by the following result of Baumgartner (cast here in modern language).

Theorem 5.1. [4] *(*BPFA*) Every two $\aleph_1$-dense*[5] *sets of reals are isomorphic.*

The conclusion of the theorem implies the existence of a single element basis for the uncountable separable linear orders.

An important class of orders isolated long ago and natural to consider in this context are the *Aronszajn lines* — those uncountable linear orders in which every suborder which is separable, a well order, or a converse well order, is countable. Such orders necessarily have cardinality $\aleph_1$. Analysis of these orders tends to be necessarily set theoretic in nature and even their existence (proved by Aronszajn) is non-trivial to establish.

For some time it was open whether the Aronszajn lines could consistently have a single element basis and that all uncountable linear orders could have a four element basis. Shelah refuted this in [14]. He proved the existence of the following object whose definition is due to Countryman.

Definition 5.2. *A linear order C is* Countryman *if C^2 is the union of countably many chains in the coordinate-wise partial order.*

Observe that if L is a linear order and f, f^* are order embeddings from L into C and C^* respectively than the set

$$A = \{(f(l), f^*(l)) : l \in L\}$$

intersects every chain in C^2 in a singleton and, since C is Countryman, L must be countable. It follows that any basis for the uncountable linear orders must contain both C and C^*.

Todorčević has given the following canonical example of a Countryman line.

Theorem 5.3. [22] *If $\langle e_\alpha : \alpha < \omega_1\rangle$ is a sequence of functions such that $e_\alpha : \alpha \to \omega$ is finite-1 for all $\alpha < \omega_1$ and $e_\alpha =^* e_\beta \restriction \alpha$ for all $\alpha < \beta$ then $\{e_\alpha : \alpha < \omega_1\}$ is Countryman when given the lexicographical ordering.*

Early on the following reduction was made (see the final section of [21] for a complete proof).

[5]Here a set of reals is $\aleph_1$-dense if it meets every interval in a set of size $\aleph_1$.

Theorem 5.4. [1] (BPFA) *Suppose that there is an Aronszajn tree T such that whenever $K \subseteq T$ there is an uncountable antichain $A \subseteq T$ such that $\wedge(A) = \{s \wedge t : s, t \in A\}$ is either contained in or disjoint from K. Then X, ω_1, ω_1^*, C, and C^* forms a five element basis whenever X is a subset of $\mathbb{R}$ of size $\aleph_1$ and C is a Countryman order.*

My contribution has been to prove the next result, thereby proving that Shelah's conjecture is an consequence of PFA.

Theorem 5.5. [12] (MRP + BPFA) *There is an Aronszajn tree T such that whenever $K \subseteq T$ there is an uncountable antichain $A \subseteq T$ such that $\wedge(A) = \{s \wedge t : s, t \in A\}$ is either contained in or disjoint from K.*

The analysis is rather lengthy and a multitude of definitions are required just to state the main lemmas. In short, the techniques used are a modification of Todorčević's method of building forcings with elementary submodels as side conditions (see [23], [24]). The proof has an interesting new feature, however. In the past, applications of PFA have involved defining a forcing and proving, in ZFC, that it is proper. PFA is then applied to supply the requisite generic absoluteness. The necessary objects — such as embeddings, homogeneous sets, isomorphisms, and uncountable discrete sets — then exist in V where the analysis is finished. In the proof of Theorem 1.2, however, PFA is needed *just to prove that the relevant forcing is proper.* Hence, while the conclusion of Theorem 5.5 is a conjunction of Σ_1-sentences in the language of $(H(\omega_2), \in)$, it is *possible* that BPFA does not imply Shelah's conjecture on grounds of consistency strength alone.

6. Questions and concluding remarks

Recently Larson, Veličković, and myself showed that PFA(ω_2) is sufficient to imply Shelah's conjecture, thus greatly reducing the consistency strength. With some additional work, we were able to reduce the consistency strength to the existence of a reflecting Mahlo cardinal (see [9] for a definition of reflecting cardinal). The following two questions, however, remain open.

Question 6.1. *Does* BPFA *imply Shelah's Conjecture?*

Question 6.2. *Does Shelah's Conjecture imply that ω_2 is either reflecting or Mahlo in L?*

If the proper class ordinal is Mahlo, then there are many reflecting cardinals. On the other hand, the least Mahlo cardinal is not reflecting. It is known that BPFA is equiconsistent with a reflecting cardinal

[9][25]. It is not known that Shelah's Conjecture has any large cardinal strength at all.

One question left conspicuously open in [13] is:

Question 6.3. *Does the Proper Forcing Axiom imply the Singular Cardinals Hypothesis?*

In light of the results mentioned above and a recent result of Shelah that stationary reflection in $[\theta]^\omega$ to sets of size $\aleph_1$ implies SCH [16] it seems likely that the answer is yes (see also the earlier results in [28]). It should be noted, however, that Assaf Sharon has recently shown that SCH can fail at κ (even for $\kappa = \aleph_\omega$) and yet all stationary subsets of κ^+ reflect. Hence it is not possible to supplement Theorem 3.4 with a ZFC theorem and resolve this question.

Another question I have become interested in is:

Question 6.4. *Does the Proper Forcing Axiom imply that there is a generic almost huge embedding with critical point ω_1?*

One approach is to show that PFA implies there is a completion of $\mathcal{P}(\omega_1)/\mathrm{NS}$ which preserves ω_2. Giving a negative answer to this question would be rather tricky since a Woodin cardinal is already sufficient to give such a completion (the stationary tower forcing) [29]. On the other hand, presently all of PFA's consistency strength at the level of a Mahlo or higher that I am aware of has come from the fact that PFA implies the failure of the combinatorial principle $\square(\kappa)$ at all levels [19].[6]

From an old result of Sierpiński's [17] one can readily deduce that Shelah's conjecture implies that $|\mathbb{R}| > \aleph_1$.

Question 6.5. *Does Shelah's conjecture imply $|\mathbb{R}| = \aleph_2$?*

The current methods require that the continuum is $\aleph_2$. On the other hand, it is known [2] that a one element basis for the separable linear orders is consistent with a large continuum. If this question has a positive answer, it would be a remarkable development as it would draw a rather unexpected connection between the theory of linear orders and the analysis of the Continuum Problem.

Finally, it seems plausible that the methods for proving Shelah's Conjecture from PFA may give some new insight into problems such as Fremlin's problem on perfectly normal compacta (see Question 2.1 in

[6] The only "$\square$-free" result that I am aware of is the proof that PFA implies that there are no Kurepa trees. A few people have pointed to Baumgartner's result that PFA implies that there are no ω_2-Aronszajn trees. In hind sight, at least, this is connected to $\square$-sequences since $\square(\omega_2)$ implies that there is an ω_2-Aronszajn tree.

[10]). The only use of MRP in [12] can be isolated to a single lemma. With a stronger assumption, this lemma can be stated in the following abstract form.

0-1 **Law for Open Set Mappings.** *(*SMRP*) Suppose that* Σ *is an open set mapping defined on a club and that* Σ *has the following properties:*

(1) *If* N *is in the domain of* Σ*, then* $\Sigma(N)$ *is closed under end extensions.*[7]
(2) *If* N *and* $\overline{N}$ *are in the domain of* Σ *and* $\overline{N}$ *is an end extension of* N *then* $\Sigma(N) = \Sigma(\overline{N}) \cap N$.

Then for a closed unbounded set of N *in the domain of* Σ*, there is a club* $E \subseteq [X_\Sigma]^\omega$ *in* N *such that* $E \cap N$ *is either contained in or disjoint from* $\Sigma(N)$.

Here SMRP is the Strong Mapping Reflection Principle obtained by replacing "club" in the statement of MRP with "projective stationary" (see [7]). It follows from Martin's Maximum by a routine modification of the proof that PFA implies MRP [11]. While at the present it does not seem that the answers to these problems will simply "fall out" of the new methods, the methods of [12] reduce these problems to new, unfamiliar difficulties.

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[7] Here we define $\overline{N}$ *end extends* N as meaning that $N \cap \omega_1 = \overline{N} \cap \omega_1$ and $N \subseteq \overline{N}$.

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