RANDOM REALS AND $\omega_1 \to (\omega_1, (\alpha : \alpha))^2$

J. TATCH MOORE

Abstract. The purpose of this note will be to extend the results of J. Barnett and S. Todorčević concerning the influence $\mathbf{MA}_{\aleph_1}$ has on random graphs. I will demonstrate that if $\mathbf{MA}_{\aleph_1}$ holds then $\omega_1 \to (\omega_1, (\alpha : \alpha))^2$ holds for every $\alpha < \omega_1$ after the addition of any number of random reals.

1. Introduction

The focus of this note is to extend J. Barnett's result in [2] that $\omega_1 \to (\omega_1, (\alpha : n))^2$ holds for all $\alpha < \omega_1$ and all $n < \omega$ after the addition of any number of random reals to a model of $\mathbf{MA}_{\aleph_1}$. Also, in [1] Barnett has shown that the stronger partition relation $\omega_1 \to (\omega_1, (\alpha : \omega_1))^2$ holds for every $\alpha < \omega_1$ after the addition of one random real to a model of $\mathbf{MA}_{\aleph_1}$. S. Todorcevic has shown in [9, Ch. 6] that $\omega_1 \to (\omega : \omega_1)^2_2$ always fails in the presence of a Sierpinski set (and in particular in models in which uncountably many random reals have been added). In light of these results it is reasonable to ask whether $\omega_1 \to (\omega_1, (\alpha : \alpha))^2$ holds for all $\alpha < \omega_1$ after the addition of an arbitrary number of random reals to a model of $\mathbf{MA}_{\aleph_1}$. The result of this note provides a positive answer to this question.

This note is intended to be self contained, modulo the standard partition calculus notation. This can be found in Erdös, Hajnal, Maté, and Rado's book [4] on partition calculus, which is also a good source of reading on the topic.

2. Partition Trees on Countable Ordinals

In this section I will introduce a class of objects which will prove useful in our analysis of objects in an extension of a model of $\mathbf{MA}_{\aleph_1}$ by random reals. The motivation for these definitions comes from a class of ultrafilters which were introduced and developed by A. Hajnal and R. Laver (see [3], [6]) to show that $\mathbf{MA}_{\aleph_1}$ implies $\omega_1 \to (\omega_1, (\alpha : \omega_1))^2$ for all $\alpha < \omega_1$.

Definition 2.1. A *partition tree* T (on a set of ordinals) is a collection closed countable sets of ordinals such that the following conditions hold

1. T is a tree under the order of reverse inclusion.
2. If A and B are incompatible elements of T then either $A < B$ (i.e. $\sup A < \inf B$) or $B < A$.
3. An element of T is a terminal node (i.e. has no successors) iff it is a singleton.
4. If A is an element of T which is not a terminal node, then A has infinitely many immediate successors, none of which contain $\max A$.
5. If A is a nonterminal node of T then $\max A$ is the unique limit point of the collection of immediate successors of A (i.e. the minimums of the immediate successors of A forms an ω-sequence converging to $\max A$).
6. If B, C are immediate successors of a node A in T such that $\max B < \min C$ then $\operatorname{otp} B \leq \operatorname{otp} C$.

Notation. The set of all α such that $\{\alpha\}$ is a node of T will be denoted by πT.

Notice that all partition trees are well founded (i.e. have no infinite branch) since the maximum function is a decreasing map into the ordinals along any branch of a partition tree. The definition of a partition tree is made to facilitate the definition and discussion of classes of objects, usually filters, which are associated with them.

Definition 2.2. Suppose T is a partition tree. A *filter* $\mathcal{F}$ on T is a collection of partition subtrees of T such that

1. the empty tree is not in $\mathcal{F}$,
2. if $\mathcal{F}_0 \subseteq \mathcal{F}$ is finite then $\cap \mathcal{F}_0$ is in $\mathcal{F}$, and
3. if S is in $\mathcal{F}$ and S' is a subtree of T which contains S, then S' is also in $\mathcal{F}$.

A filter $\mathcal{F}$ on T is *tree-like* if there is an association of a filter $\mathcal{F}_A$ on the successors of A for each nonterminal A in T such that S is in $\mathcal{F}$ iff for every A in S, A has $\mathcal{F}_A$-many successors in T. A tree-like filter is an ultrafilter if the filters $\mathcal{F}_A$ are all ultrafilters. A tree-like filter $\mathcal{F}$ is *uniform* if $\mathcal{F}_A$ contains the Frechet filter for every A in T.

Notice that every filter $\mathcal{F}$ on a partition tree T induces a filter $\pi\mathcal{F}$ on πT which is generated by $\{\pi S : S \in \mathcal{F}\}$.

Definition 2.3. The *ordertype* $\operatorname{otp}(T)$ of a partition tree T is defined by recursion on the rank of T. If T is just a singleton, then $\operatorname{otp}(T) = 1$. If $\{A_n\}_{n=0}^{\infty}$ is an increasing enumeration of the first level of T above the root of T then $\operatorname{otp}(T) = \sum_{n=0}^{\infty} \operatorname{otp}(T[A_n])$ (where $T[A] = \{B \in T : B \subseteq A\}$).

It is easily checked that, if T is a partition tree $\mathrm{otp}(T) = \min\{\mathrm{otp}(X) : X \in \mathcal{F}\}$ whenever $\mathcal{F}$ is a uniform tree-like filter on T, thus justifying the definition of $\mathrm{otp}(T)$. It is well known and easily verified that for every indecomposable ordinal α, there is a partition tree T such that $\pi T \subseteq \alpha$ and $\mathrm{otp}(T) = \alpha$. The following is essentially proven in [6].

Lemma 2.4 ($\mathbf{MA}_{\aleph_1}$). *If $\mathcal{F}$ is a uniform tree-like filter on a partition tree T and $\{A_\xi : \xi < \omega_1\}$ is a sequence of elements of $\pi\mathcal{F}$ then there is an uncountable $B \subseteq \omega_1$ such that $\bigcap_{\xi \in B} A_\xi$ has order type at least* $\mathrm{otp}(T)$.

The following standard absoluteness result will be very useful in proving the main theorem. The proof is included for completeness.

Lemma 2.5. *Suppose $M \subseteq N$ are models of ZFC which contain the same ordinals. If $\{A_\xi : \xi < \omega_1\} \subseteq M$ is a sequence of subsets of ω_1 and $\alpha < \omega_1$ is such that there are sets $A, B \subseteq \omega_1$ in N with $A \subseteq \bigcap_{\xi \in B} A_\xi$ and* $\mathrm{otp}(A) = \mathrm{otp}(B) = \alpha$ *then M contains a pair of such sets which have the same properties.*

Proof. Fix a bijection $e : \omega \to \alpha$. Let $(E, <)$ be the collection of all pairs (s, t) such that

1. s, t are injections from some set $\{0, \dots, n\}$ into ω_1,
2. $s(i)$ is in $A_{t(j)}$ for every $i, j \leq n$, and
3. for every $i < j \leq n$, $s(i) < s(j)$ iff $t(i) < t(j)$ iff $e(i) < e(j)$

and $<$ orders E by coordinatewise extension. Clearly an infinite branch through $(E, <)$ gives the desired object and $(E, <)$ is well founded in M iff it is in N. □

3. The Main Result

The following theorem was proven implicitly in [7] during the course of proving the main result.

Lemma 3.1 ($\mathbf{MA}_{\aleph_1}$). *If $\dot{G}$ is a name for a graph on ω_1 then either*

1. *$\dot{G}$ is forced to be countably chromatic or*
2. *for every $\varepsilon < \omega_1$ there are sets $A, B \subseteq \kappa$ and a $\delta > 0$ such that* $\mathrm{otp}(A) \geq \varepsilon$, $\mathrm{otp}(B) = \omega_1$, *$A < B$, and for every $\alpha \in A$, $\beta \in B$*
$$\mu([\![\{\alpha, \beta\} \in \dot{G}]\!]) \geq \delta.$$

The following lemma is a modification of Theorem 4 of [8] which is useful in the setting of tree-like filters.

Lemma 3.2. *Suppose that T is a partition tree, $\mathcal{R}$ is a measure algebra whose character is everywhere greater than $\#(I)$ for some set I, and*

$\mathcal{U}$ is a tree-like ultrafilter on T. If $\dot{f}_\beta : \pi T \to \mathcal{R}$ is indexed by I then there is a sequence $\tilde{f}_\beta : T \to \mathcal{R}$ indexed by I satisfying the following conditions for every $\beta \in I$:

1. *If ξ is in πT then $\tilde{f}_\beta(\{\xi\}) = \dot{f}_\beta(\xi)$.*
2. *If A is in T and A is nonterminal then $\mu_{\mathcal{U}_A}(\tilde{f}_\beta) = \mu(\tilde{f}_\beta(A))$.*
3. *For every A in T, every U in $\mathcal{U}_A$, and every finite $\Gamma \subseteq \omega_1$*

$$\sum_{B \in U} \prod_{\beta \in \Gamma} \tilde{f}_\beta(B) \geq \prod_{\beta \in \Gamma} \tilde{f}_\beta(A).$$

Proof. This is simply a recursion carried out on the rank of T, applying the following special case of Theorem 4 of [8]:

Theorem. Suppose that $\mathcal{R}$ is a measure algebra whose character is everywhere greater than ω_1, and $\mathcal{U}$ is a tree-like ultrafilter on a countable set S. If $\dot{f}_\beta : S \to \mathcal{R}$ is indexed by ω_1 then there is a sequence c_β indexed by ω_1 satisfying the following conditions:

1. $\mu_{\mathcal{U}}(\dot{f}_\beta) = \mu(c_\beta)$.
2. For every U in $\mathcal{U}$ and every finite $\Gamma \subseteq \omega_1$

$$\sum_{\alpha \in U} \prod_{\beta \in \Gamma} \dot{f}_\beta(\alpha) \geq \prod_{\beta \in \Gamma} c_\beta.$$

□

Theorem 3.3. *Suppose that $\mathbf{V}$ models $\boldsymbol{MA}_{\aleph_1}$ and $\{\dot{r}_\xi\}_{\xi<\kappa}$ is a reals which random over $\mathbf{V}$. In $\mathbf{V}[\dot{r}_\xi : \xi < \kappa]$ the partition relation $\omega_1 \to (\omega_1, (\alpha : \alpha))^2$ holds for all $\alpha < \omega_1$.*

Proof. Let $\alpha < \omega_1$ be given and assume without loss of generality that α is indecomposable. Suppose that $\dot{G}$ is a name for a graph on ω_1. By Theorem 3.1 either $\dot{G}$ is forced to be countably chromatic or else there are sets $A < B \subseteq \omega_1$ of ordertypes α and ω_1 respectively and $\varepsilon > 0$ such that for every α in A and β in B

$$\mu([\![\{\alpha, \beta\} \in \dot{G}]\!]) \geq \varepsilon.$$

Since the first case gives us the desired conclusion it may be assumed that we are in the second case.

Let $(\mathcal{R}, \mu)$ be the measure algebra for introducing the reals $\{\dot{r}_\xi : \xi < \kappa\}$. Fix a measure algebra $(\mathcal{S}, \mu)$ which contains $\mathcal{R}$ and has character everywhere greater then $\aleph_1$. Let T be a tree-like ultrafilter on A and set $\dot{f}_\beta(\alpha) = [\![\{\alpha, \beta\} \in \dot{G}]\!]$ for $\alpha \in A$ and β in B. Applying Lemma 3.2 fix a lifting $\tilde{f}_\beta : T \to \mathcal{S}$ satisfying the conclusion of the lemma. Define $\dot{S}_\beta$ by

$$[\![\check{A} \in \dot{S}_\beta]\!] = \tilde{f}_\beta(A)$$

and $\dot{\Lambda}$ by
$$[\![\beta \in \dot{\Lambda}]\!] = \tilde{f}_\beta(\mathrm{root}(T)).$$
It is easy to verify that $\check{\mathcal{U}} \cup \{\dot{S}_\beta : \beta \in \Lambda\}$ generates a tree-like filter $\tilde{\mathcal{U}}$ on T. Let $c \in \mathcal{R}^+$ force that Λ is uncountable. In $\mathbf{V}^{\mathcal{S}}$ fix a c.c.c. partial order $(\mathcal{P}, \leq)$ which forces $\mathbf{MA}_{\aleph_1}$. Applying Theorem 2.4 in the extension $\mathbf{V}^{\mathcal{S}*\mathcal{P}}$ to $\{\pi\dot{S}_\beta : \beta \in \dot{\Lambda}\}$, it is possible to find sets $\dot{A}_0 \subseteq \check{A}$ and $\dot{B}_0 \subseteq \dot{\Lambda}$ such that $\mathrm{otp}(\dot{A}_0) = \mathrm{otp}(\dot{B}_0) = \alpha$ and $\dot{A}_0 \times \dot{B}_0 \subseteq \dot{G}$. Now applying Theorem 2.5, the desired homogeneous set can be pulled back to $\mathbf{V}^{\mathcal{R}}$. $\square$

4. Questions

It might initially seem that there is no "reasonable" positive partition relation which lies between $\omega_1 \to (\omega_1, (\alpha : \alpha))^2$ and $\omega_1 \to (\omega_1, (\alpha : \omega_1))^2$. This, however, is not the case. Let $(\alpha : \beta : \gamma)$ denote the class of graphs of the form $(A \times B) \cup (B \times C) \cup (A \times C)$ where $A < B < C$ and $\mathrm{otp}(A) = \alpha$, $\mathrm{otp}(B) = \beta$, and $\mathrm{otp}(C) = \gamma$. It is clear that $\omega_1 \to (\omega_1, (\alpha : \omega_1))^2$ implies that $\omega_1 \to (\omega_1, (\alpha : \alpha : \alpha))^2$ which in turn implies that $\omega_1 \to (\omega_1, (\alpha : \alpha))^2$. This leads us to the following:

Question 1. *Does $\omega_1 \to (\omega_1, (\alpha : \alpha : \alpha))^2$ hold for any/all $\alpha \in [\omega, \omega_1)$ in an extension of a model of $\boldsymbol{MA}_{\aleph_1}$ by an arbitrary number of random reals?*

Being even more ambitious, it is also reasonable to ask the following:

Question 2. *Can $\omega_1 \to (\omega_1, \alpha)^2$ ever hold for all $\alpha < \omega_1$ after adding uncountably many random reals?*

It should be remarked here that it is unknown whether $\mathbf{MA}_{\aleph_1}$ implies even $\omega_1 \to (\omega_1, \omega^2 + 2)^2$. Therefore it is more natural to ask the later question under the assumption of **PFA**.

References

[1] J. H. Barnett. Effect of a Random Real on $\kappa \to (\kappa, (\alpha : \omega_1))^2$. Periodica Mathematica Hungarica **30** 1 (1995), pages 27-36.

[2] J. H. Barnett. Random Reals and the Relation $\omega_1 \to (\omega_1, (\alpha : n))^2$. Periodica Mathematica Hungarica **30** 3 (1995), pages 171-176.

[3] J. Baumgartner, A. Hajnal. A proof (involving Martin's Axiom) of a partition relation. Fund. Math. **78** (1973), pages 193-203.

[4] P. Erdös, A. Hajnal, A. Maté, R. Rado. *Combinatorial Set Theory: Partition Relations For Cardinals* North-Holland, Amsterdam 1984.

[5] D. H. Fremlin. Measure Algebras. In *Handbook of Boolean Algebras* J. D. Monk, ed. North-Holland, Amsterdam 1989.

[6] R. Laver. Partition Relations For Uncountable Cardinals $\leq 2^{\aleph_0}$. Volume II in *Finite and Infinite Sets*, A. Hajnal, R. Rado, Vera T. Sós eds. North-Holland pages 1029-1043.

[7] R. Laver. Random Reals and Souslin Trees Proc. Amer. Math. Soc. **100** (1987) pages 531-534.

[8] S. Todorčević. Random Set Mappings and Separability of Compacta Topology and its Applications **74** (1996) pages 265-274.

[9] S. Todorčević. Partition Problems in Topology, AMS (1989) pages 265-274.

Boise State University, Boise, Idaho 83725

E-mail address: `justin@math.boisestate.edu`