LOCALLY COMPACT, LOCALLY COUNTABLE SPACES AND RANDOM REALS

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Abstract. In this note I will present a proof that, assuming PFA, if $\mathcal{R}$ is a measure algebra then after forcing with $\mathcal{R}$ every uncountable locally compact locally countable cometrizable space contains an uncountable discrete set. The lemmas and techniques will be presented in a general form as they may be applicable to other problems.

1. Introduction

The purpose of this paper is to further the understanding of which topological and combinatorial consequences of $MA_{\aleph_1}$ (and forcing axioms in general) can hold after forcing with a measure algebra. This program began with Laver's result which states that, assuming $MA_{\aleph_1}$, all Aronszajn trees are special after forcing with any measure algebra[5]. This was used to establish the consistency of Suslin's hypothesis with the continuum having arbitrary cardinality.[1] Todorčević extended this result to show that, assuming $MA_{\aleph_1}$, after forcing with a measure algebra the conjecture (L) is true for any regular space with a countably tight compactification [11]. This paper can be considered as a continuation of [7] — which investigates the conjectures (S) and (L) for various classes of space in this context. The reader is assumed to have some familiarity with arguments involving $MA_{\aleph_1}$ and forcing axioms.[2]

In [9] Todorčević suggested that studying random forcing extensions of models of $MA_{\aleph_1}$ might yield a better understanding of perfectly normal compacta and in particular a solution to Katětov's problem. A corollary of his result in [11] is that in such forcing extensions all perfectly normal compacta are separable. Moreover he demonstrates in [9] that any counterexample providing a solution to Katětov's problem in a forcing extension of $MA_{\aleph_1}$ by a nonseparable measure algebra must have a square which is a compact S-space. Recently in [4] Todorčević and Larson showed that Katětov's problem is independent of the usual

[1]Laver provided the last case in which $\mathbb{R}$ has singular cardinality.

[2][10] — chapters 7 and 8 in particular — is a useful reference, both for the results it contains and for its bibliography.

axioms of set theory by using fragments of forcing axioms to analyze different types of generic extensions. Still, an analysis of random combinatorial objects using $MA_{\aleph_1}$ and other forcing axioms is of interest in that it may yield a better understanding of Katětov's problem and perfect compacta in general.

In [2] Gruenhage used CH to construct an example of compact space X with a hereditarily separable and hereditarily normal but nonmetric square. This and Nyikos's example under $MA_{\aleph_1}$ [2] are the only known consistent counterexamples to Katětov's problem. Gruenhage's construction closely followed a construction due to Kunen [3] of a locally countable locally compact strengthening of the topology on $\mathbb{R}$ whose closure operator differs from the metric closure by a countable set — commonly known as a a Kunen line. In fact a Kunen line appears as a subspace of the square of Gruenhage's example. Todorčević has constructed a Kunen line on any ω_1-sequence of reals using only the assumption that $\mathfrak{b} = \omega_1$ [10]. In this paper I will prove the following.

Theorem 1.1. *(*PFA*) After forcing with any measure algebra, every uncountable locally compact, locally countable cometrizable*[3] *space contains an uncountable discrete space. In particular there are no Kunen lines in such forcing extensions.*

This establishes the following consistency result which in particular suggests that the hypothesis $\mathfrak{b} = \omega_1$ which Todorčević uses to construct a Kunen line is somewhat optimal.[4]

Theorem 1.2. *It is relatively consistent that there are no Kunen lines, there is a set of reals of size $\aleph_1$ of positive measure, and the continuum is any cardinal greater than $\aleph_1$ having uncountable cofinality.*

It should be noted also that the hypotheses on the topological space in Theorem 1.1 are somewhat optimal since it is demonstrated in [7] that neither local compactness nor cometrizability can be removed if the measure algebra is nonseparable.

In addition to any interest in the theorem itself, the proof is of significance for two reasons. First, I will prove a set of ZFC results — Theorem 2.2 and Lemmas 3.4-3.7 — which govern the probabilistic behavior of random names for elements of separable metric spaces. Second, for the first time $MA_{\aleph_1}$ did not seem sufficient for our analysis

[3] Recall that a space X is *cometrizable* if there is a metric topology on the underlying set such that every point has neighborhood base of sets which are closed in the metric topology.

[4] The covering number for the Lebesgue null ideal is the only cardinal in Cichoń's diagram which is not either at least $\mathfrak{b}$ or at most $\mathrm{non}(\mathcal{N})$.

of the random graphs involved. Certainly there are consequence of PFA (such as the non-existence of Kurepa trees) which require PFA in the corresponding analysis of $V^{\mathcal{R}}$. In our case, though, $\mathrm{MA}_{\aleph_1}$ is sufficient to imply that Kunen's construction can't be carried out in V yet the stronger PFA seems necessary when proving the analogous theorem in $V^{\mathcal{R}}$.

2. Some notation and background

When considering forcing extensions obtained by adjoining sequences of random reals to a ground model, we will take the Boolean algebraic approach and view these as forcing extensions by measure algebras. Here a measure algebra $(\mathcal{R}, \mu)$ is defined to be a complete Boolean algebra $\mathcal{R}$ together with a strictly positive probability measure $\mu : \mathcal{R} \to [0,1]$. If there is no opportunity for confusion we will write $\mathcal{R}$ instead of $(\mathcal{R}, \mu)$.

The prototypical examples of measure algebras are the Haar algebras $(\mathcal{R}_\theta, \mu)$. Here μ is the product measure on the Baire subsets of 2^θ, where $2 = \{0,1\}$ is given the uniform probability measure. $\mathcal{R}_\theta$ is then obtained by taking the quotient by the μ-null sets. By a deep result of Maharam [6], these are the only homogeneous measure algebras. A measure algebra is separable if it is completely generated by a countable set.

We will need the following theorem due essentially to Laver and isolated as a theorem unto itself in [8].

Theorem 2.1. *($\mathrm{MA}_{\aleph_1}$) If $\mathcal{R}$ is a measure algebra and $\dot{G} : [\omega_1]^2 \to \mathcal{R}$ is a $\mathcal{R}$-name for a graph on ω_1 then either*

(1) There is a sequence $\mathcal{R}$-names $\dot{X}_n : \omega_1 \to \mathcal{R}$ indexed by ω such that for all $\alpha < \omega_1$

$$\bigvee_{n<\omega} \dot{X}_n(\alpha) = \mathbf{1}$$

and for all n and $\alpha, \beta < \omega_1$

$$\dot{X}_n(\alpha) \wedge \dot{X}_n(\beta) \wedge \dot{G}(\alpha,\beta) = \mathbf{0}$$

(i.e. $\dot{G}$ is forced to be countably chromatic) or else

(2) there is a sequence F_ξ ($\xi < \omega_1$) of disjoint finite subsets of ω_1 and a $\delta > 0$ such that for all $\xi \neq \eta$

$$\bigvee_{\alpha \in F_\xi} \bigvee_{\beta \in F_\eta} \dot{G}(\alpha,\beta)$$

has measure at least δ.

The proof of this theorem is carried out explicitly in [8]. The techniques of the proof already appear in [5] and the argument can readily be extracted from section 2 of [11]. The reader is encouraged to extract a proof from the techniques used to prove Lemma 4.5 below.

We will also need the following generalization of Theorem 3 from [11]. The proof is reproduced from [8] for completeness and due to its brevity.

Theorem 2.2. *Suppose that $\mathcal{R}$ and $\mathcal{S}$ are homogeneous measure algebras such that $\mathcal{R}$ is a subalgebra of $\mathcal{S}$. If $\mathcal{S}_0$ is a subalgebra of $\mathcal{S}$ which has character less than that of $\mathcal{R}$ then there is a measure preserving homomorphism $h : \mathcal{S}_0 \to \mathcal{R}$ such that for all a in $\mathcal{S}_0$, $h(a) \leq \pi_{\mathcal{R}}(a)$.*

Remark 2.3. Here $\pi_{\mathcal{R}}(a)$ is the projection of a in the subalgebra $\mathcal{R}$ — the meet of all b in $\mathcal{R}$ which satisfy $a \leq b$. This result reduces to Todorčević's when one considers the case that $\mathcal{S}$ is an ultrapower of $\mathcal{R}$.

Proof. Let $\mathcal{R}_0$ be the image of $\mathcal{S}_0$ under the projection map. Define h_0 so that it fixes the elements of $\mathcal{R}_0$. Using a standard lemma from the proof of Maharam's theorem (see, e.g., Lemma 3.4 of [1]), extend h_0 to a measure preserving homomorphism h defined on the algebra generated by $\mathcal{S}_0 \cup \mathcal{R}_0$. Now observe that if a is in $\mathcal{S}_0$ then $h(a) \leq \pi_{\mathcal{R}}(a) = h(\pi_{\mathcal{R}(a)})$. □

3. ω_1-sequences of random elements of $[0,1]^\omega$

Before we begin with the proof of the main result, it will be useful to prove a few lemmas which concern the behavior of sequences of $\mathcal{R}$-names for elements of a separable metric space $(\dot{X}, \dot{d})$ where $(\mathcal{R}, \mu)$ is a measure algebra. First recall that every separable metric space is homeomorphic to a subspace of $([0,1]^\omega, d)$ where d is a metric compatible with the product topology on $[0,1]^\omega$. Hence we will concentrate on $\mathcal{R}$-names for elements of $[0,1]^\omega$.

Let $(\mathcal{S}, \mu)$ be a measure algebra, $\mathcal{R}$ be a complete subalgebra of $\mathcal{S}$, and $\dot{x}$ be an $\mathcal{S}$-name such that it is forced that $\dot{x}$ is in $[0,1]^\omega$. For concreteness we will fix the following definitions.

Definition 3.1. *$\dot{x}$ is an $\mathcal{R}$-*name *if for every i in ω and rational q*

$$[\![\dot{x}(i) < q]\!]$$

is in $\mathcal{R}$. Equivalently, there is an $\mathcal{R}$-name $\dot{y}$ such that $\mathbf{1}$ forces $\dot{x} = \dot{y}$.

Definition 3.2. *An element A of $\mathcal{S}$* decides $\dot{x}$ to be an $\mathcal{R}$-name *if for every i in ω and rational q*

$$[\![\dot{x}(i) < q]\!]$$

is the meet of A with an element of $\mathcal{R}$. Equivalently, there is a $\mathcal{R}$-name $\dot{y}$ such that A forces $\dot{x} = \dot{y}$.

Definition 3.3. *$\dot{x}$ is* forced not to be an $\mathcal{R}$-name *if there is no positive A in $\mathcal{S}$ which decides $\dot{x}$ to be an $\mathcal{R}$-name.*

The following lemma provides a useful equivalence to being forced not to be an $\mathcal{R}$-name.

Lemma 3.4. *Suppose that $\mathcal{S}$, $\mathcal{R}$, and $\dot{x}$ are as above. The following are equivalent:*

(1) For every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $\mathcal{R}$-name $\dot{y}$ for an element of $[0,1]^\omega$

$$\mu([\![d(\dot{x}, \dot{y}) \geq \delta]\!]) > 1 - \varepsilon$$

(2) $\dot{x}$ is forced not to be an $\mathcal{R}$-name.

Proof. The implication "1 implies 2" is trivial since if $\dot{y}$ is an $\mathcal{R}$-name and A is a positive element of $\mathcal{S}$ which forces $\dot{x} = \dot{y}$, no $\delta > 0$ can be found for $\varepsilon = 1 - \mu(A)$.

To see that 2 implies 1, suppose that 1 fails and, for some $\varepsilon > 0$, pick a sequence of $\mathcal{R}$-names $\dot{y}_n$ and elements B_n of $\mathcal{S}$ such that $\mu(B_n) \geq \varepsilon$ and B_n forces $d(\dot{x}, \dot{y}_n) < 2^{-n}$. Define an $\mathcal{S}$-name $\dot{E}$ for a subset of ω by $[\![k \in \dot{E}]\!] = B_k$. The condition $B = \bigwedge_{n=0}^{\infty} \bigvee_{k=n}^{\infty} B_k$ forces that $\dot{E}$ is infinite and that $\{\dot{y}_k : k \in \dot{E}\}$ converges to $\dot{x}$. It is enough to prove that for some positive $A \leq B$ and some $\mathcal{R}$-name $\dot{y}$ the condition A forces $\{\dot{y}_k : k \in \dot{E}\}$ converges to $\dot{y}$.

Form a pair of homogeneous measure algebras $\mathcal{R}^* \subseteq \mathcal{S}^*$ such that

(1) $\mathcal{S}^*$ contains $\mathcal{S}$ as a subalgebra,
(2) $\mathcal{S}^*$ is generated by $\mathcal{R}^* \cup \mathcal{S}$,
(3) $\mathcal{R}^*$ is non-separable, and
(4) for all $A \in \mathcal{S}$ and $B \in \mathcal{R}^*$, A and B are independent given $\pi_{\mathcal{R}}(A) \wedge \pi_{\mathcal{R}}(B)$.

Let $\mathcal{S}_0$ be a separable subalgebra of $\mathcal{S}^*$ generated by the sequence $\{B_n\}_{n=0}^{\infty}$. Fix a measure preserving homomorphism $h : \mathcal{S}_0 \to \mathcal{R}^*$ such that $h(B) \leq \pi_{\mathcal{R}}(B) = \pi_{\mathcal{R}^*}(B)$ for all B in $\mathcal{S}_0$. Define an $\mathcal{R}^*$-name $\dot{E}^*$ for a subset of ω by putting $[\![k \in \dot{E}^*]\!] = h(B_k)$.

Set

$$C = \bigwedge_{n=0}^{\infty} \bigvee_{k=n}^{\infty} (B_k \wedge h(B_k)) \, .$$

First note that, by independence of the events B_k and $h(B_k)$ below $\pi_{\mathcal{R}}(B_k) = \pi_{\mathcal{R}}(h(B_k))$,

$$\mu(B_k \wedge h(B_k)) = \frac{\mu(B_k) \cdot \mu(h(B_k))}{\mu(\pi_{\mathcal{R}}(B_k))} \geq \varepsilon^2 > 0$$

and hence C is positive. Similarly if $A \leq B$ is a positive element of $\mathcal{S}$ then $C \wedge A$ is positive. Furthermore C forces that $\dot{E} \cap \dot{E}^*$ is infinite and that $\{\dot{y}_n : n \in \dot{E}^*\}$ converges to $\dot{z}$ for some $\mathcal{R}^*$-name $\dot{z}$. The only way for this to happen is if there is no $A \leq B$ which forces that $\dot{x}$ is not a $\mathcal{R}$-name (if G is $\mathcal{S}^*$-generic then $V[G \cap \mathcal{R}^*] \cap V[G \cap \mathcal{S}] = V[G \cap \mathcal{R}]$). Hence there must be a positive $A \leq B$ and a $\mathcal{R}$-name $\dot{y}$ such that A forces $\{\dot{y}_n : n \in \dot{E}\}$ converges to $\dot{y}$. □

We will need the following definition.

Definition 3.5. *An increasing chain $\mathcal{R}_\alpha$ $(\alpha < \omega_1)$ of measure algebras is said to be* continuous *if for every limit ordinal α, $\mathcal{R}_\alpha$ is completely generated by $\bigcup_{\gamma<\alpha} \mathcal{R}_\gamma$.*

For the next two lemmas $\mathcal{R}_\alpha$ $(\alpha < \omega_1)$ will be an increasing continuous chain of separable measure algebras, each of which are subalgebras of $\mathcal{R}$. For each $\alpha < \omega_1$, $\dot{x}_\alpha$ is an $\mathcal{R}_\alpha$-name for an element of $[0,1]^\omega$. We will need the following two lemmas.

Lemma 3.6. *For every stationary $S \subseteq \omega_1$ and every $\varepsilon > 0$ there is a stationary set $S' \subseteq S$ such that for all α, β in S'*

$$\mu([\![d(\dot{x}_\alpha, \dot{x}_\beta) < \varepsilon]\!]) > 1 - \varepsilon.$$

Lemma 3.7. *If $S, T \subseteq \omega_1$ are stationary sets then for every $\varepsilon > 0$ there are stationary sets $S' \subseteq S$ and $T' \subseteq T$ and a $\delta > 0$ such that for all α in S' and β in T'*

$$\mu([\![d(\dot{x}_\alpha, \dot{x}_\beta) \geq \delta]\!]) > 1 - \varepsilon.$$

Proof. (of Lemma 3.6) For each $\gamma < \omega_1$ pick a countable set Z_γ of $\mathcal{R}_\gamma$-names for elements of $[0,1]^\omega$ such that if $\dot{x}$ is an $\mathcal{R}_\gamma$-name for an element of $[0,1]^\omega$ and $\varepsilon_0 > 0$ then there is a $\dot{z}$ in Z_γ such that

$$\mu([\![d(\dot{x}, \dot{z}) < \varepsilon_0]\!]) > 1 - \varepsilon_0.$$

Now let $\varepsilon > 0$ be given. Notice that if $\alpha < \omega_1$ is a limit ordinal then there is a $\gamma < \alpha$ and a $\dot{z}_\alpha$ in Z_γ with

$$\mu([\![\dot{d}(\dot{x}_\alpha, \dot{z}_\alpha) < \varepsilon/2]\!]) > 1 - \varepsilon/2.$$

Now, by applying the pressing down lemma, it is possible to find a single $\gamma < \omega_1$ and $\dot{z}$ in Z_γ such that

$$S' = \{\alpha \in S : \dot{z}_\alpha = \dot{z}\}$$

is stationary. It is now easily checked that the set S' satisfies the conclusion of the lemma. □

Proof. (of Lemma 3.7) Use Lemma 3.6 to select a decreasing sequence S_k ($k < \omega$) of stationary subsets of S such that for each $k < \omega$ and α, α' in S_k

$$\mu([\![d(\dot{x}_\alpha, \dot{x}_{\alpha'}) < 2^{-k}]\!]) > 1 - 2^{-k}.$$

Let $\dot{z}$ be the $\mathcal{R}_{\omega_1}$-name for the element of $[0,1]^\omega$ to which the sets

$$\{\dot{x}_\alpha : \alpha \in S_k\}$$

converge. Now pick a $\delta > 0$ such that

$$T' = \{\beta \in T : \mu([\![d(\dot{z}, \dot{x}_\beta) \geq 2\delta]\!]) > 1 - \varepsilon/2\}$$

is stationary. Now let $S' = S_k$ where k is large enough so that

$$2^{-k} < \min(\delta, \varepsilon/2).$$

It is now easily checked that δ, S', and T' satisfy the conclusion of the theorem. □

4. The main result

The focus of this section will be to prove the following result.

Theorem 4.1. *(*PFA*) If $(\mathcal{R}, \mu)$ is any measure algebra and $\dot{K}$ is an $\mathcal{R}$-name for an uncountable locally compact, locally countable, cometrizable space then $\dot{K}$ is forced to contain an uncountable discrete set.*

Let $(\mathcal{R}, \mu)$ and $\dot{K}$ be as in the hypothesis of the theorem. Observe that $\dot{K}$ contains a subspace of size $\aleph_1$ which is also locally compact, locally countable, and cometrizable. Hence we may assume without loss of generality that $\dot{K}$ is forced to have size $\aleph_1$. Since $\dot{K}$ is forced to be locally compact, it must be a strengthening of the metric topology. Also, since we are clearly finished if $\dot{K}$ refines a nonseparable metric topology, we will assume that $\mathbf{1}$ forces that $\dot{K}$ refines the metric topology on a subspace of $[0,1]^\omega$. Let $\dot{x}_\alpha$ ($\alpha < \omega_1$) be a sequence of $\mathcal{R}$-names which is forced to be an enumeration (without repetition) of the elements of $\dot{K}$. Our proof will break into cases depending on the nature of $\dot{X} = \{\dot{x}_\alpha : \alpha < \omega_1\}$. These are handled by Lemmas 4.2 and 4.5. Let $\dot{E}_\alpha$ be an $\mathcal{R}$-name for a countable compact subset of $\dot{X}$ such that $\dot{x}_\alpha$ is forced to be in $\dot{E}_\alpha$ and $\dot{E}_\alpha$ is a neighborhood of $\dot{x}_\alpha$ in $\dot{K}$. Fix an increasing continuous chain of complete separable subalgebras $\mathcal{R}_\alpha$ ($\alpha < \omega_1$) such that if $\gamma < \alpha$ then both $\dot{x}_\gamma$ and $\dot{E}_\gamma$ are added by $\mathcal{R}_\alpha$.

Lemma 4.2. *($\mathrm{MA}_{\aleph_1}$) If there is a stationary set of $\upsilon < \omega_1$ such that for all $\beta \geq \upsilon$ $\dot{x}_\beta$ is forced not to be an $\mathcal{R}_\upsilon$-name then $\dot{K}$ is forced to be σ-discrete.*

Remark 4.3. Notice that if $\dot{X}$ is an ω_1-sequence of random reals then it satisfies the hypothesis of this lemma.

Proof. By Theorem 2.1 it suffices to prove the following claim.

Claim 4.4. *If F_ξ ($\xi < \omega_1$) is a sequence of disjoint finite subsets of ω_1 and $\varepsilon > 0$, there is a pair $\xi < \eta$ such that if α is in F_ξ and β is in F_η then*

$$\mu([\![\dot{x}_\alpha \in \dot{E}_\beta]\!]) < \varepsilon.$$

Proof. Suppose that $\upsilon < \omega_1$ satisfies

(1) for all $\beta > \upsilon$ it is forced that $\dot{x}_\beta$ is not an $\mathcal{R}_\upsilon$-name and
(2) if $\gamma < \upsilon$ then there is a ξ such that $\gamma < F_\xi < \upsilon$ and if $\alpha \in F_\xi$ then $\dot{x}_\alpha$ is forced not to be an $\dot{\mathcal{R}}_\gamma$-name[5].

Now let $\eta < \omega_1$ be arbitrary such that $\upsilon < F_\eta$. Define $\dot{E}$ to be the $\mathcal{R}$-name for the union of all $\dot{E}_\beta$ such that β is in F_η. Let ζ_i ($i < \omega$) enumerate all ordinals $\zeta \geq \nu$ with the property that

$$[\![\dot{x}_\zeta \in \dot{E}]\!] \neq \mathbf{0}$$

(since $\mathcal{R}$ is c.c.c. and $\dot{E}$ is forced to be countable, the set of such ζ's is countable). Using 1 and Lemma 3.4, pick a sequence δ_i ($i < \omega$) such that for all $\alpha < \upsilon$

$$\mu([\![d(\dot{x}_\alpha, \dot{x}_{\zeta_i}) \geq \delta_i]\!]) > 1 - \varepsilon/2^{i+2}.$$

Define $\dot{U}$ to be $\mathcal{R}$-name for the open set consisting of all $\dot{y}$ such that for some i it is forced that $d(\dot{y}, \dot{x}_{\zeta_i}) < \delta_i$. Notice that for all $\alpha < \upsilon$

$$\mu([\![\dot{x}_\alpha \in \dot{U}]\!]) < \varepsilon/2.$$

Also, since $\dot{E} \setminus \dot{U}$ is forced to be compact and contained in $\dot{X}$, by 2 it is forced for be contained in

$$\{\dot{x}_\alpha : \alpha < \dot{\gamma}\}$$

for some $\mathcal{R}$-name $\dot{\gamma}$ for an ordinal less than υ. Now find a $\xi < \omega_1$ such that

$$\mu([\![\dot{\gamma} < F_\xi < \upsilon]\!]) > 1 - \varepsilon/2.$$

It is now easily verified that $\xi < \eta$ are as desired. □

□

[5]Here $\gamma < F_\xi < \nu$ abbreviates $\gamma < \min F_\xi$ and $\max F_\xi < \nu$.

Lemma 4.5. *(*PFA*) Suppose that for a closed unbounded set C of $\upsilon < \omega_1$ there is a $\beta_\upsilon \geq \upsilon$ and a condition B_υ in $\mathcal{R}$ which decides $\dot{x}_{\beta_\upsilon}$ to be an $\mathcal{R}_\upsilon$-name. Then there is a positive element of $\mathcal{R}$ which forces $\dot{K} \cap \{\dot{x}_{\beta_\upsilon} : \upsilon \in C\}$ to contain an uncountable discrete set.*

Proof. For simplicity we will reenumerate our set of $\dot{x}_{\beta_\upsilon}$'s, letting $\dot{y}_\upsilon = \dot{x}_{\beta_\upsilon}$. Also, set $\dot{F}_\upsilon = \dot{E}_{\beta_\upsilon}$. Let $\mathcal{P}$ be the collection of all $(\mathcal{N}, \rho, B)$ which satisfy

1. $\mathcal{N}$ is a finite $\in$-chain of countable elementary submodels of $H(\theta)$ for θ sufficiently large so that $\mathcal{R}$, C, $\langle \dot{y}_\alpha : \alpha \in C\rangle$, $\langle \dot{F}_\alpha : \alpha \in C\rangle$, and $\langle B_\alpha : \alpha \in C\rangle$ are in every member of $\mathcal{N}$.
2. ρ is a map from $\mathcal{N}$ into the rationals in $(0,1)$.
3. B is a map from $\mathcal{N}$ into $\mathcal{R}$ such that $\mu(B(N)) > \rho(N)$ and $B(N) \leq B_{N\cap\omega_1}$ for all N in $\mathcal{N}$.
4. If N is in $\mathcal{N}$ then the restriction of B to N is in N.
5. If N_1, N_2 are in $\mathcal{N}$ with $N_1 \in N_2$ then $B(N_1) \wedge B(N_2)$ forces that $\dot{y}_{N_1\cap\omega_1}$ is not a member of $\dot{F}_{N_2\cap\omega_1}$.

Define an order $\leq$ on $\mathcal{P}$ by $(\mathcal{N}_p, \rho_p, B_p) \leq (\mathcal{N}_q, \rho_q, B_q)$ iff

1. $\mathcal{N}_p$ contains $\mathcal{N}_q$,
2. the restriction of ρ_p to $\mathcal{N}_q$ is ρ_q, and
3. if N is in $\mathcal{N}_q$ then $B_p(N) \leq B_q(N)$.

We will now show that $\mathcal{P}$ is proper and in the process see that $D_\alpha = \{p \in \mathcal{P} : \alpha \in \cup\mathcal{N}_p\}$ is dense for all $\alpha < \omega_1$. This is sufficient since if G meets D_α for all $\alpha < \omega_1$, set $\mathcal{N}$ to be the union of all $\mathcal{N}_p$ for p in G and

$$\bar{B}_\nu = \bigcap\{B_p(N) : p \in G \text{ and } N \in \mathcal{N}_p \text{ and } \nu = N \cap \omega_1\}$$

for $\nu < \omega_1$ of the form $N \cap \omega_1$ for some N in $\mathcal{N}$. Each $\bar{B}_\nu$ is a positive element of $\mathcal{R}$ and if $\dot{A}$ is defined by putting $[\![\nu \in \dot{A}]\!] = \bar{B}_\nu$ then $\dot{A}$ is forced by some condition to be uncountable and satisfy $\{\dot{y}_\nu : \nu \in \dot{A}\}$ is discrete in $\dot{K}$.

Claim 4.6. $(\mathcal{P}, \leq)$ *is a proper partial order.*

Proof. Let M be an elementary submodel of some large enough $H(\lambda)$ containing all of the objects mention thus far in the proof and let $p = (\mathcal{N}_p, B_p)$ be a condition in $\mathcal{P} \cap M$. By extending p if necessary, we may assume that

$$\mu\left(\bigvee_{N\in\mathcal{N}_p} B(N)\right) < 1.$$

Set $\bar{p}$ to be $(\mathcal{N}_p \cup \{M \cap H(\theta)\}, \rho_{\bar{p}}, B_{\bar{p}})$ where $\rho_{\bar{p}} \upharpoonright M = \rho_p$, $B_{\bar{p}} \upharpoonright M = B_p$,

$$0 < \rho_{\bar{p}}(M \cap H(\theta)) \leq \frac{1 - \mu\left(\bigvee_{N \in \mathcal{N}_p} B(N)\right)}{2}$$

$$B_{\bar{p}}(M \cap H(\theta)) = \mathbf{1} - \bigvee_{N \in \mathcal{N}_p} B(N).$$

It is easily checked that $\bar{p}$ is a condition in $\mathcal{P}$. We will now see that $\bar{p}$ is $(M, \mathcal{P})$-generic. To this end, let $D \subseteq \mathcal{P}$ be a dense open set in M and let r be an extension of $\bar{p}$ which is in D. Define T_0 to be the set of all elements $s = (\mathcal{N}_s, \rho_s, B_s)$ of $\mathcal{P}$

(1) there is a condition $\bar{s}$ in D which extends s such that $|\mathcal{N}_{\bar{s}}| = |\mathcal{N}_r|$,
(2) $\mathcal{N}_r \cap M$ is an initial part of $\mathcal{N}_s$,
(3) ρ_r and ρ_s agree on $\mathcal{N}_r \cap M$, and
(4) B_r and B_s agree on the $\mathcal{N}_r \cap M$.

We will consider T_0 as a tree when ordered by end extension on all three coordinates. Notice that, by an elementarity argument, T_0 contains a subtree T such that if s is a nonterminal node of T then there are stationarily many $\nu < \omega_1$ such that for some immediate successor $\bar{s}$ of s in T

$$\nu = \max(\mathcal{N}_{\bar{s}}) \cap \omega_1.$$

The following subclaim will be key to our argument.

Subclaim 4.7. *Suppose that $S \subseteq \omega_1$ is a stationary set in M. Then there is a sequence α_n in $M \cap S$ which converges to $\upsilon = M \cap \omega_1$ such that $\dot{y}_{\alpha_n}$ is forced to converge to an element of $[0,1]^\omega \setminus \dot{X}$.*

Proof. Fix a sequence ξ_n $(n < \omega)$ which is cofinal in υ. Using Lemmas 3.6 and 3.7, build a sequence of stationary sets S_σ $(\sigma \in 2^{<\omega})$ which are elements of M and positive rationals $\delta_k > 0$ such that:

(1) $S = S_{\langle\rangle}$.
(2) If σ is an initial part of τ then $S_\sigma \supseteq S_\tau$.
(3) If σ and τ are incomparable then S_σ and S_τ are disjoint.
(4) If $|\sigma| = k$ and α, α' are in S_σ then

$$\mu([\![d(\dot{x}_\alpha, \dot{x}_{\alpha'}) < \delta_{-k-1}]\!]) > 1 - 2^{-k-1}.$$

(5) $\delta_k < 2^{-k}$.
(6) If $|\sigma| = k$ and α is in $S_{\sigma^\frown 0}$ and β is in $S_{\sigma^\frown 1}$ then

$$\mu([\![d(\dot{x}_\alpha, \dot{x}_\beta) \geq 2\delta_k]\!]) > 1 - 2^{-k-1}.$$

(7) If $|\sigma| = k$, $\min(S_\sigma) > \xi_k$.

For a given b in 2^ω, let $\alpha_k(b)$ be the least element of $S_{b\restriction k}$. Notice that for each b, it is forces that $\{\dot{y}_{\alpha_k(b)}\}_{k=0}^{\infty}$ converges and that for distinct b and b', these sequences are forced to converge to different elements of $[0,1]^\omega$. Since PFA implies that $2^{\aleph_0} > \aleph_1$, there must be a b in 2^ω such that $\{\dot{y}_{\alpha_k(b)}\}_{k=0}^{\infty}$ is forced to converge to something outside of $\dot{X}$. □

We are finished once we prove the following subclaim.

Subclaim 4.8. *If s is a nonterminal node of T which is compatible with r, then there is an immediate successor $\bar{s}$ of s in T such that $\bar{s}$ is compatible with r.*

Proof. Let q be an extension of s and r. Pick an $\varepsilon > 0$ such that for all N in $\mathcal{N}_r \setminus M$, $\mu(B_q(N)) > \rho_q(N) + \varepsilon$. Let S be the set of all ν such that for some immediate successor $\bar{s}$ of s in T

$$\nu = \max(\mathcal{N}_{\bar{s}}) \cap \omega_1.$$

By the definition of T, S is stationary. Therefore it is possible, using Subclaim 4.7, to find a sequence $\bar{s}_n$ $(n < \omega)$ of immediate successors of s in T such that

$$\max(\mathcal{N}_{\bar{s}_n}) \cap \omega_1 \to \upsilon \text{ and}$$

$$y_{\max(\mathcal{N}_{\bar{s}_n})} \to \dot{z}$$

where $\dot{z}$ is forced to be outside of $\dot{X}$. Let $\dot{F}$ be the $\mathcal{R}$-name for the union of all $\dot{F}_{N\cap\omega_1}$ for N in $\mathcal{N}_r$. Since $\dot{F}$ is forced to be compact and contained in $\dot{X}$, there is an $\mathcal{R}$-name $\dot{m}$ for an element of ω such that $\dot{y}_{\max(\mathcal{N}_{\bar{s}_{\dot{n}}})}$ is forced not to be in $\dot{F}$ for any $\dot{n}$ forced to be larger than $\dot{m}$. Now find an n such that

$$\mu([\![\check{n} > \dot{m}]\!]) > 1 - \varepsilon.$$

Finally, define $\mathcal{N}_{\bar{q}} = \mathcal{N}_{\bar{s}} \cup \mathcal{N}_r$ and $\rho_{\bar{q}} = \rho_q \cup \rho_r$. Define $B_{\bar{q}}$ piecewise. If N is in $\mathcal{N}_{\bar{s}}$, then $B_{\bar{q}}(N) = B_{\bar{s}}(N)$. If N is in $\mathcal{N}_r \setminus M$, define

$$B_{\bar{q}}(N) = B_r(N) \setminus [\![\dot{y}_{\max(\mathcal{N}_{\bar{s}_n})} \notin \dot{F}]\!].$$

Since for all N in $\mathcal{N}_r \setminus M$ the measure of $[\![\dot{y}_{\max(\mathcal{N}_{\bar{s}_n})} \notin \dot{F}]\!]$ is less than $\mu(B_r(N)) - \rho(N)$, $\bar{q}$ is a condition in $\mathcal{P}$. Then $\bar{q}$ is a condition in $\mathcal{P}$ and an extension of r and $\bar{s}_n$ as required. □

□

□

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