

## Subspaces of the Sorgenfrey Line

by

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**Abstract.** We study three problems which involve the nature of subspaces of the Sorgenfrey Line  $\mathbb{S}$ . It is shown that no integer power of an uncountable subspace of  $\mathbb{S}$  can be embedded in a smaller power of  $\mathbb{S}$ . We review the known results about the existence of uncountable  $X \subseteq \mathbb{S}$  where  $X^2$  is Lindelöf. These results about Lindelöf powers are quite set-theoretic. A descriptive characterization is given of those subspaces of  $\mathbb{S}$  which are homeomorphic to  $\mathbb{S}$ . We show that a nonempty subspace  $Z \subseteq \mathbb{S}$  is homeomorphic to  $\mathbb{S}$  if and only if  $Z$  is dense-in-itself and is both  $F_\sigma$  and  $G_\delta$  in  $\mathbb{S}$ .

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## 1. Introduction

The Sorgenfrey Line  $\mathbb{S}$  is an elementary example [S] of a topological space almost always introduced and studied in the first basic topology course. Many of the interesting properties of this space can be discussed with a minimum exposure to topology so it is interesting that there are still open questions concerning the Sorgenfrey Line. In this note we answer two questions about subspaces of  $\mathbb{S}$  and review the solution to a third.

It was shown in [BuL] that, for distinct positive integers  $n$  and  $m$ , the powers  $\mathbb{S}^n$  and  $\mathbb{S}^m$  are not homeomorphic and the question was asked whether there existed an uncountable subspace  $X \subseteq \mathbb{S}$  such that  $X^2 \approx X^3$ . We answer this question negatively by showing that if  $X \subseteq \mathbb{S}$  is uncountable and  $0 < n < m$  then  $X^m$  cannot be embedded in  $\mathbb{S}^n$ .

The question of whether there exist uncountable subspaces  $X$  of  $\mathbb{S}$  such that  $X^2$  is Lindelöf is interesting by itself but also turns out to be related to the first question. In Section 3 we summarize and add a little to the known results which are quite set-theoretic. The answers are: yes, under **CH**, there are “many” such subspaces; but, under **PFA** there are no uncountable subspaces  $X \subseteq \mathbb{S}$  with  $X^2$  Lindelöf.

It is clear that certain subspaces of  $\mathbb{S}$  are actually homeomorphic to  $\mathbb{S}$  itself. For example, the subspaces  $[0, 1)$  and  $\{0\} \cup \bigcup_{n>0} (\frac{1}{n+1}, \frac{1}{n})$  are both homeomorphic to  $\mathbb{S}$  but the subspace  $\mathbb{T}$  of irrationals is not [vD]. This last fact is not obvious and it is natural to ask whether there could be a useful characterization of those subspaces homeomorphic to  $\mathbb{S}$ . This turns out to have a nice descriptive-set-theoretic answer. In Section 4 we show that a subspace  $X \subseteq \mathbb{S}$  is homeomorphic to  $\mathbb{S}$  if and only if  $X$  is a dense-in-itself subspace which is both an  $F_\sigma$  and  $G_\delta$  set in  $\mathbb{S}$ .

As above,  $\mathbb{S}$  will be used throughout the paper to denote the Sorgenfrey Line. That is,  $\mathbb{S}$  is the set  $\mathbb{R}$  of real numbers with a base for the topology given by  $\{[a, b) : a, b \in \mathbb{R}, a < b\}$ .  $\mathbb{T}$  is the Sorgenfrey subspace of irrational numbers. We write  $X \approx Y$  to say the spaces  $X$  and  $Y$  are homeomorphic. Most of the notation is standard as can be found in [K]. In particular, we use  $\omega$  for the set of non-negative integers and  $\mathbb{N}$  for the set of positive integers. Any special notation will be defined and introduced as needed.

## 2. Uniqueness of Products

It was shown in [vDP] that no positive integer power of  $\mathbb{S}$  could be homeomorphic to a power of  $\mathbb{T}$  and it was shown in [BuL] that all powers  $\mathbb{S}^n$ ,  $n \in \mathbb{N}$ , and  $\mathbb{T}^n$ ,  $n \in \mathbb{N}$ , are topologically distinct. This certainly contrasts with the real line  $\mathbb{R}$  where all powers  $\mathbb{R}^n$ ,  $n \in \mathbb{N}$ , are topologically distinct but the powers  $\mathbb{P}^n$ ,  $n \in \mathbb{N}$ , of the irrational numbers, are all homeomorphic. Of course, the real line also has other uncountable subspaces, such as the Cantor set, with all countable powers homeomorphic. There remained a question posed in [BuL] of whether there existed any uncountable subspace  $X \subseteq \mathbb{S}$  where  $X^n \approx X^m$  whenever  $2 \leq n < m < \omega$ . (It is known [BuL], that  $X \not\approx X^2$ , for uncountable  $X \subseteq \mathbb{S}$ .)

This is not an unreasonable question as there exist uncountable subspaces  $X$  of  $\mathbb{R}$  (and of the Michael Line) such that  $X \not\approx X^2$  but  $X^n \approx X^m$  for  $2 \leq n < m < \omega$ .

The main result of this section gives a strong negative answer to the above question. That is, for uncountable  $X \subseteq \mathbb{S}$ , an integer power  $X^n$  cannot live in a smaller power of  $X$  or even of  $\mathbb{S}$ .

It will be convenient to introduce some notation. An element  $x \in \mathbb{S}^n$  is viewed as a finite sequence  $x = (x_i)_{i \in n}$ . For  $0 \leq k \leq n$ ,  $x \in \mathbb{S}^n$  and  $V \subseteq \mathbb{S}^n$  let

$$\partial_k^n(V, x) = \{y \in V : |\{i \in n : x_i \neq y_i\}| = k\}.$$

This will be used when  $V$  is a basic neighborhood of  $x$  of the form  $B_n[x, \epsilon) = \prod_{i \in n} [x_i, x_i + \epsilon)$  for  $\epsilon > 0$ . Notice that for such  $V$ ,  $\{\partial_k^n(V, x) : 0 \leq k \leq n\}$  is a partition of  $V$  such that  $\bigcup_{i=k}^n \partial_i^n(V, x)$  is open in  $\mathbb{S}^n$ . Also, for  $1 \leq k \leq n$ ,  $\partial_k^n(V, x)$  embeds in  $\mathbb{S}^k$  (and  $\partial_0^n(V, x) = \{x\}$ ).

**2.1. Theorem.** If  $X_0, X_1, \dots, X_n$  are uncountable subspaces of  $\mathbb{S}$  then  $\prod_{i=0}^n X_i$  does not embed in  $\mathbb{S}^n$ .

**Proof.** Assume for contradiction that there does exist a smallest  $n \in \mathbb{N}$  such that for some collection  $\{X_i : 0 \leq i \leq n\}$ , of uncountable subspaces of  $\mathbb{S}$ , the product  $\prod_{i=0}^n X_i$  embeds in  $\mathbb{S}^n$ . Since  $\mathbb{S}$  is hereditarily Lindelöf we assume, without loss of generality, that every nonempty open subset of each  $X_i$  is uncountable. Note that  $(\prod_{i=0}^n X_i) \times \mathbb{S}$  embeds in  $\mathbb{S}^{n+1}$  and let  $\phi : (\prod_{i=0}^n X_i) \times \mathbb{S} \rightarrow \mathbb{S}^{n+1}$  be an embedding map. Now,

$$\mathcal{E} = \left\{ \left( \prod_{i=0}^{n-1} X_i \right) \times \{(x, -x)\} : x \in X_n \right\}$$

is an uncountable relatively discrete collection of copies of  $\prod_{i=0}^{n-1} X_i$ , each of which embeds in  $\mathbb{S}^{n+1}$  by  $\phi$ . Let  $\mathcal{F} = \{F_\alpha\}_{\alpha \in \Lambda}$  be the collection of images of elements of  $\mathcal{E}$  under  $\phi$  and for each  $\alpha \in \Lambda$  pick  $x_\alpha \in F_\alpha$ . Since  $\mathcal{F}$  is also a relatively discrete collection in  $\mathbb{S}^{n+1}$  we can find, for each  $\alpha \in \Lambda$ , some  $\epsilon_\alpha > 0$  such that  $V_\alpha = B_{n+1}[x_\alpha, \epsilon_\alpha)$  is disjoint from  $F_\beta$  for all  $\beta \neq \alpha$ .

Continuing the proof of this result will take advantage of the fact that, for any  $n \in \mathbb{N}$ ,  $\mathbb{S} \times \mathbb{R}^n$  is hereditarily Lindelöf [M1]. For  $0 \leq j \leq n$ , we let  $\sigma_j$  denote the topology on the product space  $\prod_{i=0}^n Z_i$  where  $Z_j = \mathbb{S}$  and  $Z_i = \mathbb{R}$ , for  $i \neq j$ . These spaces are all hereditarily Lindelöf and homeomorphic of course but there is a need for a formal distinction. In fact, we see that, for every  $0 \leq j \leq n$ , the hereditarily Lindelöf topology  $\sigma_j$  tells us that  $(\text{int}_{\sigma_j} V_\alpha) \cap F_\alpha = \emptyset$  for all but at most countably many  $\alpha$ . So, we can find  $\beta \in \Lambda$  such that the set  $F_\beta$  is disjoint from the union  $\bigcup_{j=0}^n (\text{int}_{\sigma_j} V_\beta)$ . Observe that

$$V_\alpha \setminus \bigcup_{j=0}^n (\text{int}_{\sigma_j} V_\alpha) = \bigcup_{i=0}^{n-1} \partial_i^{n+1}(V_\alpha, x_\alpha)$$

so that we must have

$$V_\beta \cap F_\beta \subseteq \bigcup_{i=0}^{n-1} \partial_i^{n+1}(V_\beta, x_\beta).$$

Now, for this  $\beta$ , pick the largest  $k < n$  such that  $F_\beta \cap \partial_k^{n+1}(V_\beta, x_\beta) \neq \emptyset$ . Since  $F_\beta \cap \bigcup_{i=k}^{n+1} \partial_i^{n+1}(V_\beta, x_\beta) = F_\beta \cap \partial_k^{n+1}(V_\beta, x_\beta)$  is open in  $F_\beta$  we see that

$$W = \phi^{-1}[F_\beta \cap \partial_k^{n+1}(V_\beta, x_\beta)]$$

is open in  $\phi^{-1}[F_\beta]$  (one of the elements of  $\mathcal{E}$ ). Hence, there exists a collection  $\{Y_i : 0 \leq i \leq n-1\}$ , of uncountable subsets of  $\mathbb{S}$  such that a copy of  $\prod_{i=0}^{n-1} Y_i$  is contained in  $W$ . But since  $W$  embeds in  $\partial_k^{n+1}(V_\beta, x_\beta)$  (by  $\phi$ ) and  $\partial_k^{n+1}(V_\beta, x_\beta)$  embeds in  $\mathbb{S}^k$  we have that  $\prod_{i=0}^{n-1} Y_i$  embeds in  $\mathbb{S}^{n-1}$ , contradicting the minimality of  $n$ . (The case  $n = 1$  would simply say that  $Y_1$  embeds in the singleton  $\partial_0^{n+1}(V_\beta, x_\beta)$ , also a contradiction.) This completes the proof.

### 3. Lindelöf Sorgenfrey Products.

The authors of this note became interested in the existence of large subspaces  $X \subseteq \mathbb{S}$  with  $X^2$  Lindelöf while initially looking at the question finally answered in Section 2. This was because it was essentially shown in [BuL] that, for  $X \subseteq \mathbb{S}$ ,  $X^2 \approx X^3$  only if  $X^2$  is Lindelöf. While Section 2 certainly voids this reason for looking at such subspaces there is, perhaps, independent interest to consider looking at this question a little more. In this section we review some of the known results and add a bit more.

When you consider the standard reason why  $\mathbb{S}^2$  is not Lindelöf, the existence of the closed discrete anti-diagonal, it may seem strange at first to even consider asking whether there can exist uncountable  $X \subseteq \mathbb{S}$  such that  $X \times X$  is Lindelöf. But, “maybe”  $X^2$  does not have to contain any subsets like the anti-diagonal. E. Michael [M2] was the first to realize that **CH** could be used to help construct uncountable  $X \subseteq \mathbb{S}$  with  $X^2$  Lindelöf. In fact, he was able to show existence of more remarkable examples as detailed in the following theorem from [M2].

**3.1. Theorem.** (**CH**) For every  $n \in \mathbb{N}$  there exists uncountable  $X \subseteq \mathbb{S}$  such that  $X^n$  is Lindelöf but  $X^{n+1}$  is not normal.

The construction technique used in the above theorem relied on the essential fact that  $\mathbb{S}$  is a Baire space. In the  $n = 2$  case the points of  $X$  are cleverly chosen so that in the end,  $X^2$  is concentrated about the Lindelöf subspace  $(X \times \mathbb{Q}) \cup (\mathbb{Q} \times X)$ . This clearly would make  $X^2$  itself Lindelöf.

Below, we use another technique (still under **(CH)**) to find other subspaces by showing that for any uncountable  $Y \subseteq \mathbb{S}$  there exists uncountable  $X \subseteq Y$  such that  $X^n$  is Lindelöf. Of course, the non-normality condition on  $X^{n+1}$  is not possible in this case since  $Y^{n+1}$  could be hereditarily normal to begin with. Actually, it will be clear from considerations below that, under **(CH)**,  $Y^m$  is normal if and only if  $Y^m$  is Lindelöf.

Before moving on to the construction of the above mentioned Lindelöf products it will be convenient to characterize exactly when  $X^n$  is Lindelöf. For this purpose we say that

a set  $A \subseteq \mathbb{S}^n$  is a *discrete surface* if, for all distinct  $x, y \in A$ , there exist  $i, j \in n$  such that  $x_i < y_i$  and  $x_j > y_j$ . It is clear that such a set is closed and discrete in  $\mathbb{S}^n$ .

**3.2. Lemma.** If  $n \in \mathbb{N}$  and  $X \subseteq \mathbb{S}$  then  $X^n$  is Lindelöf if and only if  $X^n$  contains no uncountable discrete surfaces.

**Proof.** If  $X^n$  is Lindelöf it is trivial that  $X^n$  contains no uncountable closed discrete set and hence no uncountable discrete surface. For the converse, suppose that  $X^n$  is not Lindelöf. Since  $X^n$  is subparacompact [L2] it must contain an uncountable discrete set  $E$ . We show that  $E$  contains an uncountable discrete surface  $F$ . For every  $m \in \mathbb{N}$  let  $E_m$  denote the set of all  $x \in E$  such that  $E \cap B_n[x, \frac{1}{m}] = \{x\}$  and pick some  $k \in \mathbb{N}$  such that  $E_k$  is uncountable. Now there exists some  $y \in E_k$  such that every  $\mathbb{R}^n$  open ball about  $y$  has uncountable intersection with  $E_k$ . Let  $F = \{z \in E_k : d(y, z) < \frac{1}{2k}\}$ , where  $d$  is the usual metric on  $\mathbb{R}^n$ . For distinct  $x, z \in F$ ,  $x \notin B_n[z, \frac{1}{k}]$  and  $d(x, z) < \frac{1}{k}$  implies there exists  $i$  such that  $x_i < z_i$ . Similarly, there exists  $j$  such that  $z_j < x_j$ . This shows  $F$  is a discrete surface.

**3.3. Lemma.** If  $Y \subseteq \mathbb{S}$  is uncountable,  $n \in \mathbb{N}$ , and  $\{R_\alpha\}_{\alpha < \omega_1}$  is a collection of discrete subsets of  $\mathbb{S}^n$  then there exists uncountable  $X \subseteq Y$  such that, for all  $\alpha < \omega_1$ ,  $|X^n \cap R_\alpha| < \omega$ .

**Proof.** We choose the elements of  $X = \{x_\alpha : \alpha < \omega_1\}$  by induction. Suppose  $\beta < \omega_1$  and  $X_\beta = \{x_\alpha : \alpha < \beta\} \subseteq Y$  has been chosen. For  $a \subseteq n$  let

$$E_a = \{z \in Y^n : z_i = z_j (\forall i, j \in a) \text{ and } z_k \in X_\beta (\forall k \in n \setminus a)\}.$$

Notice that if  $\pi_i : Y^n \rightarrow Y$  denotes the  $i$ th projection mapping and  $\alpha \leq \beta$  it follows that  $\pi_i[R_\alpha \cap E_a]$  is countable. Hence we can pick

$$x_\beta \in Y \setminus \bigcup_{i \in n} \pi_i \left[ \bigcup_{\alpha \leq \beta} R_\alpha \cap \left( \bigcup \{E_a : a \subseteq n\} \right) \right].$$

Now, if  $\gamma \leq \omega_1$ , we see that  $R_\gamma \cap X^n \subseteq X_\gamma^n$ , which is countable.

**3.4. Theorem. (CH)** If  $Y \subseteq \mathbb{S}$  is uncountable then there exists uncountable  $X \subseteq Y$  such that  $X^n$  is Lindelöf.

**Proof.** Since  $\mathbb{R}^n$  is second countable, it has only  $\mathfrak{c} = \omega_1$  closed subsets. Also observe that if  $A \subseteq \mathbb{S}^n$  is a discrete surface then  $\text{cl}_{\mathbb{R}^n}(A)$  is discrete in  $\mathbb{S}^n$ . Let  $\{A_\alpha\}_{\alpha < \omega_1}$  be a listing of all discrete subsets of  $\mathbb{S}^n$  which are closed in  $\mathbb{R}^n$ . As promised by Lemma 3.3 there is an uncountable subset  $X \subseteq Y$  such that  $X^n \cap A_\alpha$  is countable for all  $\alpha \leq \omega_1$ . Since each discrete surface in  $\mathbb{S}^n$  is contained in some  $A_\alpha$  it follows that  $X^n$  does not contain an uncountable discrete surface and so, Lemma 3.2 says that  $X^n$  must be Lindelöf.

The previous discussion makes it clear that, assuming **CH**, there are many uncountable subsets  $X$  of  $\mathbb{S}$  with  $X \times X$  Lindelöf. In contrast, we now mention that under **PFA** there are no such subsets of  $\mathbb{S}$ . This was probably first noticed by S. Todorćević [T] but

also follows easily from Baumgartner’s theorem that, under **PFA**, any two  $\aleph_1$ -dense subsets of  $\mathbb{R}$  are order isomorphic [Ba2]. In fact, using the Open Coloring Axiom (**OCA**), Todorćević [T] shows that if  $X$  and  $Y$  are two uncountable sets of reals then there is a strictly increasing mapping from an uncountable subset of  $X$  into  $Y$ . So, for uncountable  $X \subseteq \mathbb{S}$ , let  $\psi$  be a strictly increasing mapping from some uncountable  $Z \subseteq X$  into  $-X$ . The set  $\{(z, -\psi(z)) : z \in Z\}$  is an uncountable discrete surface in  $X \times X$  so  $X^2$  cannot be Lindelöf. A proof using Baumgartner’s theorem (actually using the condition that “any two  $\aleph_1$ -dense subsets of  $\mathbb{R}$  are order-isomorphic”) is slightly more involved but similar. The only additional complication is showing that an uncountable subspace of  $\mathbb{S}$  must contain a subspace order isomorphic to an  $\aleph_1$ -dense subset of  $\mathbb{R}$ . For completeness, we state this result as a theorem. A proof similar to that of Todorćević, using **OCA**, can also be found in [Mo].

**3.5. Theorem.** [Ba2] (**PFA**), [T] (**OCA**) For every uncountable  $X \subseteq \mathbb{S}$ ,  $X \times X$  is not Lindelöf.

It is natural to ask whether the result in Theorem 3.5 would follow using Martin’s Axiom (**MA**) plus the negation of the Continuum Hypothesis. Because the property of  $X^2$  (for  $X \subseteq \mathbb{S}$ ) being not Lindelöf is equivalent to the existence of large decreasing partial functions from  $X$  into  $X$  (Lemma 3.2) it is possible verify that **MA**+ $\neg$ **CH** is consistent with the statement of Theorem 3.5 and is also consistent with the negation of that statement. Baumgartner shows in [Ba1] the consistency of **MA**+ $\neg$ **CH**+“any two  $\aleph_1$ -dense subsets of  $\mathbb{R}$  are order-isomorphic.” In this model, the statement from Theorem 3.5 would be true. On the other hand, Abraham and Shelah have shown in [AS] the consistency of **MA**+ $\neg$ **CH**+“there is a 2-entangled set  $Z \subseteq \mathbb{R}$  with  $|Z| = \aleph_1$ .” For our purpose it suffices to know that the 2-entangled set  $Z$  has the property that, for every pair  $A, B$  of disjoint uncountable subsets of  $Z$ , there is no uncountable, decreasing partial function from  $A$  into  $B$ . From this it follows that  $Z^2$  (as a subspace of  $\mathbb{S}^2$ ) is Lindelöf. So, in the Abraham-Shelah model it is true that  $\mathbb{S}$  contains an uncountable subspace  $Z$  with  $Z^2$  Lindelöf.

A more involved study of uncountable real order types can be found in [ARS]. This includes other models in which there are no uncountable subspaces of  $\mathbb{S}$  with a Lindelöf square.

#### 4. Copies of the Sorgenfrey Line within itself.

The characterization presented in this section was inspired by an observation of the authors that  $\mathbb{S}$  is homeomorphic to each of its nonempty closed subspaces which have no isolated points. The question arose: Is every subspace of  $\mathbb{S}$  with a scattered complement homeomorphic to  $\mathbb{S}$ ? The answer was eventually discovered to be “yes,” and with some modifications to the conditions, a characterization was found:  $G \subseteq \mathbb{S}$  is homeomorphic to  $\mathbb{S}$  if and only if  $G$  has no isolated points and is both a  $G_\delta$  and an  $F_\sigma$  subset of  $\mathbb{S}$ .

We begin with the lemmas used for the result about the dense-in-itself closed subspace. Along the way toward the proof of the main result, Theorem 4.6, other lemmas and results may have independent interest.

**4.1. Lemma.** If  $X$  and  $Y$  are subspaces of  $\mathbb{S}$  which are dense-in-themselves and  $Y$  is closed in  $\mathbb{S}$  (and has a least element if  $X$  has a least element) then there exists  $\hat{\phi} : X \rightarrow Y$ , an order monomorphism, such that  $\hat{\phi}[X]$  is dense in  $Y$ .

**Proof.** Since  $\mathbb{S}$  is hereditarily separable there exist countable dense sets  $D_X, D_Y$  in  $X, Y$  respectively, such that neither  $D_Y$  nor  $D_X$  have a least element. These two countable subsets of  $\mathbb{R}$ , having no isolated points and neither a first or last element are order isomorphic, so let  $\phi : D_X \rightarrow D_Y$  be an order isomorphism. Define  $\hat{\phi} : X \rightarrow Y$  by  $\hat{\phi}(x) = \inf\{\phi(z) : z \in D_X \wedge x < z\}$ . It can be verified that  $\hat{\phi}$  is the desired order monomorphism.

The next lemma is certainly intuitive and fairly easy to prove, but since the corresponding result is not true in the real line it is worth giving an explicit statement. The proof amounts to checking several details and is left to the reader.

**4.2. Lemma.** Let  $X, Y \subseteq \mathbb{S}$  have no isolated points. If  $\phi : X \rightarrow Y$  is an order isomorphism then  $\phi$  is also a homeomorphism.

The next proposition will be superseded by the main result but does need to be proved first. It will be used as a lemma later in this section. This begins to show how to find nontrivial subspaces of  $\mathbb{S}$  which are homeomorphic to  $\mathbb{S}$ .

**4.3. Theorem.** If  $X \subseteq \mathbb{S}$  is closed and dense-in-itself then  $X \approx \mathbb{S}$ .

**Proof.** By Lemma 4.1 there exists an order monomorphism  $\phi : X \rightarrow [0, 1)$  such that  $\phi[X]$  is dense in  $[0, 1)$  and, by Lemma 4.2,  $X \approx \phi[X]$ . Since the subspaces  $(0, 1)$  and  $[0, 1)$  of  $\mathbb{S}$  are both known to be homeomorphic to  $\mathbb{S}$  it suffices to show  $(0, 1) \subseteq \phi[X]$ . Pick  $s \in (0, 1)$ . Define  $A_s = \{x \in X : s < \phi(x)\}$  and let  $z = \inf(A_s)$ . Note that  $z \in X$  since  $X$  is closed and  $s \neq 0$ . Clearly, by definition of  $A_s$ ,  $s \leq \phi(z)$ . If  $s < \phi(z)$ , pick  $r \in \phi[X] \cap (s, \phi(z))$ . But this says that  $\phi^{-1}(r) \in A_s$  and  $\phi^{-1}(r) < z$ , contradicting our choice of  $z$ . So,  $(0, 1) \subseteq \phi[X] \subseteq [0, 1)$  and  $X \approx \mathbb{S}$ .

We can now begin to see other subspaces of  $\mathbb{S}$  homeomorphic to  $\mathbb{S}$ . Let  $C_{\mathbb{S}}$  denote the usual Cantor set  $C$  with the Sorgenfrey topology after the isolated points (right end points) have been removed. Then  $C_{\mathbb{S}} \approx \mathbb{S}$ . As mentioned in the introduction, the space  $\mathbb{T}$  of Sorgenfrey irrationals is not homeomorphic to  $\mathbb{S}$  [vD]. However, there is a subspace  $K$  of the irrationals  $\mathbb{P}$  which is homeomorphic to the Cantor set and  $K$ , of course, is closed in both  $\mathbb{P}$  and  $\mathbb{T}$ . If  $K_{\mathbb{S}}$  is the set  $K$  with the Sorgenfrey topology and isolated points removed then  $K_{\mathbb{S}} \approx \mathbb{S}$ . So, while  $\mathbb{T} \not\approx \mathbb{S}$ , we see that  $\mathbb{T}$  contains a copy of  $\mathbb{S}$ .

Perhaps it is at least worth a remark that the homeomorphism between  $X$  and  $\mathbb{S}$ , as promised in Theorem 4.3, can actually be made an order-preserving homeomorphism if  $X$  does not have a least element.

Before we further consider homeomorphic images of  $\mathbb{S}$ , however, we want to first look at the continuous images of  $\mathbb{S}$  within  $\mathbb{S}$ . As with continuous functions from  $\mathbb{R}$  to  $\mathbb{R}$ , continuous functions from  $\mathbb{S}$  to  $\mathbb{S}$  seem to have more properties than just continuity would suggest on the surface.

**4.4. Lemma.** If  $X \subseteq \mathbb{S}$  and  $f : X \rightarrow \mathbb{S}$  is continuous, then there exists a countable cover  $\mathcal{G}$  of  $X$  where, for all  $G \in \mathcal{G}$ ,  $G$  is bounded and closed in  $X$  and  $f$  restricted to  $G$  is non-decreasing.

**Proof.** For  $m \in \mathbb{N}$ , let  $\{I_{m,n}\}_{n \in \mathbb{N}}$  be a collection of clopen intervals of the form  $[z, z + \frac{1}{m})$  which partitions  $\mathbb{S}$ . Define

$$E_{m,n} = \{x \in X \cap I_{m,n} : \forall y \in X \cap I_{m,n} (x < y \Rightarrow f(x) \leq f(y))\}.$$

By construction,  $E_{m,n}$  is bounded from below and  $f$  is non-decreasing over  $E_{m,n}$ . A straightforward continuity argument can be used to show that  $E_{m,n}$  is closed in  $X$  for all  $m$  and  $n$ . Let  $\mathcal{G} = \{E_{m,n} : m, n \in \mathbb{N}\}$ . Using the continuity of  $f$  it is routine to verify that  $\mathcal{G}$  covers  $X$ .

It is worth pointing out now that if  $f : E \rightarrow \mathbb{S}$  is continuous and non-decreasing, where  $E \subseteq \mathbb{S}$  is closed and bounded from below, then  $f[E]$  is also closed in  $\mathbb{S}$ . The proof of this is rather basic and left to the reader. It is clear from this and the previous result that the continuous images of  $\mathbb{S}$  in  $\mathbb{S}$  must be  $F_\sigma$  subsets of  $\mathbb{S}$ . This provides the proof of one direction of the following theorem which characterizes the continuous images of  $\mathbb{S}$  in  $\mathbb{S}$ . Notice that this gives another way to see why  $\mathbb{S} \not\approx \mathbb{T}$  (since  $\mathbb{T}$  cannot even be a continuous image of  $\mathbb{S}$ ).

**4.5. Theorem.** If  $\emptyset \neq Z \subseteq \mathbb{S}$ , there exists a continuous onto function  $f : \mathbb{S} \rightarrow Z$  if and only if  $Z$  is a  $F_\sigma$  set in  $\mathbb{S}$ .

**Proof.** We have already seen from comments above that such a continuous image  $Z$  of  $\mathbb{S}$  is an  $F_\sigma$  set. For the converse, suppose  $Z = \bigcup_{n \in \omega} E_n$ , where each  $E_n$  is closed in  $\mathbb{S}$ . Pick  $\{I_n\}_{n \in \omega}$  to be a partition of  $\mathbb{S}$  with clopen sets. Note that it can be assumed every  $E_n$  is either dense-in-itself or a singleton. If  $E_n$  is dense-in-itself define  $f : I_n \rightarrow E_n$  to be the homeomorphism guaranteed by Theorem 4.3. If  $E_n$  is a singleton  $\{z_n\}$ , let  $f(x) = z_n$  for all  $x$  in  $I_n$ . Since  $f$  is piecewise continuous on clopen sets,  $f$  itself is continuous.

We are now ready to state and prove the main characterization of this section. We supply two different views of the copies of  $\mathbb{S}$  within  $\mathbb{S}$ . The proof will consist of the introduction of some notation, three lemmas and a final argument.

**4.6 Theorem.** For a subspace  $X$  of  $\mathbb{S}$  the following are equivalent:

- (i)  $X$  is homeomorphic to  $\mathbb{S}$ .
- (ii)  $X$  is a dense-in-itself  $F_\sigma$ -subset with every closed subspace Baire.
- (iii)  $X$  is a dense-in-itself  $F_\sigma$ -subset that is also a  $G_\delta$ -set.

Before we proceed, let us define some notation. All closures, unless otherwise noted, are taken with respect to  $\mathbb{S}$ . Let  $G \subseteq \mathbb{S}$  be a nonempty  $F_\sigma$  set with no isolated points such that every closed subset of  $G$  is Baire (we should note that if  $G \subseteq \mathbb{S}$  is homeomorphic to  $\mathbb{S}$ , then  $G$  has this property). Define the set  $\Gamma_\beta(G)$  inductively for such a set  $G$ .

Let  $\Gamma_0(G) = G$  and assume  $\Gamma_\alpha(G)$  has been defined for all  $\alpha < \beta$ .

If  $\beta$  is a limit ordinal, then  $\Gamma_\beta(G) = \bigcap_{\alpha < \beta} \Gamma_\alpha(G)$ .

If  $\beta = \alpha + 1$  then  $\Gamma_{\alpha+1}(G) = \Gamma_\alpha(G) \setminus \text{int}_{\overline{\Gamma_\alpha(G)}}(\Gamma_\alpha(G))$ .

We point out that the following conditions hold:

- (1)  $\Gamma_\alpha(G) \subseteq G$  is closed in  $G$ .
- (2) For all  $\gamma < \alpha$ ,  $\Gamma_\alpha(G) \subseteq \Gamma_\gamma(G)$  and, if  $\Gamma_\gamma(G) \neq \emptyset$  then  $\Gamma_\alpha(G) \neq \Gamma_\gamma(G)$ .



(3) There is a least ordinal  $\mu$  for which  $\Gamma_\mu(G) = \emptyset$ . Since  $\mathbb{S}$  is hereditarily Lindelöf,  $\mu$  is countable. We shall indicate  $\mu$  by  $\mu(G)$ .

Strict containment in (2) is the only verification which is non-trivial. By (1),  $\Gamma_\gamma(G)$  is closed in  $G$  and hence Baire. Since  $G$  is an  $F_\sigma$  set in  $\mathbb{S}$ ,  $\Gamma_\gamma(G)$  is as well. Since  $\Gamma_\gamma(G)$  is Baire it follows that there is a set  $E \subseteq \Gamma_\gamma(G)$  which is closed in  $\mathbb{S}$  and has nonempty interior in  $\Gamma_\gamma(G)$ . Because this set is closed in  $\mathbb{S}$ , it also has interior in  $\overline{\Gamma_\gamma(G)}$ . Thus containment in (2) is strict if  $\Gamma_\gamma(G) \neq \emptyset$ .

**4.7. Lemma.** If  $G \subseteq \mathbb{S}$  is homeomorphic to  $\mathbb{S}$  then  $G$  is a  $G_\delta$ -set in  $\mathbb{S}$ .

**Proof.** As noted above,  $G$  is an  $F_\sigma$  set such that every closed subset is Baire. Let

$$F_\alpha = \left( \overline{\Gamma_\alpha(G)} \setminus \overline{\Gamma_{\alpha+1}(G)} \right) \setminus G.$$

We want to argue that every  $F_\alpha$  is an  $F_\sigma$ -set. To see this, remember that  $\Gamma_\alpha(G)$  is closed relative to  $G$  so  $\overline{\Gamma_\alpha(G)} \setminus \Gamma_\alpha(G) \subseteq \overline{G} \setminus G$ . This says

$$\left( \overline{\Gamma_\alpha(G)} \setminus \overline{\Gamma_{\alpha+1}(G)} \right) \setminus \text{int}_{\overline{\Gamma_\alpha(G)}}(\Gamma_\alpha(G)) \subseteq \overline{G} \setminus G.$$

In fact, it follows from this that

$$F_\alpha = \left( \overline{\Gamma_\alpha(G)} \setminus \overline{\Gamma_{\alpha+1}(G)} \right) \setminus \text{int}_{\overline{\Gamma_\alpha(G)}}(\Gamma_\alpha(G)).$$

Now, it is clear that  $F_\alpha$  is closed relative to  $\overline{\Gamma_\alpha(G)} \setminus \overline{\Gamma_{\alpha+1}(G)}$ . Since  $\overline{\Gamma_\alpha(G)} \setminus \overline{\Gamma_{\alpha+1}(G)}$  is open in  $\overline{\Gamma_\alpha(G)}$ , a closed subset of  $\mathbb{S}$ , it is true that  $\overline{\Gamma_\alpha(G)} \setminus \overline{\Gamma_{\alpha+1}(G)}$  is an  $F_\sigma$ -set; thus, the relatively closed subset  $F_\alpha$  must also be an  $F_\sigma$ -set in  $\mathbb{S}$ .

To finish showing that  $G$  is a  $G_\delta$ -set in  $\mathbb{S}$  recall that  $\overline{\Gamma_0(G)} = \overline{G}$ ,  $\overline{\Gamma_{\mu(G)}(G)} = \emptyset$ , and  $\overline{\Gamma_\alpha(G)} \subseteq \overline{\Gamma_\gamma(G)}$  for all  $\gamma < \alpha \leq \mu(G)$ . Hence,

$$\overline{G} \setminus G = \bigcup_{\alpha < \mu(G)} \left( \overline{\Gamma_\alpha(G)} \setminus \overline{\Gamma_{\alpha+1}(G)} \right) \setminus G = \bigcup_{\alpha < \mu(G)} F_\alpha$$

and is therefore an  $F_\sigma$ -set in  $\mathbb{S}$  as well. Since  $\overline{G}$  is a  $G_\delta$ -set,  $G = \overline{G} \setminus \bigcup_{\alpha < \mu(G)} F_\alpha$  is also a  $G_\delta$  set.

The next two lemmas give the final machinery needed to complete the characterization of homeomorphic copies of  $\mathbb{S}$ .

**4.8. Lemma.** Let  $G = \bigcup_{n \in \omega} U_n \subseteq \mathbb{S}$  be such that  $U_n$  is open in  $G$ ,  $U_n \subseteq U_{n+1}$ , and  $U_n \approx \mathbb{S}$  for all  $n \in \omega$ . Then  $G \approx \mathbb{S}$ .

**Proof.** Let  $\{W_n\}_{n \in \omega}$  be a countable collection of non-empty disjoint clopen subsets of  $G$  such that each  $W_n$  is contained in some  $U_k$  and  $\{W_n\}_{n \in \omega}$  covers  $G$ . Note that each  $W_n$  is clopen (and contained) in some  $U_k$  and therefore homeomorphic to  $\mathbb{S}$ . Let  $\{I_k\}_{k \in \omega}$  be a countable collection of disjoint clopen intervals which cover  $\mathbb{S}$ . Then there exists a function  $\varphi : G \rightarrow \mathbb{S}$  such that for all  $n \in \omega$ ,  $\varphi$  restricted to  $W_n$  is a homeomorphism sending  $W_n$

onto  $I_n$ . It follows that, because it is a homeomorphism when restricted to each  $W_n$ ,  $\varphi$  is a homeomorphism over all of  $G$  and hence  $G \approx \mathbb{S}$ .

**4.9. Lemma.** Let  $G \subseteq \mathbb{S}$  be dense-in-itself and let  $E \subseteq \overline{G}$  be closed in  $\mathbb{S}$ . If  $G \setminus E \approx \mathbb{S}$ , then  $G \cup E \approx \mathbb{S}$ .

**Proof.** Let  $\{I_n\}_{n \in \omega}$  be a collection of disjoint clopen intervals which covers  $\overline{G} \setminus E$  such that  $G \cap I_n \neq \emptyset$  and  $E \cap I_n = \emptyset$  for all  $n$ . Since  $G \setminus E \approx \mathbb{S}$ , we also know by Theorem 4.3 that  $I_n \cap G \approx I_n \cap \overline{G}$ . As in Lemma 4.7, we can find a homeomorphism  $\varphi : G \setminus E \rightarrow \overline{G} \setminus E$  such that  $\varphi[I_n \cap G] = I_n \cap \overline{G}$ . Now let  $\hat{\varphi} : G \rightarrow \overline{G}$  be defined to be the same as  $\varphi$  on  $G \setminus E$ . On  $E$  define  $\hat{\varphi}$  to be the identity map. Verifying that  $\hat{\varphi}$  is a homeomorphism is somewhat lengthy, but straightforward, and is left to the reader.

Putting the previous lemmas together, it is now possible to finish proving the main characterization of this section, Theorem 4.6. We show (i)  $\rightarrow$  (iii)  $\rightarrow$  (ii)  $\rightarrow$  (i) for a subset  $G \subseteq \mathbb{S}$ .

To show (i)  $\rightarrow$  (iii), we see that if  $G \approx \mathbb{S}$  then clearly  $G$  is dense-in-itself. Theorem 4.5 says that  $G$  is an  $F_\sigma$  and Lemma 4.7 says that  $G$  is a  $G_\delta$  set in  $\mathbb{S}$ .

To show (iii)  $\rightarrow$  (ii) suppose  $G$  is dense-in-itself and both an  $F_\sigma$  and  $G_\delta$  in  $\mathbb{S}$ . To verify that every closed subspace  $A$  of  $G$  is a Baire space, it really suffices to show every closed subspace of  $G$  is of second category. If a closed subspace  $A$  has any isolated points then  $A$  certainly is of second category. If  $A$  is dense-in-itself and closed in the  $G_\delta$  set  $G$  then  $A$  is also a  $G_\delta$  set. Since  $\overline{A}$  is homeomorphic to  $\mathbb{S}$  by Theorem 4.3,  $\overline{A}$  is a Baire space. We have  $A$  as a dense  $G_\delta$  subset of the Baire space  $\overline{A}$ . Hence  $A$  must also be a Baire space.

To show (ii)  $\rightarrow$  (i) we really finish the proof by induction on the ordinal  $\mu(G)$  introduced earlier. If  $\mu(G) = 1$  then it is clear that  $G \approx \mathbb{S}$  since in this case  $G$  is open in  $\overline{G}$  and  $\overline{G} \approx \mathbb{S}$ . Assume that for all appropriate  $H \subseteq S$  with  $\mu(H) < \mu(G)$ ,  $H \approx \mathbb{S}$ . (“Appropriate” means dense-in-itself  $F_\sigma$  subset of  $\mathbb{S}$  with every closed subset a Baire space.)

Case 1:  $\mu(G)$  is a limit ordinal. Let  $\{\alpha_n\}_{n \in \omega}$  be a cofinal sequence in  $\mu(G)$ . Observe that if  $U_n = G \setminus \Gamma_{\alpha_n}(G)$  then because  $\Gamma_{\alpha_n}(G)$  is closed in  $G$ ,  $U_n$  is open in  $G$  for all  $n \in \omega$ . Since  $U_n$  is a dense-in-itself  $G_\delta$ ,  $F_\sigma$  subset of  $\mathbb{S}$  and  $\mu(U_n) < \mu(G)$  we know  $U_n \approx \mathbb{S}$ . Also, since  $\Gamma_{\alpha_{n+1}}(G) \subseteq \Gamma_{\alpha_n}(G)$  and  $\bigcap_{n \in \omega} \Gamma_{\alpha_n}(G) = \emptyset$ ,  $U_n \subseteq U_{n+1}$  and  $\bigcup_{n \in \omega} U_n = G$ . By Lemma 4.8,  $G \approx S$ .

Case 2:  $\mu(G) = \beta + 1$  for some ordinal  $\beta$ . Since  $\mu(G \setminus \Gamma_\beta(G)) = \beta < \mu(G)$ , we have  $G \cup \overline{\Gamma_\beta(G)} \approx \mathbb{S}$  by Lemma 4.9. Since  $\Gamma_{\beta+1}(G) = \emptyset$  we know that  $\Gamma_\beta(G) = \text{int}_{\overline{\Gamma_\beta(G)}}(\Gamma_\beta(G))$  and therefore  $G = (G \setminus \Gamma_\beta(G)) \cup \Gamma_\beta(G)$  is open relative to  $G \cup \overline{\Gamma_\beta(G)}$ . So  $G$  is an open subspace of a homeomorph of  $\mathbb{S}$  and  $G$  itself must be homeomorphic to  $\mathbb{S}$ .

That completes the proof of Theorem 4.6.

The results in this section brings up an example related to the question of how the Sorgenfrey irrationals  $\mathbb{T}$  embed in  $\mathbb{S}$ . As before, let  $C$  denote the usual Cantor set and  $C_\mathbb{S}$  the set  $C$  minus the right endpoints. Now,  $\mathbb{Q} + C = \mathbb{Q} + C_\mathbb{S}$  is a dense subspace of  $\mathbb{S}$  and a countable union of closed nowhere dense subspaces. Certainly,  $\mathbb{Q} + C$  is not a  $G_\delta$  in  $\mathbb{S}$  and cannot be homeomorphic to  $\mathbb{S}$ ; also, its complement  $Z = \mathbb{S} \setminus (\mathbb{Q} + C)$  is not homeomorphic to  $\mathbb{S}$ . But, since  $Z$ , with the Real Line topology, is homeomorphic to the space  $\mathbb{P}$  of

irrationals the question arises as to whether Sorgenfrey  $Z$  must be homeomorphic to the Sorgenfrey irrationals  $\mathbb{T}$ . Perhaps it is mildly surprising that this is not true.

**4.10. Example.** Let  $Z = \mathbb{S} \setminus (\mathbb{Q} + C)$ . Then,  $Z$ , with the Real Line subspace topology, is homeomorphic to the irrationals  $\mathbb{P}$  but  $Z$ , with the Sorgenfrey Line subspace topology, is not homeomorphic to the Sorgenfrey irrationals  $\mathbb{T}$ .

**Proof.** The first part is just a reflection of the fact that  $Z$ , in  $\mathbb{R}$ , is a dense  $G_\delta$  with a dense complement. Any such space is a zero-dimensional, nowhere compact, topologically complete, separable metric space and is therefore homeomorphic to  $\mathbb{P}$ . For the last part assume, for contradiction, that  $f : Z \rightarrow \mathbb{T}$  is a homeomorphism. By Lemma 4.4 there is a countable cover  $\mathcal{G}$  of subsets of  $Z$  such that every  $G \in \mathcal{G}$  is closed and bounded in  $Z$  and  $f$  is nondecreasing on  $G$ . Since  $Z$  is a Baire space (as a subspace of  $\mathbb{S}$ ) there must be some  $H \in \mathcal{G}$  with nonempty interior in  $Z$ . Notice that  $(\mathbb{Q} + C) \cap \overline{H}$  is uncountable. For every  $a \in (\mathbb{Q} + C) \cap \overline{H}$  pick a monotone decreasing sequence  $\{s(a, n)\}_{n \in \omega}$  in  $H$  converging to  $a$ . In each case the image sequence  $\{f(s(a, n))\}_{n \in \omega}$  is a decreasing bounded sequence in  $\mathbb{T}$  and must converge to some  $t(a) \in \mathbb{Q}$ . Clearly, distinct  $a, b \in (\mathbb{Q} + C) \cap \overline{H}$  give distinct  $t(a), t(b) \in \mathbb{Q}$ . This is impossible since  $\mathbb{Q}$  is countable.

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