WHAT MAKES THE CONTINUUM $\aleph_2$

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This article is dedicated to Hugh Woodin on the occasion of his 60th birthday.

1. Introduction

The question of how many real numbers there are traces all the way back to the beginning of set theory. It has played an important role in the development and modernization of the subject throughout the last century. Cantor, who was the first to consider this problem [23], was most interested in the question of whether or not $|\mathbb{R}| = \aleph_1$, also known as the *Continuum Hypothesis (CH)*. Of course one can more generally ask how the cardinality of the continuum relates to any particular value of the $\aleph$ function. A remarkable empirical discovery over the last 50 years has been that the assertion $|\mathbb{R}| \neq \aleph_2$ is quite rich in its consequences, while up to the present, the same cannot be said for any other $\aleph_\xi$. It is this phenomenon which will occupy our attention for the duration of this article.

The starting point and inspiration for this article is [112] and in many ways the present survey can be regarded as an expansion and update of that article. The intention of this article is *not* to give a philosophical argument in favor of $|\mathbb{R}| = \aleph_2$. Instead it will collect as many proofs of $\aleph_2 \leq 2^{\aleph_0}$ and $2^{\aleph_0} \leq \aleph_2$ from different hypotheses as possible and place them in their mathematical context. Whether this is "evidence" of anything is left to the reader, although it is worth pointing out that in some cases the hypotheses used in these proofs are not consistent with each other. The article does not consider the arguments in favor of, for instance, CH or try to weigh the "evidence" for it against the "evidence" for any other value of the continuum. The reader interested in such a philosophical discussion is referred to, for

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instance, [44], Maddy's [74] [75], Woodin's [127] [128] and Foreman's rebuttal of this argument in [46] (see also [47], [129]).

This article will begin with an examination the relationship between the Continuum Hypothesis, the Perfect Set Property, and the Open Coloring Axiom. The classical consequences of CH will be discussed in Section 3, followed by Woodin's Σ^2_1-Absoluteness Theorem in Section 4. Section 5 will present a more detailed discussion of OCA and Section 6 will outline a Ramsey-theoretic proof of $2^{\aleph_0} = \aleph_2$. Martin's Maximum and its consequences will be over-viewed in Section 7. Stationary reflection and its impact on the continuum will be described in Section 8. In Section 9, the conjectures of Chang and Rado will be introduced and related to both the Continuum Problem and stationary reflection. Woodin's $\mathbb{P}_{\max}$-extension will be briefly described in Section 10; Section 11 contains related information concerning well orderings of $\mathbb{R}$ which are definable over $H(\omega_2)$. Section 12 will discuss some more sophisticated consequences of CH which are related to iterated forcing. The Semifilter Trichotomy and the principle of Near Coherence of Filters will be over-viewed in Section 13, along with a discussion of how these principles may imply $2^{\aleph_0} = \aleph_2$. The conclusion of the article contains a long list of open problems.

One omission worth noting is Gödel's outline of a proof that $2^{\aleph_0} = \aleph_2$ in [56]. The ingredients of this proof are both diverse and unconventional: the perfect set properties of uncountable Boolean combinations of open subsets of $\mathbb{R}$, an analysis of strong measure 0 sets, and scales for $\mathbb{R}^\omega$. Gödel's argument was analyzed in detail recently in [21] and the interested reader is referred there for more information. Many of these concepts are still poorly understood in spite of the advances in modern set theory and the "proof" remains incomplete in the sense that it is not known that the hypotheses needed to carry it out are consistent (although see [21, §3] for a discussion of this point). The reference [21] also contains a number of questions which are interesting but tangential to the present discussion and thus are not included in the final section.

The notation and terminology in this article is mostly standard. If X is a set and κ is a (possibly finite) cardinal, then $[X]^\kappa$ will denote the collection of subsets of X of cardinality κ. If $f, g \in \omega^\omega$, then $f <^* g$ means that $f(n) < g(n)$ except for finitely many $n \in \omega$ (similarly one defines $f \leq^* g$). I will use ω^ω/fin to denote the structure $(\omega^\omega, <^*)$. Suppose that X is an uncountable set. A subset $C \subseteq [X]^\omega$ is *club* if it is the set of all f-closed sets for some $f : X^{<\omega} \to X$ (here $M \subseteq X$ is *f-closed* if f maps $M^{<\omega}$ into M). A subset $S \subseteq [X]^\omega$ is *stationary* if S intersects every club. Notice that $\omega_1 \subseteq [\omega_1]^\omega$ is club. In particular, the stationary subsets of $[X]^\omega$ are a generalization of the stationary subsets of ω_1.

2. The Continuum Hypothesis, the Perfect Set Property, and the Open Coloring Axiom

A common philosophical justification for CH is that we cannot *effectively demonstrate* the existence of a subset of $\mathbb{R}$ of cardinality strictly between $|\mathbb{N}|$ and $|\mathbb{R}|$. In fact the reason why we cannot effectively produce an example of such an X is that those X which are *Borel* or even *analytic* have what is known as the *perfect set property*: they are either at most countable or else contain a homeomorphic copy of the Cantor set, also known as a *perfect set.* In fact, in the presence of large cardinals, every reasonably definable subset of a Polish space has the perfect set property. Recall that $L(\mathbb{R})$ is the smallest transitive model of Zermelo-Frankel set theory which contains all of the reals and all of the ordinals.

Theorem 2.1. [100][1] *If there is a proper class of Woodin cardinals, then every set of reals in $L(\mathbb{R})$ has the perfect set property.*

Now consider the following graph-theoretic assertion for a given separable metric space X:

> $\mathrm{OCA}^*(X)$**:** If G is a graph with vertex set X and whose incidence relation is open when regarded as a symmetric subset of X^2, then G either admits a countable vertex coloring or else contains a perfect clique.

This principle first appeared in [45] and was derived from Todorcevic's *Open Coloring Axiom (OCA)* which asserts that every open graph on a separable metric space is either countably chromatic or else contains an uncountable clique. Notice that by considering the complete graph on X, $\mathrm{OCA}^*(X)$ trivially implies the perfect set property. Theorem 2.1 admits the following strengthening.

Theorem 2.2. [45] *If there is a proper class of Woodin cardinals, then* $\mathrm{OCA}^*(X)$ *is true whenever X is a separable metric space in $L(\mathbb{R})$.*

In fact both CH and OCA follow from the Axiom of Determinacy (see [63] for an introduction to AD). It is therefore quite interesting that in the presence of the Axiom of Choice, these two principles take on a quite different character from each other. CH is equivalent to the assertion that $|\mathbb{R}| = \aleph_1$ — i.e. that there is a well ordering of $\mathbb{R}$ in

[1] Actually this theorem is proved from the hypothesis of a supercompact cardinal in [100]. The refinement stated here is due to Woodin and can be found in [72].

which every proper initial segment of the well order is countable. On the other hand, we will see that not only does OCA refute CH, but also many of its typical consequences. In fact it is unknown whether OCA implies $|\mathbb{R}| = \aleph_2$.

3. Consequences of the Continuum Hypothesis

There are many uses of CH in mathematics. It is often utilized to build ultrafilters with special properties, nontrivial morphisms between structures, and pathological topological spaces. In many cases the objects constructed have properties which are, to quote Gödel [55], "highly implausible." The vast majority of these arguments involve a recursive construction of length continuum in order to build some mathematical object of interest. The hypothesis $|\mathbb{R}| = \aleph_1$ allows one to arrange that at each stage of the construction, there have only been countably many previous stages. This section collects a number of such consequences of CH. The reader is referred to [101] for additional examples.

The first example is perhaps overly simplistic, but it illustrates the basic idea at the core of the more complex constructions.

Theorem 3.1. *Assume CH. There is an uncountable subset of* $\mathbb{R}$ *in which every Lebesgue measure* 0 *subset is countable and there is an uncountable subset of* $\mathbb{R}$ *in which every first category subset is countable.*

For instance, by enumerating all Borel sets of measure 0, it is straightforward to recursively construct a set X which is uncountable but such that every measure 0 subset of X is countable.

Similarly, one can build the ultrafilter described in the next theorem of Choquet by enumerating the functions $f : \omega \to \omega$ in a sequence of length ω_1 and then building a base for the ultrafilter by recursion.

Theorem 3.2. [24] [25] *Assume CH. There is an ultrafilter* $\mathcal{U}$ *on* ω *such that if* $f : \omega \to \omega$ *then* f *is either constant or one-to-one on a set in* $\mathcal{U}$.

Ultrafilters with the property stated in this theorem are known as *selective ultrafilters* and are well studied in the literature [16] [34] [37] [66] [77].

The next result, independently proved by Christensen and Mokobodzki (who was inspired by Choquet's construction), asserts that CH implies the existence of so called *medial limits*.

Theorem 3.3. *(see* [26]*,* [78]*) Assume CH. Then there exists a universally measurable* $\mu : \mathscr{P}(\omega) \to [0, 1]$ *which is a finitely additive probability measure such that* $\mu(F) = 0$ *whenever* $F \subseteq \omega$ *is finite.*

Actually CH is a much stronger hypothesis than needed for both of these theorems; Martin's Axiom (MA) is sufficient.

The next result shows that CH can have an influence on the phenomenon of *automatic continuity* and provides a consistent answer to an old problem of Kaplansky [64].

Theorem 3.4. [30] *Assume CH. If X is an infinite compact space then there is a discontinuous homomorphism of $C(X)$ into a Banach algebra.*

Solovay and Woodin later proved that it is consistent that if X is a compact space, then every homomorphism of $C(X)$ into a Banach algebra is continuous; in fact this conclusion follows from Martin's Maximum discussed in Section 7 (see [31]). See also [39] for a general discussion of these results.

In some cases, consequences of CH can actually be shown to be equivalents of CH.

Theorem 3.5. [85] *CH is equivalent to the existence of a surjection $\gamma : \mathbb{R} \to \mathbb{R}^2$ with the property that for every t at least one of the coordinate functions γ_1 and γ_2 is differentiable at t.*

Theorem 3.6. [59] *CH is equivalent to the assertion that there is an uncountable subset of $[0,1]^\omega$ with the property that every finite dimensional subset is at most countable.*

Theorem 3.7. [101] *CH is equivalent to the existence of an uncountable subset $X \subseteq [0,1]^\omega$ with the property that if $Y \subseteq X$ is uncountable, then Y is projected onto the interval by all but finitely many of the coordinate projections.*

The following are two typical examples of how CH can be used to influence the existence of morphisms between structures.

Theorem 3.8. [36] *Assume CH. If $X \subseteq \mathbb{R}$ has cardinality continuum, then there exists a $Y \subseteq X$ of cardinality continuum such that if $f \subseteq Y \times Y$ is a partial monotone injective function, then*

$$|\{x \in \mathrm{dom}(f) : x \neq f(x)\}| < 2^{\aleph_0}.$$

In particular, CH implies that there are $|\mathscr{P}(\mathbb{R})|$-many pairwise nonisomorphic $\aleph_1$-dense suborders of $\mathbb{R}$.

Theorem 3.9. [62] *Assume CH. If $\mathscr{I}$ and $\mathscr{J}$ are two proper F_σ-ideals on ω, then $\mathscr{P}(\omega)/\mathscr{I}$ and $\mathscr{P}(\omega)/\mathscr{J}$ are isomorphic.*

These two theorems contrast each other in an interesting way. The first demonstrates that under CH, there are nonisomorphic suborders of

$\mathbb{R}$ for which there is no apparent obstruction to an isomorphism. In fact Baumgartner has shown that if CH is true then there is a c.c.c. forcing extension in which all $\aleph_1$-dense suborders of $\mathbb{R}$ are isomorphic [8]. Thus the reason for the failure of the isomorphism to exist under CH is not absolute; it simply reflects that there are "not enough" monotone functions from $\mathbb{R}$ to $\mathbb{R}$ in models of CH.

On the other hand, in the case of the Theorem 3.9 there is no obvious reason why there should be an isomorphism between the quotients $\mathscr{P}(\omega)/\mathrm{fin}$ and $\mathscr{P}(\omega)/\mathscr{I}_{1/n}$ (here $\mathscr{I}_{1/n}$ is the collection of subsets of ω for which the sum of the reciprocals is convergent). CH allows the construction of the isomorphism via a back and forth argument of length continuum — at each stage of the construction one extends a countable partial isomorphism between the two Boolean algebras. Indeed it is possible to show that there is no isomorphism which is induced by a function from ω to ω or even an isomorphism between these quotients which has a Baire-measurable lifting $\Phi : \mathscr{P}(\omega) \to \mathscr{P}(\omega)$ [60]. Moreover, starting from any model of set theory, there is a forcing extension in which $\mathscr{P}(\omega)/\mathrm{fin}$ and $\mathscr{P}(\omega)/\mathscr{I}_{1/n}$ are not isomorphic [60]. In this case, the existence of an isomorphism between these two structures can be "blamed" on there not being enough subsets of ω to *prevent* the existence of an isomorphism.

A relative of Theorem 3.9 is the following result of W. Rudin.

Theorem 3.10. [93] *Assume CH. There are $2^{2^{\aleph_0}}$ autohomeomorphisms of the Čech-Stone remainder of ω. Equivalently there are $2^{2^{\aleph_0}}$ automorphisms of $\mathscr{P}(\omega)/\mathrm{fin}$. In particular, there is an automorphism which is not induced by a function from ω to ω.*

As in the proof of Theorem 3.9, the elements of $\mathscr{P}(\omega)$ are enumerated in length ω_1 and partial automorphisms are constructed recursively. The construction allows a subset of ω_1 as a parameter, so that each choice of parameter results in a different automorphism.

Unlike in the case of Theorem 3.8, there is no general means to continue such an inductive construction following uncountable limit stages. In fact while every countable partial automorphism of $\mathscr{P}(\omega)/\mathrm{fin}$ can always be extended to an arbitrary element not in its domain, it is a theorem of ZFC that there are partial automorphisms of $\mathscr{P}(\omega)/\mathrm{fin}$ of cardinality $\aleph_1$ which cannot be extended to certain subsets of ω.

This is perhaps best understood in terms of model theory. Recall that a structure $\mathfrak{A}$ is *homogeneous* if every partial automorphism of cardinality less than the domain of $\mathfrak{A}$ can be extended to an automorphism of $\mathfrak{A}$ (note that this is considerably stronger than what is meant by homogeneity outside of model-theoretic literature). A structure $\mathfrak{A}$

is *saturated* if every partial type of cardinality less than the domain of $\mathfrak{A}$ is realized in $\mathfrak{A}$. Here a *type* is a collection of n-ary formulas in the language of $\mathfrak{A}$ which is finitely satisfiable. We now have the following theorem.

Theorem 3.11. *The following are equivalent:*

1. *The Continuum Hypothesis.*
2. $\mathscr{P}(\omega)/\mathrm{fin}$ *is a* saturated *Boolean algebra.*
3. $\mathscr{P}(\omega)/\mathrm{fin}$ *is a* homogeneous *Boolean algebra.*

The CH implication in this theorem is closely related to a reformulation of following topological result of Parovičenko.

Theorem 3.12. [88] *Assume CH. The Čech-Stone remainder of ω is the unique* 0*-dimensional compact Hausdorff space having no isolated points such that if F and G are two disjoint open F_σ-sets, then there exists a closed and open set U such that $F \subseteq U$ and $U \cap G = \emptyset$.*

What is counter-intuitive about this theorem is that it implies that many compact spaces are homeomorphic even when there is no apparent homeomorphism: For instance the Čech-Stone remainders of the infinite countable limit ordinals are all homeomorphic assuming CH (contrast this with [42]).

CH also produces a wealth of exotic examples of topological spaces. I will mention only a few; it is impossible to adequately give even a brief survey.

Theorem 3.13. *(*[53]*; see also* [106]*) If CH is true, then for each n there is a topological space X such that X^{n+1} contains an uncountable family of pairwise disjoint open sets but such that X^n does not.*

Theorem 3.14. [65] *If CH is true, then there is a nonseparable compact Hausdorff space K which does not contain an uncountable discrete subspace. This space moreover supports a Borel probability measure μ for which the following are equivalent for $Y \subseteq K$:*

- *Y is nowhere dense;*
- *Y is first category;*
- $\mu(Y) = 0$*;*
- *Y is second countable;*
- *Y is separable.*

(Similar constructions were carried out independently by Haydon [58] and Talagrand [104].)

Theorem 3.15. [43] *If CH is true, then for each $d > 0$ there is an infinite compact Hausdorff space K_d in which every infinite closed subspace has dimension d. In particular, K_d contains neither convergent sequences nor a copy of $\beta\omega$.*

4. Woodin's Σ^2_1-Absoluteness Theorem

CH can be phrased as asserting the existence of a binary relation $\triangleleft$ on $\mathscr{P}(\omega)$ such that:

- $(\mathscr{P}(\omega), \triangleleft)$ is a linear ordering;
- for each $y \in \mathbb{R}$, $\{x \in \mathscr{P}(\omega) : x \triangleleft y\}$ is countable.

In particular, CH is equivalent to a Σ^2_1-sentence — a sentence of the form $\exists X \subseteq \mathscr{P}(\omega)\phi(X)$ where ϕ involves only quantification over $\mathscr{P}(\omega)$. In fact many of the consequences of CH can be phrased as Σ^2_1-sentences.

Moreover, it was noticed that, empirically at least, CH tended to be sufficient to settle the truth of Σ^2_1-sentences which would come up in practice. Quite remarkably, there is a general result due to Woodin which explains this.

Theorem 4.1. *(see* [72]*) Assume there is a proper class of measurable Woodin cardinals and that CH is true. Any Σ^2_1-sentence which can be forced is true.*

It should be noted that some care is needed in stating this theorem. For instance the assertion that there is a nonconstructible real is trivially a Σ^2_1-sentence (in fact this assertion can be formalized using only quantification over $\mathscr{P}(\omega)$). By Cohen's work it is consistent and yet it is not true in L, which satisfies CH. Thus we cannot hope to have a result which says that if a Σ^2_1-sentence is consistent with ZFC, then it is a consequence of CH.

5. Todorcevic's Open Coloring Axiom

We will now return to the Open Coloring Axiom introduced in Section 2 above. This axiom has a strong influence on structures closely related to separable metric spaces and often helps provide an alternative mathematical theory to that imposed by CH. This axiom's name was based on a family of similar Ramsey-theoretic principles in Abraham, Rubin, and Shelah's [2]. The following are examples of open graphs.

Example 5.1. Suppose that $A, B \subseteq \mathbb{R}$ and define a graph on $A \times B$ by connecting (x_0, y_0) to (x_1, y_1) whenever $x_0 < x_1$ and $y_0 < y_1$ or $x_1 < x_0$ and $y_1 < y_0$. Observe that this graph is open. It is countably chromatic only when either A or B is countable. On the other hand, a clique is clearly the graph of a partial increasing function from A to B. Thus OCA implies that if $A, B \subseteq \mathbb{R}$ are uncountable, then they have uncountable isomorphic suborders. In particular, by Theorem 3.8, OCA refutes CH.

The next examples are more typical of the use of OCA in applications. They provided Todorcevic's motivation isolating his formulation of OCA from the similar forms initially studied in [2].

Example 5.2. Suppose that $A \subseteq \mathscr{P}(\omega)$ and define a graph on A where $\{a, b\}$ is an edge if neither $a \subseteq b$ nor $b \subseteq a$. In particular, OCA implies that if A is uncountable, then either A contains an uncountable chain or antichain with respect to containment (here *antichain* means pairwise incomparable). This conclusion was originally deduced by Baumgartner from PFA [10].

Example 5.3. Let $X \subseteq \omega^\omega \times \omega^\omega$ consist of all pairs (a, b) for which $a \leq b$ coordinate-wise. Define $\{(a_0, b_0), (a_1, b_1)\}$ to be an edge of the graph if there is an i such that

$$\max(a_0(i), a_1(i)) > \min(b_0(i), b_1(i))$$

Typically one considers induced subgraphs of this graph as follows. Suppose that $A, B \subseteq \omega^\omega$ have the property that $a \leq^* b$ whenever $a \in A$ and $b \in B$ and that A and B are closed under making finite modifications to their elements. Consider the subgraph in which the vertices come from $A \times B$. This graph is countably chromatic exactly when there exist sets c_n $(n \in \omega)$ such that for every $a \in A$ and $b \in B$, there is an n such that $a \leq^* c_n \leq^* b$.

This second example readily gives the following classification of *gaps* in ω^ω/fin in the presence of OCA. Here we recall that a (κ, λ^*)-gap in ω^ω/fin is a pair of sequences a_ξ $(\xi \in \kappa)$ and b_η $(\eta \in \lambda)$ such that $a_\xi <^* a_{\xi'} <^* b_{\eta'} <^* b_\eta$ for all $\xi < \xi' < \kappa$ and $\eta < \eta' < \lambda$ and yet there is no c such that $a_\xi <^* c <^* b_\eta$ for all ξ and η.

Theorem 5.4. [108] *Assume OCA. If κ and λ are regular cardinals and there is a (κ, λ^*)-gap, then either* $\min(\kappa, \lambda) = \omega$ *or else* $\kappa = \lambda = \omega_1$.

This complements the following classical results of Hausdorff.

Theorem 5.5. [57] *There is an (ω_1, ω_1^*)-gap in ω^ω/fin.*

Theorem 5.6. [57] *The following are equivalent for a regular cardinal κ:*

- *There is an unbounded chain in ω^ω/fin of ordertype κ.*

- *There is a* (κ, ω^*)*-gap in* ω^ω/fin.
- *There is a* (ω, κ^*)*-gap in* ω^ω/fin.

In particular, there exists a regular cardinal κ *for which there exist both* (κ, ω^*)*-gaps and* (ω, κ^*)*-gaps.*

We will finish this section by mentioning some consequences of OCA and MA concerning quotient algebras which contrast the influence of CH detailed in Section 3. Building on work of Just, Shelah, Steprāns, and Veličković, Farah proved the following general results.

Theorem 5.7. [41] *Assume OCA and* MA*. If* $\mathscr{I}$ *and* $\mathscr{J}$ *are analytic ideals and at least one is either a* P*-ideal or countably generated, then every isomorphism between* $\mathscr{P}(\omega)/\mathscr{I}$ *and* $\mathscr{P}(\omega)/\mathscr{J}$ *has a Baire measurable lifting.*

Theorem 5.8. [41] *If* $\mathscr{I}$ *and* $\mathscr{J}$ *are analytic ideals, at least one is either a nonpathological* P*-ideal or countably generated, and* $\mathscr{P}(\omega)/\mathscr{I}$ *and* $\mathscr{P}(\omega)/\mathscr{J}$ *are isomorphic via a map admitting a Baire measurable lifting, then* $\mathscr{I}$ *and* $\mathscr{J}$ *are isomorphic.*

In particular the conjunction of OCA and MA implies that $\mathscr{P}(\omega)/\text{fin}$ and $\mathscr{P}(\omega)/\mathscr{I}_{1/n}$ are nonisomorphic, in contrast to Theorem 3.9. (This instance of Theorems 5.7 and 5.8 was first proved by Just [60] [61].) Farah later adapted these techniques to the context of C*-algebras [40], complementing a previous CH construction of Phillips and Weaver [89].

Recall that the *measure algebra* is the quotient of the algebra of Borel subsets of $[0, 1]$ by the ideal of Lebesgue measure 0 sets. In the presence of CH, $\mathscr{P}(\omega)/\text{fin}$ is saturated and hence any Boolean algebra of cardinality at most $2^{\aleph_0}$ can be embedded inside it. On the other hand, we have the following result of Dow and Hart.

Theorem 5.9. [35] *Assume OCA. The measure algebra does not embed into* $\mathscr{P}(\omega)/\text{fin}$.

6. A Ramsey-theoretic proof that the continuum is $\aleph_2$

The only precursor of OCA considered in [2] which is not a formal weakening of OCA is $\text{OCA}_{[\text{ARS}]}$:

> If X is a separable metric space of cardinality $\aleph_1$ and $c : [X]^2 \to \{0, 1\}$ is a continuous function, then X can be decomposed into countably many c-homogeneous sets.

Both OCA and $\text{OCA}_{[\text{ARS}]}$ are consequences of Martin's Maximum (discussed in Section 7 below) and both refute CH. Formally they are unrelated — it is consistent that either hold but not the other. On one hand $\text{OCA}_{[\text{ARS}]}$ only makes assertions about graphs which are closed and open and in which the underlying vertex set has cardinality $\aleph_1$. On the other hand, unlike with OCA, the conclusion of $\mathrm{OCA}_{[\mathrm{ARS}]}$ is "global."

While it remains unknown whether either of these assertions decides the value of the continuum, we have the following result — a purely Ramsey-theoretic proof that the continuum is $\aleph_2$.

Theorem 6.1. [81] *The conjunction of OCA and* $\mathrm{OCA}_{[\mathrm{ARS}]}$ *implies that* $|\mathbb{R}| = \aleph_2$.

We have already seen that OCA implies that all gaps are of one of the following forms: (κ, ω^*), (ω, κ^*), or (ω_1, ω_1^*). In fact this conclusion already implies that $\mathfrak{b} \leq \aleph_2$, where $\mathfrak{b}$ is the minimum cardinality of an unbounded subset of $\omega^\omega/\mathrm{fin}$. To see this, suppose that $\mathfrak{b} > \aleph_2$ and fix a bounded chain a_ξ $(\xi < \omega_2)$ in $\omega^\omega/\mathrm{fin}$. It is routine to show that this is the lower part of an (ω_2, λ^*)-gap for some λ. By Theorem 5.6, $\lambda > \omega$ and thus by Theorem 5.4, OCA fails.

It is also the case that OCA implies that $\mathfrak{b} > \aleph_1$ and thus that $\mathfrak{b} = \aleph_2$. This is proved via the following ZFC result.

Proposition 6.2. [117] *Suppose that* $X \subseteq \omega^\omega/\mathrm{fin}$ *consists of monotone increasing functions, is unbounded and countably directed. For every* n*, there exists* $x, y \in X$ *such that* $\mathrm{osc}(x, y) = n$ *(in particular, there exist* $x \neq y \in X$ *with* $x \leq y$*).*

Here $\mathrm{osc}(x, y) < n$ exactly when ω can be covered by n intervals I such that $x \leq y$ on I or $y \leq x$ on I. Notice that if $\mathfrak{b} = \aleph_1$, then there is an unbounded chain X in $\omega^\omega/\mathrm{fin}$ of length ω_1 which moreover consists of monotone functions. If G is the positive oscillation graph on X, then G is open and is neither countably chromatic nor has an uncountable clique.

Corollary 6.3. [108] *OCA implies that* $\mathfrak{b} = \aleph_2$.

The remainder of the proof of Theorem 6.1 hinges on the following variation on Proposition 6.2.

Proposition 6.4. [81] *There is a continuous partial function* $\sigma : [\omega^\omega]^2 \to 2^{<\omega}$ *with an open domain such that:*

- *if* $x <^* y$*, then* $\sigma(x, y)$ *is defined and has the same length as the common initial part of* x *and* y*;*
- *if* X *is unbounded and countably directed in* $\omega^\omega/\mathrm{fin}$ *and* r *is in* 2^ω*, then there exists an* $x \neq y$ *in* X *such that* $\sigma(x, y)$ *is an initial part of* r*.*

If r is in 2^ω, then define an open graph G_r on ω^ω by putting $\{x, y\} \in G_r$ if $\sigma(x, y)$ is defined and is an initial part of r. A *code* for r is an uncountable clique in G_r. Observe that a given H can be a code for at most one r. It is immediate from the previous proposition that OCA implies that every unbounded countably directed subset of $\omega^\omega/\mathrm{fin}$ contains codes for all elements of 2^ω. The proof is then finished by Corollary 6.3 and the following proposition.

Proposition 6.5. $\mathrm{OCA}_{[\mathrm{ARS}]}$ *implies that if* $X \subseteq \omega^\omega/\mathrm{fin}$ *is a chain of cardinality* $\aleph_1$*, then* X *contains codes for at most* $\aleph_1$ *elements of* 2^ω*.*

This is achieved by showing that $\mathrm{OCA}_{[\mathrm{ARS}]}$ implies that for each n, any $X \subseteq \omega^\omega/\mathrm{fin}$ of cardinality $\aleph_1$ can be partitioned into sets X_σ for σ a length n binary sequence so that if $H \subseteq X$ codes $r \in 2^\omega$, then H is contained in $X_{r\restriction n}$ modulo a countable set. If we define $X_r = \bigcap_n X_{r\restriction n}$, then $\{X_r : r \in 2^\omega\}$ is a disjoint family and hence only $\aleph_1$ many can be uncountable (or even nonempty). The proposition then follows from the observation that any code for r is contained in X_r modulo a countable set.

7. Forcing Axioms and Generic Absoluteness

Both forms of the Open Coloring Axiom follow from a much more general set-theoretic hypothesis known as *Martin's Maximum.* MM was first formulated and proved consistent relative to the existence of a supercompact cardinal by Foreman, Magidor, and Shelah [50]. It is the strongest example from an important class of axioms known as *forcing axioms.*

Forcing axioms are generalizations of the Baire Category Theorem and grew out of a line of research which started with Solovay and Tennenbaum's proof that Souslin's Hypothesis is consistent [102]. A stratification of these axioms has been included below mostly for historical accuracy. The reader who is not interested in fine details will lose little of the general picture by assuming MM in place of whatever forcing axiom is needed as a hypothesis (this may, however, mean that the result was proved earlier and possibly by someone else).

We will now review some of the basic definitions associated to these axioms. If θ is a cardinal and $\mathfrak{Q}$ is a class of partially ordered sets, then $\mathrm{FA}_\theta(\mathfrak{Q})$ is the assertion that if Q is in $\mathfrak{Q}$ and $\mathcal{A}$ is a family of maximal antichains in Q with $|\mathcal{A}| \leq \theta$, then there is a filter $G \subseteq Q$ which meets each element of $\mathcal{A}$ — such a filter is said to be $\mathcal{A}$-*generic.* (Here *antichain* means in the sense of forcing — a family of elements of the poset such that no pair of distinct elements has a common lower bound.) It is not difficult to show that if $\mathfrak{Q}$ is the class of all posets, then

$FA_{\aleph_0}(\mathfrak{Q})$ is a theorem of ZFC. This is closely related to the assertion that no compact Hausdorff space can be covered by countably many nowhere dense sets, which is itself a variation of the Baire Category Theorem. On the other hand, it is not difficult to see that if Q is the poset of all finite partial functions from ω to ω_1, then there is a family $\mathcal{A}$ of $\aleph_1$ maximal antichains in Q such that there is no $\mathcal{A}$-generic filter. In particular $FA_{\aleph_1}(\mathfrak{Q})$ is false if $\mathfrak{Q}$ is the class of all partial orders (or even all partial orders of cardinality at most $\aleph_1$).

If $\mathfrak{Q}$ is the class of c.c.c. partial orders (those in which all antichains are countable), then $FA_\theta(\mathfrak{Q})$ is denoted MA_θ and known as *Martin's Axiom for θ antichains*. This was historically the first forcing axiom. It was isolated by D. A. Martin from the proof that Souslin's Hypothesis is consistent [102] and provided the template for the stronger forcing axioms which followed later. Unlike Souslin's Hypothesis, Martin's Axiom turned out to be widely applicable in analysis and point-set topology. The reader is referred to the encyclopedic [52] for a full account of the consequences of MA_θ. While it is trivial that MA_θ implies that $2^{\aleph_0} > \theta$, MA_θ has no other influence on the cardinality of the continuum other than that $2^{\aleph_0} = 2^\theta$ and hence that $\mathrm{cf}(2^{\aleph_0}) > \theta$ (see Theorem 11.1 below).

Forcing Axioms for broader classes of posets than c.c.c. posets are typically inconsistent if $\theta > \aleph_1$ and it is therefore common to use $FA(\mathfrak{Q})$ to denote $FA_{\aleph_1}(\mathfrak{Q})$. (Note however that MA typically denotes the assertion that MA_θ holds for all $\theta < 2^{\aleph_0}$.) A poset Q is said to be *proper* if forcing with it preserves stationary subsets of $[X]^\omega$ whenever X is an uncountable set. The *Proper Forcing Axiom (PFA)* is FA(proper). Martin's Maximum is the forcing axiom for posets which preserve stationary subsets of ω_1. These axioms were proved consistent relative to the existence of a supercompact cardinal by Baumgartner and by Foreman, Magidor and Shelah, respectively.

Theorem 7.1. *(see [32]) If there is a supercompact cardinal κ, then there is a forcing extension which satisfies PFA.*

Theorem 7.2. [50] *If there is a supercompact cardinal κ, then there is a forcing extension which satisfies MM.*

MM is the strongest forcing axiom in the sense that if Q is a poset which does not preserve some stationary subset of ω_1, then there is a family $\mathcal{A}$ of $\aleph_1$ maximal antichains such that there does not exist an $\mathcal{A}$-generic filter.

The theory of MM has been the subject of extensive study, both for its set-theoretic consequences and for its applications to other branches of mathematics — see for instance [52], [82], [115]. The following are some of its most important early consequences.

Theorem 7.3. [50] *MM implies that the nonstationary ideal on ω_1 is* saturated*: if S_ξ $(\xi < \omega_2)$ are stationary subsets of ω_1, then there exist $\xi \neq \eta < \omega_2$ such that $S_\xi \cap S_\eta$ is stationary.*

Theorem 7.4. [50] *Assume MM and let $\kappa > \aleph_1$ be a regular cardinal. Suppose that $\langle A_\xi : \xi \in \omega_1 \rangle$ are pairwise disjoint stationary subsets of ω_1 and $\langle B_\xi : \xi \in \omega_1 \rangle$ are stationary subsets of κ which each consist of ordinals of countable cofinality. There is a continuous increasing sequence $\langle \gamma_\xi : \xi \in \omega_1 \rangle$ in κ such that for all $\xi \in \omega_1$, if ξ is in A_ξ, then γ_ξ is in B_ξ.*

An immediate consequence of the second result is that, assuming MM, $\kappa^{\aleph_1} = \kappa$ whenever $\kappa > \aleph_1$ is a regular cardinal. In particular, MM implies that $2^{\aleph_1} \leq {\aleph_2}^{\aleph_1} = \aleph_2$; this is historically the first use of a forcing axiom to produce a bound on the cardinality of the continuum. Since it is trivial that MM refutes CH, we also have that $2^{\aleph_0} = \aleph_2$ is a consequence of MM.

Woodin showed that Theorem 7.3 leads to a more dramatic failure of CH.

Theorem 7.5. [126, Ch.3] *If the ideal of nonstationary subsets of ω_1 is saturated and $\mathscr{P}(\omega_1)^\sharp$ exists, then $\underset{\sim}{\delta}^1_2 = \omega_2$.*

Here $\underset{\sim}{\delta}^1_2$ is the supremum of the lengths of all $\underset{\sim}{\delta}^1_2$ pre-wellorderings of $\mathbb{R}$. The hypothesis $\mathscr{P}(\omega_1)^\sharp$ *exists* is a consequence of MM as well; it also follows from the existence of a measurable cardinal. In particular, MM not only refutes CH, but in fact gives an *effective failure* of CH — one at the level of descriptive set theory. This addressed question raised in [49].

PFA, which is a weakening of MM, plays an important role in classifying structures of cardinality $\aleph_1$.

Theorem 7.6. [8] *If CH is true and X and Y are $\aleph_1$-dense suborders of $\mathbb{R}$, then there is a c.c.c. forcing extension in which X and Y are isomorphic as linear orders. In particular PFA implies that any two $\aleph_1$-dense suborders of $\mathbb{R}$ are isomorphic.*[2]

Theorem 7.7. [107] *Assume PFA. Every directed system of cardinality at most $\aleph_1$ is cofinally equivalent to one of the following: 1, ω, ω_1, $\omega \times \omega_1$, $[\omega_1]^{<\omega}$.*

[2]PFA had not been formulated at the time of [8].

This classification was extended to the transitive relations on ω_1 in [111].

Next recall that an *Aronszajn line* is an uncountable linear order in which all separable and scattered suborders are countable. Such linear orders were first constructed by Aronszajn and Kurepa (see [67]). A *Countryman line* is an uncountable linear order with the property that its Cartesian square is a countable union of chains. Such orders are necessarily Aronszajn and were first constructed by Shelah [95] (see also [116]). The following results show that there is a rather strong analogy between the Aronszajn lines and the countable linear orderings (with Countryman lines playing the roles of ω and $-\omega$).

Theorem 7.8. *Assume PFA. The following are true:*

- [1] *Every pair of normal Countryman lines are isomorphic or reverse isomorphic.*
- [79] *Every Aronszajn line contains a Countryman suborder.*
- [84] *If C is a Countryman line, then*

$$\operatorname{dirlim} C \times (-C) \times \cdots \times (-C)$$

  is a universal Aronszajn line.
- [76] *The Aronszajn lines are well quasi-ordered by embeddability.*

Much of the theory of MM can be developed through the combinatorial consequences of PFA. We have already noted that both OCA and $\mathrm{MA}_{\aleph_1}$ follow from PFA. Another useful consequence of PFA is Todorcevic's P-Ideal Dichotomy (PID):

If S is any set and $\mathscr{I} \subseteq [S]^\omega$ is a P-ideal, then either:
- there is an uncountable $X \subseteq S$ such that $[X]^\omega \subseteq \mathscr{I}$ or else
- there is a decomposition $S = \bigcup_{n=0}^{\infty} S_n$ such that for each n, no infinite subset of S_n is in $\mathscr{I}$.

(Here $\mathscr{I}$ is a *P-ideal* on S if it is an ideal which includes all finite subsets of S and which is countably directed under $\subseteq^*$.) PID was analyzed in detail for sets of size $\aleph_1$ in [3]. That paper also briefly mentions the general case and how to prove its consistency based on methods in [108], although a detailed analysis of its consequences and a proof of its consistency were first given in [113]. This axiom has a similar influence on gaps as OCA does:

Theorem 7.9. [113] *Assume PID. If κ and λ are regular cardinals and there is a (κ, λ^*)-gap in $\omega^\omega/\mathrm{fin}$, then either $\min(\kappa, \lambda) = \omega$ or else $\kappa = \lambda = \omega_1$.*

In particular, PID implies $\mathfrak{b} \leq \aleph_2$. On the other hand, unlike OCA, PID is consistent with CH [113]. PID also influences the combinatorics of sets which are far removed from $\mathbb{R}$.

Theorem 7.10. [3] *PID implies that every uncountable tree either contains an uncountable chain or an uncountable antichain.*

Theorem 7.11. [124] *PID implies that if κ is a singular strong limit cardinal, then $2^\kappa = \kappa^+$.*

Another general consequence of MM which we will discuss is the *Strong Reflection Principle (SRP):*

> If $\omega_1 \subseteq X$ and $S \subseteq [X]^\omega$, then there is a continuous $\subseteq$-chain $\langle N_\xi : \xi \in \omega_1 \rangle$ such that $\xi \subseteq N_\xi$ for all $\xi \in \omega_1$ and if $\nu \in \omega_1$ is a limit ordinal and there is an ω_1-end extension of N_ν which is in S, then N_ν is in S.

Here N is an ω_1-*end extension* of M if $M \subseteq N$ and $M \cap \omega_1 = N \cap \omega_1$.

SRP was first formulated by Todorcevic who abstracted it from arguments in [50]. SRP captures a number of consequence of MM which were originally directly deduced in [50], including the saturation of the ideal of nonstationary subsets of ω_1 (NS_{ω_1}), Chang's Conjecture, and the conclusion of Theorem 7.4 [13]. SRP was also the inspiration for the *Mapping Reflection Principle (MRP)*, another related consequence of PFA [83] which played an important role in the solution of the *basis problem for uncountable linear orders* [79]. We will discuss some of the consequences of SRP in the next section; see [13] for more details on how to derive its consequences.

We will finish this section with two more variants on the theme of forcing axioms which will be needed the sections which follow. If membership to $\mathfrak{Q}$ is an invariant of forcing equivalence, then BFA($\mathfrak{Q}$) is the weakening of FA($\mathfrak{Q}$) in which the families $\mathcal{A}$ consist only of antichains of cardinality at most $\aleph_1$. By work of Bagaria [6], this is equivalent to the assertion that $H(\aleph_2)$ is a Σ_1-elementary substructure of any generic extension by a poset from $\mathfrak{Q}$. The assertion $\mathrm{FA}^+(\mathfrak{Q})$ is the following strengthening of FA($\mathfrak{Q}$): If Q is in $\mathfrak{Q}$, $\mathcal{A}$ is a family of at most $\aleph_1$ maximal antichains in Q and $\dot{S}$ is a Q-name for a stationary set, then there is an $\mathcal{A}$-generic filter $G \subseteq Q$ such that

$$\{\xi \in \omega_1 : \exists p \in G(p \Vdash \check{\xi} \in \dot{S}\}$$

is stationary; $\mathrm{FA}^{++}(\mathfrak{Q})$ is the strengthening in which $\aleph_1$-many names for stationary sets are allowed.

8. Stationary set reflection and $2^{\aleph_0} \leq \aleph_2$

This section contains an outline of why the *reflection of stationary sets* in $[\omega_2]^\omega$ implies that $2^{\aleph_0} \leq \aleph_2$. Let $\theta \geq \omega_2$ be a cardinal. A stationary subset $S \subseteq [\theta]^\omega$ is said to *reflect* if there exists an $X \subseteq \theta$ of cardinality $\aleph_1$ such that $\omega_1 \subseteq X$ and $S \cap [X]^\omega$ is stationary in $[X]^\omega$. This notion was perhaps first considered by Baumgartner, who proved the following results.

Theorem 8.1. *(see* [9, §8]*) If there is supercompact cardinal, then there is a forcing extension in which* PFA^{++} *holds.*

Theorem 8.2. *(see* [9, §8]*)* PFA^{++} *implies that every stationary subset of* $[X]^\omega$ *reflects.*

The assertion that stationary subsets of $[\theta]^\omega$ reflect is a natural hypothesis and one which has been studied extensively in the literature [27], [28], [29], [99], [103], [110]. We will begin by noting that it follows from MM both directly and also via the principle SRP.

Theorem 8.3. [50] *MM implies that if* $\theta \geq \aleph_2$ *is a cardinal, then every stationary subset of* $[\theta]^\omega$ *reflects.*

Theorem 8.4. [13] *SRP implies that if* $\theta \geq \aleph_2$ *is any cardinal, then every stationary subset of* $[\theta]^\omega$ *reflects.*

In fact if $S \subseteq [\theta]^\omega$ and N_ξ $(\xi < \omega_1)$ satisfies the conclusion of SRP for S, then

$$\{M \in S : M \cap \omega_1 \notin \{\xi \in \omega_1 : N_\xi \in S\}\}$$

is nonstationary. Since the set of all $M \in [\theta]^\omega$ which ω_1-end extend an element of $\{N_\xi : \xi \in \omega_1\}$ is club, it follows that $\{\xi \in \omega_1 : N_\xi \in S\}$ must be stationary if S is stationary.

Quite remarkably, we have the following result.

Theorem 8.5. [118] *If every stationary subset of* $[\omega_2]^\omega$ *reflects, then* $2^{\aleph_0} \leq \aleph_2$.

The reason for this is twofold. On one hand, it is a result of Baumgartner and Taylor that every club in $[\omega_2]^\omega$ has cardinality at least $2^{\aleph_0}$ [12]. On the other, Todorcevic proved that there is a subset F of $[\omega_2]^\omega$ of cardinality $\aleph_2$ whose complement is either nonstationary or else is equal to a nonreflecting stationary set modulo the club filter [118].

In fact by appealing to a result of Gitik [54], Todorcevic was able to draw an even stronger conclusion.

Theorem 8.6. [110] *Suppose that every stationary subset of* $[\omega_2]^\omega$ *reflects. If* N *is an inner model such that* ${\aleph_2}^N = {\aleph_2}^V$, *then* $\mathbb{R} \subseteq N$. *In particular* $|\mathbb{R}| \leq \aleph_2$.

Todorcevic's set F can be described as follows. Fix for a moment a sequence $\vec{e} = \langle e_\beta : \beta \in \omega_2 \rangle$ such that e_β is an injection from β into ω_1. A subset X of ω_2 is *$\vec{e}$-closed* if $X \cap \omega_1$ is an ordinal and whenever β is in X and $\alpha \in \beta$, $e_\beta(\alpha)$ is in X if and only if α is in X. If we define E to be the collection of all $\vec{e}$-closed elements of $[\omega_2]^\omega$, then it is easily checked that E is club. Also, observe that if $\omega_1 \subseteq X \subseteq \omega_2$ and X is $\vec{e}$-closed, then X is an ordinal. Define

$$F_{\alpha,\beta} = \{\xi \in \beta : e_\beta(\xi) < \alpha\}$$

$$F = \{F_{\alpha,\beta} : \alpha < \omega_1 \leq \beta < \omega_2\}$$

and let A be the set of all $\vec{e}$-closed subsets of ω_2 which are not in F. Notice that if $\delta \in \omega_2$, then $\{e_\delta^{-1}\xi : \xi \in \omega_1\}$ is club in $[\delta]^\omega$. In particular, if $\omega_1 \subseteq X \subseteq \omega_2$ and X has cardinality $\aleph_1$, then $A \cap [X]^\omega$ is nonstationary in $[X]^\omega$.

Gitik's result can be described as follows. Suppose that a, b_0, b_1 are countable subsets of ω_2 and $\xi \in a$ and define $r \in 2^\omega$ and $(\xi_k)_k \in \omega_2^\omega$ recursively as follows:

$$r(k) = 1 \text{ if and only if } \min(b_0 \setminus \xi_k) > \min(b_1 \setminus \xi_k)$$

$$\xi_0 = \xi \qquad \text{and} \qquad \xi_{k+1} = \min(a \setminus \min(b_{r(k)} \setminus \xi_k))$$

Set $h_\xi(a, b_0, b_1) = r$. Gitik proved that for every club $E \subseteq [\omega_2]^\omega$ and $r \in 2^\omega$, there exist $a, b_0, b_1 \in E$ and $\xi \in a$ such that $h_\xi(a, b_0, b_1) = r$. Thus if an inner model contains a club subset of $[\omega_2]^\omega$, that inner model also contains all the reals.

It is worth noting that Todorcevic has a different method for proving that club subsets of $[\omega_2]^\omega$ have cardinality at least $2^{\aleph_0}$ which is particularly elegant. First recall that, by work of Shelah [94], there is a sequence $\langle C_\delta : \delta \in \lim_\omega(\omega_2) \rangle$ such that:

- each C_δ is a cofinal ω-sequence in δ and
- if $E \subseteq \omega_2$ is closed and unbounded, then there exists a δ such that $C_\delta \subseteq E$.

Fix such a sequence and if $x \subseteq \theta$ has no last element, define

$$\operatorname{pat}(x) = \{|\alpha \cap C_\delta| : \alpha \in x\}$$

where $\delta = \sup(x)$. For each infinite $r \subseteq \omega$, it can be verified that

$$S_r = \{N \in [\omega_2]^\omega : \operatorname{pat}_{\omega_2}(N) = r\}$$

is stationary; clearly $\{S_r : r \subseteq \omega\}$ are pairwise disjoint. The proof that S_r is stationary was generalized considerably by Foreman and Todorcevic in [51].

9. The Conjectures of Chang and Rado

This section will introduce two conjectures which arose outside of set theory and which relate to the phenomenon of stationary reflection described in the previous section. Each implies $2^{\aleph_0} \leq \aleph_2$ and moreover that the conclusion of Theorem 8.6 holds. The first is a strengthening of a well known model-theoretic transfer principle known as *Chang's Conjecture*. Chang's Conjecture (CC) is the assertion that if $\mathfrak{A}$ is a structure of cardinality $\aleph_2$ and $|X^{\mathfrak{A}}| = \aleph_1$ for some unary relation X, then there is an elementary substructure $\mathfrak{B} \prec \mathfrak{A}$ such that $|\mathfrak{B}| = \aleph_1$ and $|X^{\mathfrak{B}}| = \aleph_0$. Chang's Conjecture has been well studied in the literature; see [63] for further reading. The following strengthening allows one to readily build the elementary substructure $\mathfrak{B}$ via a recursive process (see Theorem 1.3 of [97]).

CC***:** If $\theta \geq \omega_2$ is a sufficiently large regular cardinal, $\lhd$ is a well ordering of $H(\theta)$, and $M \prec (H_\theta, \in, \lhd)$ is countable, then there is an $\overline{M} \prec H(\theta)$ such that $M \subseteq \overline{M}$, $M \cap \omega_1 = \overline{M} \cap \omega_1$, but $M \cap \omega_2 \neq \overline{M} \cap \omega_2$.

The relevance to the present discussion is the following result.

Theorem 9.1. [50] *If every stationary subset of* $[2^{\omega_1}]^\omega$ *reflects, then* CC* *is true.*

The hypothesis CC* implies that the set F described above contains a club regardless of the choice of the sequence $\vec{e} = \langle e_\beta : \beta \in \omega_2 \rangle$ used to define it. In particular, we have the following result.

Theorem 9.2. [110] CC* *implies* $2^{\aleph_0} \leq \aleph_2$.

Next we turn to a conjecture made by Richard Rado who was motivated by an analogous theorem which he proved in finite combinatorics. Recall that an *interval graph* is the intersection graph of a collection of intervals in a linear order. Rado proved that if the chromatic number of an interval graph is greater than n, then there is a (necessarily complete) subgraph on $n+1$ vertices which already has chromatic number greater than n [90]. He then made the following conjecture [91]:

RC**:** If an interval graph is not countably chromatic, then it has a subgraph of cardinality $\aleph_1$ which is not countably chromatic.

Todorcevic showed that Rado's Conjecture is equivalent to the assertion that every nonspecial tree has a subset of cardinality $\aleph_1$ which is nonspecial. Here we recall that a tree of height ω_1 is *special* if it is a union of countably many antichains. Notice that there are many trees of cardinality continuum which have no uncountable branch and yet are non-special — the well ordered subsets of $\mathbb{Q}$ ordered by end extension and the tree of all closed subsets of a stationary co-stationary subset of ω_1 are two standard examples. Thus while Rado's Conjecture implies a number of consequences of Martin's Maximum, it is inconsistent with $\mathrm{MA}_{\aleph_1}$, which implies that all trees of cardinality $\aleph_1$ without uncountable branches are special [11].

Theorem 9.3. [105] *If there is a supercompact cardinal, then there is a forcing extension in which Rado's Conjecture is true. Moreover, the forcing extension can be arranged to satisfy either* $2^{\aleph_0} = \aleph_1$ *or* $2^{\aleph_0} = \aleph_2$.

Theorem 9.4. [110] *Rado's Conjecture implies* CC^* *and that* $\theta^{\aleph_0} = \theta$ *whenever* $\theta \geq \aleph_2$ *is a regular cardinal. In particular, Rado's Conjecture implies* $2^{\aleph_0} \leq \aleph_2$.

10. Woodin's $\mathbb{P}_{\max}$-extension of $L(\mathbb{R})$

The hypothesis Martin's Maximum discussed in Section 7 above has been highly successful in proving consistency results and developing a rich mathematical theory extending that axiomatized by ZFC. It has long been empirically observed that the theory of MM is quite close to being complete, at least as far as the theory of $H(\aleph_2)$ is concerned. This is partially explained by the following recent result of Viale.

Theorem 10.1. [125] *There is a strengthening* MM^{+++} *of* MM^{++} *which holds in the standard models for MM and with the property that any two generic extensions by* NS_{ω_1}*-preserving forcings which satisfy* MM^{+++} *have the same theory for* $H(\aleph_2)$.

It has long been a mystery whether a result such as this can be proved if the assumption of preserving stationary sets is dropped. There is no known canonical model of MM and there is no known sense in which its theory is optimal or canonical.

Woodin's $\mathbb{P}_{\max}$-extension of the model $L(\mathbb{R})$ is motivated in part by these sorts of philosophical considerations. The next theorem is the starting point for the development of $\mathbb{P}_{\max}$.

Theorem 10.2. [100][3] *If there are a proper class of Woodin cardinals, then* $L(\mathbb{R})^V$ *is elementarily embedded in* $L(\mathbb{R})^{V[G]}$ *whenever* G *is* V*-generic for a set forcing. In particular, if there is a proper class of Woodin cardinals, then the theory of* $L(\mathbb{R})$ *cannot be changed by forcing.*

Notice that the inner model $L(\mathscr{P}(\omega_1))$ is correct about CH and that many mathematical statements which are independent of ZFC are actually assertions about what is true in this model. In light of Theorem

[3]The comment in the footnote to Theorem 2.1 applies here as well.

10.2, it is natural to speculate whether the theory of $L(\mathbb{R})$ might be used to generate a canonical theory of $L(\mathscr{P}(\omega_1))$ which extends ZFC. Woodin has shown that this is indeed the case.

Theorem 10.3. [126] *Assume there is a proper class of Woodin cardinals. In $L(\mathbb{R})$ there is a forcing $\mathbb{P}_{\max}$ with the following properties:*

- $\mathbb{P}_{\max}$ *is homogeneous;*
- $\mathbb{P}_{\max}$ *is σ-closed;*
- *If $G \subseteq \mathbb{P}_{\max}$ is $L(\mathbb{R})$-generic, then $L(\mathbb{R})[G]$ satisfies* AC*;*
- *If $G \subseteq \mathbb{P}_{\max}$ is $L(\mathbb{R})$-generic and $A \subseteq \omega_1$ is in $L(\mathbb{R})[G] \setminus L(\mathbb{R})$, then $L(\mathbb{R})[A] = L(\mathbb{R})[G]$.*
- *If ϕ is a Π_2-sentence in the language of the structure*

$$(H(\omega_2), \in, \omega_1, \mathrm{NS}_{\omega_1})$$

and ϕ is Ω-consistent, then $\mathbb{P}_{\max}$ forces $H(\omega_2)$ satisfies ϕ.

I will not define Ω-logic here; see [126] and also [7]. Suffice it to say that if ZFC proves that ϕ can be forced, then ϕ is Ω-consistent. Notice that since $\mathbb{P}_{\max}$ is σ-closed, forcing with it does not change $L(\mathbb{R})$. Since $\mathbb{P}_{\max}$ is homogeneous, the theory of the generic extension coincides with the set of sentences which are forced by every condition. In particular, the theory of the generic extension does not depend on the generic filter and can be computed in the ground model. Furthermore, in the presence of a proper class of Woodin cardinals the theory of $L(\mathbb{R})$ is fixed by forcing and hence the theory of the $\mathbb{P}_{\max}$-extension of $L(\mathbb{R})$ is fixed as well.

Much of the theory of this generic extension has been worked out and it extends the known consequences of MM for the structure $L(\mathscr{P}(\omega_1))$. In particular, both theories include the equality $2^{\aleph_0} = \aleph_2$; this is explored in more detail in the next section. It remains an open problem whether the technical strengthening MM^{++} of Martin's Maximum implies that $L(\mathscr{P}(\omega_1))$ is a $\mathbb{P}_{\max}$-extension of $L(\mathbb{R})$; see [4], [70], [71], [125].

11. Simply definable well orderings of $\mathbb{R}$

A highly nontrivial feature of the $\mathbb{P}_{\max}$-extension of $L(\mathbb{R})$ is that it satisfies the Axiom of Choice. By contrast, in the presence of large cardinals, $L(\mathbb{R})$ satisfies the Axiom of Determinacy which negates AC in a number of essential ways. (For instance AD implies all subsets of $\mathbb{R}$ satisfy the perfect set property, are Lebesgue measurable, and have the Baire Property.) Since the $\mathbb{P}_{\max}$-extension is of the form $L(\mathscr{P}(\omega_1))$, the task of verifying the Axiom of Choice reduces to exhibiting a well ordering of $\mathscr{P}(\omega_1)$ which is *definable* in $L(\mathscr{P}(\omega_1))$.

Before proceeding, let us first note that in the presence of $\mathrm{MA}_{\aleph_1}$, there is a bijection between $\mathscr{P}(\omega)$ and $\mathscr{P}(\omega_1)$ which is definable over $H(\omega_2)$. This often reduces the task of defining a well ordering of $\mathscr{P}(\omega_1)$ to the task of defining a well ordering of $\mathscr{P}(\omega)$. There are many means of achieving this; one is Solovay's *almost disjoint coding.*

Theorem 11.1. *(see* [52, 23A]*) Assume* MA_θ*. If* $\{A_\xi : \xi \in \theta\}$ *is a family of infinite subsets of* ω *which have pairwise finite intersection and* $X \subseteq \theta$*, then there is a* $Y \subseteq \omega$ *such that*

$$X = \{\xi \in \theta : A_\xi \cap Y \textit{ is infinite}\}.$$

Woodin was the first to show that MM implied that $L(\mathscr{P}(\omega_1))$ contained a well ordering of $\mathbb{R}$. This is done by mimicking the proof of Theorem 7.4 but utilizing a definable family of stationary subsets of ω_2. Suppose for a moment that $S \subseteq \omega_1$ is stationary. Define $\tilde{S} \subseteq \omega_2$ to consist of all ordinals γ such that $\omega_1 \leq \gamma < \omega_2$ and if E is a well ordering of ω_1 isomorphic to γ, then

$$\{\nu \in \omega_1 : \mathrm{otp}(E \restriction \nu) \in S\}$$

contains a club. That is, $\tilde{S}$ is the set of all $\gamma \in [\omega_1, \omega_2)$ such that every condition in $\mathscr{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ forces that $\gamma \in j(S)$, where j is the induced generic elementary embedding. Observe that if S and T are disjoint, then so are $\tilde{S}$ and $\tilde{T}$.

Theorem 11.2. [126, Ch.5] *Assume MM. If* $S \subseteq \omega_1$ *is stationary, then* $\{\gamma \in \tilde{S} : \mathrm{cof}(\gamma) = \omega\}$ *is also stationary.*

Woodin's coding principle can now be described as follows.

ϕ_{AC}**:** Whenever $\langle S_i : i \in \omega\rangle$ and $\langle T_i : i \in \omega\rangle$ are sequences of pairwise disjoint sets such that each T_i is stationary, there exists a continuous increasing sequence $\langle \gamma_\xi : \xi \in \omega_1\rangle$ in ω_2 such that if ξ is in S_i, then γ_ξ is in $\tilde{T}_i$.

An immediate consequence of ϕ_{AC} is that if $\langle T_i : i \in \omega\rangle$ is a sequence of pairwise disjoint stationary sets, then for every $x \subseteq \omega$, there is a $\delta < \omega_2$ of cofinality ω_1 such that

$$x = \{i \in \omega : \tilde{T}_i \cap \delta \text{ is stationary}\}.$$

Let δ_x denote the least such ordinal. If we define $x \lhd y$ to mean that $\delta_x < \delta_y$, then $\lhd$ is a well ordering of $\mathscr{P}(\omega)$ which is definable from the parameter $\langle T_i : i \in \omega\rangle$.

Theorem 11.3. [126, Ch.5] *MM implies* ϕ_{AC} *and* ϕ_{AC} *is true in the* $\mathbb{P}_{\max}$*-extension of* $L(\mathbb{R})$*.*

The following is a variation of ϕ_{AC} which has technical advantages in certain contexts.

> **ψ_{AC}:** Suppose that $S, T \subseteq \omega_1$ are stationary and co-stationary. There exists an uncountable $\gamma < \omega_2$ such that T corresponds to the truth value of the formula $\gamma \in j(S)$ in the generic extension by the forcing $\mathscr{P}(\omega_1)/\mathrm{NS}_{\omega_1}$.

Woodin has also shown that, like ϕ_{AC}, ψ_{AC} both is a consequence of MM and holds in the $\mathbb{P}_{\max}$-extension of $L(\mathbb{R})$ [126, Ch.5].

A subtle point in the case of both ϕ_{AC} and ψ_{AC} is that, even though both sentences are in the language of $(H(\omega_2), \in, \omega_1)$ and both follow from MM, it is not clear if ZFC proves that an instance of these assertions can always be forced with a NS_{ω_1}-preserving forcing (Woodin has showed that this is the case if a measurable cardinal exists [126, 10.95]). Todorcevic developed the following coding for this purpose, which relies instead on an ω_1-sequence of reals as a parameter.

> **θ_{AC}:** If $\langle r_\xi : \xi \in \omega_1 \rangle$ is a sequence of distinct reals, then for every $A \subseteq \omega_1$, there exist $\alpha < \beta < \gamma < \omega_2$ such that for some continuous $\subseteq$-increasing sequence $\langle N_\xi : \xi < \omega_1 \rangle$ of countable sets which covers γ we have that for all $\xi \in \omega_1$, $\xi \in A$ if and only if
>
> $$\Delta(r_{N_\xi \cap \alpha}, r_{N_\xi \cap \beta}) > \Delta(r_{N_\xi \cap \beta}, r_{N_\xi \cap \gamma})$$

Here $\Delta(r, s) = \min\{n : r(n) \neq s(n)\}$ and $r_N = r_{\mathrm{otp}(N)}$ if N is a countable set of ordinals.

Theorem 11.4. [114] *Any instance of* θ_{AC} *can be forced with an* NS_{ω_1}-*preserving forcing. In particular, BMM implies* θ_{AC} *holds.*

Previously Woodin had proved that BMM together with the existence of a measurable cardinal implies ψ_{AC} [126, 10.95]; Asperó previously proved (unpublished) that BMM implies $\mathfrak{d} = \aleph_2$.

The statements ϕ_{AC}, ψ_{AC}, and θ_{AC} all (seemingly) require the use of forcings which are *improper* — they destroy stationary subsets of $[\theta]^\omega$ for some uncountable cardinal θ. They also *a priori* require substantial a large cardinal hypothesis for their consistency. The next combinatorial statement was isolated in [83]. It follows from the *Proper Forcing Axiom* and can be forced assuming only the existence of an inaccessible cardinal. In order to formulate the coding principle, we need the following specialized notation. For each countable limit ordinal δ, fix a ladder $C_\delta \subseteq \delta$. If $M \subseteq N$ are countable sets of ordinals of limit ordertype and M is bounded in N, then define $w(M, N) = |C_\nu \cap \pi(M)|$ where $\pi : N \to \nu = \mathrm{otp}(N)$ is the transitive collapse.

υ_{AC}: For every ladder system $\vec{C}$ on ω_1 and every $A \subseteq \omega_1$ there is an uncountable $\delta < \omega_2$ and a continuous $\subseteq$-chain $\langle N_\xi : \xi \in \omega_1 \rangle$ cofinal in $[\delta]^\omega$ such that for all limit $\nu < \omega_1$, there is a $\bar{\nu} < \nu$ such that if $\bar{\nu} < \xi < \nu$, then

$$w(N_\xi \cap \omega_1, N_\nu \cap \omega_1) < w(N_\xi, N_\nu) \text{ iff } N_\nu \cap \omega_1 \in A.$$

This coding principle was the inspiration for the *Mapping Reflection Principle (MRP)* [83]. MRP implies υ_{AC} and played a central role in the solution to the basis problem for uncountable linear orders [79]. It also implies $\kappa^{\aleph_0} = \kappa$ for regular $\kappa > \aleph_1$ [123] and the failure of $\square(\kappa)$ for all regular $\kappa > \aleph_1$ [83].

A careful examination of the sentences ϕ_{AC}, ψ_{AC}, θ_{AC}, and υ_{AC} reveals that all yield a well ordering which is Σ_1-definable over the structure $(H(\omega_1), \in, \mathrm{NS}_{\omega_1})$. These sentences naturally define well orderings of $\mathscr{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ and the the transference of this well ordering to one of $\mathscr{P}(\omega_1)$ depends on a fixed partition of ω_1 into disjoint stationary sets. In [22], Caicedo and Veličković modified the ideas of [83] and combined them with those from [54] and [122] in order to remove the reference to the predicate NS_{ω_1}. This improvement has important structural implications concerning models of MM and other forcing axioms.

Theorem 11.5. [22] *BPFA implies there is a well ordering of* $\mathbb{R}$ *which is* Δ_1*-definable with a parameter which is a ladder system on* ω_1.

The details require somewhat more specialized notation and rather than develop the coding here, we refer the reader to [22]. The Caicedo-Veličković coding also played an important role in the proof of Theorem 12.2 below.

12. Iterated forcing and the Continuum Hypothesis

It is somewhat rare to encounter consequences of CH which require an elaborate proof. Typically the argument proceeds by a diagonalization argument of length ω_1 in which a continuum of tasks are handled by appropriate book keeping. In some cases, however, the argument is more subtle and less effective.

The following theorem is a special case of the main result of [33].

Theorem 12.1. *Suppose that if* $\langle C_\delta : \delta \in \lim(\omega_1) \rangle$ *is a ladder system and* $f : \omega_1 \to 2$, *then there is a function* $g : \omega_1 \to 2$ *such that for all limit ordinals* δ,

$$g \restriction C_\delta \equiv^* f(\delta)$$

(here $g \equiv^* n$ *means that* $\{\xi \in \mathrm{dom}(g) : g(\xi) \neq n\}$ *is finite). Then* $2^{\aleph_0} = 2^{\aleph_1}$ *and in particular CH is false.*

To see why this is true, assume the hypothesis of the theorem and fix functions $p, q : \omega \to \omega$ such that $n \mapsto (p(n), q(n))$ is a surjection onto $\omega \times \omega$ and so that $p(n+1) \leq n$ for all n. Given an $f : \omega_1 \to 2$, construct a sequence $\langle g_n : n \in \omega \rangle$ by recursion such that $g_0 = f$ and $g_{n+1} : \omega_1 \to 2$ satisfies:

$$g_{n+1} \restriction C_\delta \equiv^* g_{p(n)}(\delta + q(n)).$$

It is now readily verified that

$$f \mapsto \langle g_n \restriction \omega : n \in \omega \rangle$$

is a one-to-one function.

It can be shown, however, that for any ladder system and any f, there is a forcing extension with the same reals (and hence the same ω_1) in which there is a g satisfying the conclusion of Theorem 12.1. This theorem therefore represents an obstruction to being able to iterate within a certain class of forcings while not introducing new reals. Theorem 7.5, which implies that if NS_{ω_1} is saturated and there is a measurable cardinal then CH is false, is another example of this phenomenon: the forcing to seal antichains in $\mathscr{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ does not add new reals.

The theory of iterated forcing in the presence of CH is much more complex than the theory of iterated forcing which preserves ω_1; see [98] and also [38]. One reason for this is that, unlike in the case of preserving ω_1, there is no heuristically largest class of forcings which we can iterate while preserving that no reals are added. This was first formally demonstrated in the course of proving the following result, which is in contrast to Theorem 10.3.

Theorem 12.2. [5] *There are two Π_2-sentences ψ_0 and ψ_1 in the language of $(H(\omega_2), \in, \omega_1)$ such that for $i = 0, 1$*

$$(H(\omega_2), \in, \omega_1) \models \psi_i \wedge \mathrm{CH}$$

can be forced if there is an inaccessible limit of measurable cardinals but such that $\psi_0 \wedge \psi_1$ implies $2^{\aleph_0} = 2^{\aleph_1}$.

Since both ψ_0 and ψ_1 can be regarded as fragments of Martin's Maximum, this result can be interpreted as saying that there is no strongest forcing axiom which is consistent with CH. The next result gives another consequence of CH which is also related to the breakdown of a theory of iterated forcing for not adding reals.

Theorem 12.3. [80] *Assume CH. There is a tree T of height ω_1 with no cofinal branch such that T is completely proper as a forcing but such* *that*

$$\{(s,t) \in T^2 : \mathrm{ht}(s) = \mathrm{ht}(t) \text{ but } s \neq t\}$$

is a countable union of antichains (i.e. T^2 is special off the diagonal*).*

Here complete properness is a condition which rules out the coding of Devlin and Shelah described in the beginning of this section. Shelah has shown [98] that this condition can be supplemented by quite different hypotheses in order to prove that reals are not introduced in iterated forcing constructions.

Theorem 12.3 shows in particular that CH implies that there is a *Baire tree* whose square is special off the diagonal; it is not known whether this is a theorem of ZFC. (A tree is *Baire* if the intersection of any countable family of dense open subsets is dense.)

13. The Semifilter Trichotomy

Next we will examine a combinatorial statement whose origins lie outside of set theory. While this statement has a wide variety of consequences and may imply $2^{\aleph_0} = \aleph_2$, it is incompatible with Martin's Axiom (and in particular with MM). Consider the following two questions:

Question 13.1. *Does the Čech-Stone remainder of* $[0,\infty)$ *have only one composant?*

Question 13.2. *Is it impossible to express the ideal of compact operators on a separable Hilbert space as the sum of two smaller ideals?*

(Recall here that a *composant* of a point in a continua is the union of all proper subcontinua which contain the point.) These questions have no apparent relationship to each other and indeed come from different areas of mathematics — continua theory and functional analysis, respectively. M.E. Rudin proved that under CH, the remainder of the half-line has more than one component [92]. Similarly, Blass and Weiss demonstrated that question 13.2 also has a negative answer under CH [20].

Blass and Weiss [20] isolated the following statement — now known as the Near Coherence of Filters (NCF) — which is equivalent to each of the above questions having a positive answer [15]:

> NCF**:** If p and q are two nonprincipal ultrafilters on ω, then there is a finite-to-one map $f : \omega \to \omega$ such that $\beta f(p) = \beta f(q)$.

This statement was soon after proved consistent by Shelah [18]. In fact, NCF holds in any forcing extension of a model of CH obtained by iterating rational perfect set forcing $\aleph_2$ times with countable supports [19].

It turns out that NCF is a consequence of an inequality involving cardinal invariants of the continuum. Recall that a family $\mathcal{G}$ of infinite subsets of ω is said to be *groupwise dense* if $\mathcal{G}$ is closed under modulo finite containment and whenever $f : \omega \to \omega$ is a finite-to-one function, there is an infinite set $X \subseteq \omega$ such that $f^{-1}X$ is in $\mathcal{G}$. The cardinal $\mathfrak{g}$ is the minimum cardinality of a family of groupwise dense sets which have empty intersection. Recall that the cardinal $\mathfrak{u}$ is the minimum cardinality of a base for a nonprincipal ultrafilter on ω.

Theorem 13.3. [17] *The inequality* $\mathfrak{u} < \mathfrak{g}$ *implies* NCF*.*

This inequality holds in the models originally used to prove that NCF is consistent as well as the model obtained by iterating rational perfect set forcing. In fact $\mathfrak{u} < \mathfrak{g}$ is equivalent to a strengthening of NCF known as the *Semifilter Trichotomy*: if $\mathscr{S} \subseteq [\omega]^\omega$ is closed under taking almost supersets then there is a finite-to-one function $f : \omega \to \omega$ such that $\{f''S : S \in \mathcal{S}\}$ is either a family of co-finite sets, an ultrafilter, or $[\omega]^\omega$ [14] [68].

It is worth noting at this point that Larson has proved the following result, contrasting Theorem 3.3 above.

Theorem 13.4. [69] *The Semifilter Trichotomy implies that medial limits do not exist.*

Quite remarkably, $\mathfrak{u} < \mathfrak{g}$ places rather strong restrictions on the other cardinal invariants of the continuum.

Theorem 13.5. [14] *Assume* $\mathfrak{u} < \mathfrak{g}$*. Then* $\mathfrak{u} = \mathfrak{b} < \mathfrak{g} = \mathfrak{d} = \mathfrak{c}$*.*

In [120], Todorcevic posed the following question in the context of Theorem 13.5.

Question 13.6. *Does* $\mathfrak{u} < \mathfrak{g}$ *imply* $\mathfrak{u} = \aleph_1$*?*

Recently Shelah proved the following result:

Theorem 13.7. [96] *The inequality* $\mathfrak{g} \leq \mathfrak{b}^+$ *always holds.*

Therefore Todorcevic's question is equivalent to asking whether $\mathfrak{u} < \mathfrak{g}$ implies $2^{\aleph_0} = \aleph_2$.

14. Open Problems

I will conclude this paper with a collection of open problems which in some way connect to the relationship between the continuum and the second uncountable cardinal. There are many themes represented in these problems:

- What are the simplest hypotheses which imply $2^{\aleph_0} = \aleph_2$?
- Are there foundational principles which entail $2^{\aleph_0} = \aleph_2$?
- Does classification ever necessitate $2^{\aleph_0} = \aleph_2$?
- Are there consequences of $2^{\aleph_0} = \aleph_2$ which are *verifiable*?

The bulk of these questions concern the inequality $2^{\aleph_0} < \aleph_3$, although I have included some questions relating to $\aleph_1 < 2^{\aleph_0}$ as well.

Problem 14.1. *(Todorcevic* [108]*) Does Todorcevic's Open Coloring Axiom imply that the continuum is $\aleph_2$?*

Problem 14.2. *(Todorcevic; see* [115]*) Does the P-Ideal Dichotomy imply that the continuum is at most $\aleph_2$?*

As noted above, both OCA and PID imply that $\mathfrak{b} \leq \aleph_2$ because of their influence on gaps in ω^ω/fin.

In Section 6, we saw that it is possible to supplement OCA with another, related Ramsey-theoretic hypothesis $\text{OCA}_{[\text{ARS}]}$ in order to prove $2^{\aleph_0} = \aleph_2$ [81]. It is similarly unknown if $\text{OCA}_{[\text{ARS}]}$ implies that the continuum is $\aleph_2$, although the difficulties in this problem are of a different nature than in the case of OCA.

Problem 14.3. *(Abraham, Rubin, Shelah* [2]*) Does the* $\text{OCA}_{[\text{ARS}]}$ *imply that the continuum is $\aleph_2$?*

It is natural to ask whether there is an analog of Theorem 12.2 for $2^{\aleph_0} > \aleph_2$.

Problem 14.4. *Are there two Π_2-sentences in the language of the structure $(H(\omega_2), \in, \omega_1)$ which are each forcibly consistent with $2^{\aleph_0} > \aleph_2$ but whose conjunction implies $2^{\aleph_0} \leq \aleph_2$?*

Presumably if this question has a negative answer, then the reason for this is the existence of a $\mathbb{P}_{\max}$-like forcing extension of $L(\mathbb{R})$ in which $2^{\aleph_0}$ is large.

Problem 14.5. *(Foreman, Magidor* [49]*) Assume that $L(\mathbb{R})$ satisfies the Axiom of Determinacy. Is $\Theta^{L(\mathbb{R})} \leq \omega_3$?*

The relationship between the $\mathbb{P}_{\max}$-extension of $L(\mathbb{R})$ and more conventional forcing extensions is still not well understood. For instance, the following problem remains unresolved.

Problem 14.6. *(Woodin* [126, Ch.11]*) Is it possible to force, starting from a large cardinal hypothesis, that $L(\mathscr{P}(\omega_1))$ is a $\mathbb{P}_{\max}$-extension of $L(\mathbb{R})$?*

Woodin conjectured that the existence of ω^2 Woodin cardinals sufficed for a positive answer. It is also interesting to ask whether this can be achieved by a semiproper forcing; this is closely related to the following question (compare to [126, 11.15]).

Problem 14.7. *(Woodin* [126, Ch.10]*) Does MM^{++} imply that $L(\mathscr{P}(\omega_1))$ is a $\mathbb{P}_{\max}$-extension of $L(\mathbb{R})$?*

The reader is referred to [4], [70], [71] for partial results concerning this problem.

Problem 14.8. *(Woodin* [126, §3.2]*) Is it consistent with CH that the nonstationary ideal on ω_1 is saturated?*

Problem 14.9. *(Woodin* [126, 11.7]*) Assume that the nonstationary ideal on ω_1 is $\aleph_1$-dense. Must the continuum be $\aleph_2$?*

Set-theoretic hypotheses which are needed to prove classification results about structures of cardinality $\aleph_1$ are closely related to those which imply that $2^{\aleph_0} = \aleph_2$.

Problem 14.10. *(Moore* [82]*) Is there a consistent classification of structures which entails that the continuum is $\aleph_2$?*

A candidate is the classification of Aronszajn lines. MM implies that this class is in many ways analogous to the class of countable linear orderings (see [73]): it has a two element basis [79], it has a universal element [84], and it is well quasi-ordered by embeddability [76]. Moreover, this development was precipitated by the discovery of a new proof that MM implies that $2^{\aleph_0} = \aleph_2$ [83].

Problem 14.11. *(Moore* [82]*) Suppose the following are true: (a) every two $\aleph_1$-dense nonstationary Aronszajn lines are isomorphic or reverse isomorphic, (b) every Aronszajn line can be embedded into η_C, and (c) the class of Aronszajn lines is well quasi-ordered by embeddability. Must the continuum be $\aleph_2$?*

The following problem is naturally suggested by the results of [14].

Problem 14.12. *Does the Semi-Filter Trichotomy imply that the continuum is $\aleph_2$?*

As noted above, this is equivalent to Todorcevic's question which asks whether $\mathfrak{u} < \mathfrak{g}$ implies $\mathfrak{u} = \aleph_1$.

While it is comparatively easy to influence the combinatorics of $[\omega_1]^{\omega_1}$, more powerful forcings are required to alter the properties of the club filter on ω_1. This is illustrated, for instance, by the fact that the club filter in V generates the club filter in any generic extension

$V[G]$ by a c.c.c. forcing. By contrast, $\mathrm{MA}_{\aleph_1}$ has a substantial impact on the combinatorics of $[\omega_1]^{\omega_1}$ and can always be forced by a c.c.c. forcing. Since methods for producing models in which $2^{\aleph_0} \neq \aleph_2$ via iterated forcing are more limited, it may be that $2^{\aleph_0} \neq \aleph_2$ impacts the combinatorial properties of the club filter. This is the motivation behind the following question.

Problem 14.13. *(Moore) Suppose that whenever $\langle D_\alpha : \alpha \in \omega_1 \rangle$ satisfies that $D_\alpha \subseteq \alpha$ is closed for each $\alpha \in \omega_1$, then there is a club $E \subseteq \omega_1$ such that for all limit $\alpha \in \omega_1$, there is a $\bar{\alpha} < \alpha$ such that $(\bar{\alpha}, \alpha) \cap E$ is either contained in or disjoint from D_α. Must the continuum be $\aleph_2$?*

The combinatorial assertion (μ) in this problem is an immediate consequence of MRP which itself does imply $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$. Whether (μ) is consistent with CH seems to be a good test question for extending the theory of iterated forcing in the context of CH.

Next we turn to a pair of questions which concern whether certain consequences of $2^{\aleph_0} \leq \aleph_2$ are *verifiable*. The first is a topological problem posed by Peter Nyikos.

Problem 14.14. *(Nyikos* [86]*) Does there exist a separable, first countable, countably compact, noncompact Hausdorff space?*

Notice that ω_1, with the order topology, satisfies all of the properties in the problem except separability. It is known that this question has a positive answer if either $\mathfrak{t} = \aleph_1$ or if $\mathfrak{b} = 2^{\aleph_0}$. Since $\mathfrak{t} \leq \mathfrak{b}$, any model in which this problem has a negative answer must necessarily satisfy $2^{\aleph_0} > \aleph_2$. A survey of this question can be found in [87].

The next question comes from partition calculus and Todorcevic's and Veličković's Ramsey-theoretic analysis of Martin's Axiom [109] [121].

Problem 14.15. *(Todorcevic* [119, 9.3.2]*) Is it true that $\binom{\omega_3}{\omega_3} \rightarrow \binom{\omega}{\omega}^{1,1}_{\omega}$?*

Here we recall that $\binom{\theta}{\theta} \rightarrow \binom{\omega}{\omega}^{1,1}_{\omega}$ means that if $f : \theta \times \theta \rightarrow \omega$ is any function, then f is constant on the Cartesian product of two infinite sets. The least θ for which this is true is at most $(2^{\aleph_0})^+$. In fact any ZFC bound on this θ which does not involve cardinal exponentiation would be of interest. It is know that if $\binom{\kappa}{\kappa} \not\rightarrow \binom{\omega}{\omega}^{1,1}_{\omega}$, then MA_κ is equivalent to the assertion that every κ sized subset of a c.c.c. partial order has a centered subset of cardinality κ [109]. It has also been shown that, in the presence of $\mathrm{MA}_{\aleph_2}$, $\binom{\omega_2}{\omega_2} \rightarrow \binom{\omega}{\omega}^{1,1}_{\omega}$ is equivalent to Chang's Conjecture [109].

I will finish the article with the following open-ended question: *Is there a natural hypothesis which implies that the continuum is $\aleph_3$?* The cardinal $\aleph_3$ is of course somewhat arbitrary — what is relevant is that it is accessible and greater than $\aleph_2$. It would not be of particularly great interest to adapt combinatorial statements which imply $2^{\aleph_0} = \aleph_2$ so that they imply $2^{\aleph_0} = \aleph_3$. (This may, however, be interesting at a technical level.) The point is that the proofs that $2^{\aleph_0} = \aleph_2$, from various hypotheses, all employ combinatorial ideas which cannot be modified to produce a proof of $2^{\aleph_0} = \aleph_1$. Typically these ideas concern the combinatorial properties of $\aleph_1$. Are there transcendent combinatorial phenomena at $\aleph_2$ which are related to proofs that $2^{\aleph_0} = \aleph_3$?

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