NONEXISTENCE OF IDEMPOTENT MEANS ON FREE BINARY SYSTEMS

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Abstract. Free binary systems are shown not to admit idempotent means. This refutes a conjecture of the author. It is also shown that the extension of Hindman's theorem to nonassociative binary systems formulated and conjectured in [5] is false.

1. Introduction

Recall that a *binary system*, or *magma*, is a nonempty set equipped with a binary operation. If $(S, *)$ is a binary system, then $*$ can be extended to the set $M(S)$ of *means* on S by:

$$\mu * \nu(f) = \int \left(\int f(s * t) \ d\nu(t) \right) d\mu(s).$$

Here a *mean* on a set S is an element of $\ell^\infty(S)^*$ such that $\mu(f) \geq 0$ if $f \geq 0$ and $\mu(\chi_S) = 1$ — i.e. μ is a finitely additive probability measure on S. If a mean μ takes only values in $\{0, 1\}$, then we say that μ is an *ultrafilter*.

Answering a question of Galvin, Glazer noted that if $*$ is an associative operation on S, then Ellis's Lemma [1] implies there is an idempotent ultrafilter μ on S. Galvin had already noted that the existence of idempotent ultrafilters on $(\mathbf{N}, +)$ could be used to give a short proof of Hindman's Theorem [3]. In fact idempotent ultrafilters on semigroups have found extensive applications in Ramsey theory — see [4] for a more detailed account of both the history and the applications.

Possible extensions of this theory to nonassociative binary systems was considered by the author in [5]. It was noted there that idempotent ultrafilters do not exist on free binary systems. On the other hand, it was shown that the existence of an idempotent mean on any free binary system implies Richard Thomspon's group F is amenable. In fact it was demonstrated there that the existence of such idempotent means implies a version of Hindman's theorem for the free binary system on one generator which in turn implies that F is amenable.

The amenability problem for F is a long standing problem in group theory first considered by Richard Thompson [8] but rediscovered and first popularized by Ross Geoghegan; the problem first appeared in the literature in [2, p. 549]. It is arguably the most notorious problem concerning the amenability of a specific group. The author previously (and incorrectly) claimed to have proved the existence of an idempotent mean on the free binary system [6].

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In this note I will prove that the free binary systems does not support idempotent means, refuting Conjecture 1.4 of [5]. I will also refute Conjecture 1.3 of [5], which was an extension of Hindman's theorem to nonassociative binary systems. The question of F's amenability remains open. While the refutation of Conjecture 1.3 is a stronger result, the proof of the nonexistence of an idempotent mean on free binary systems is more straightforward and thus will be proved first as a warm up.

While this article is self contained, the reader will find more motivation in [5] (and for that matter in [6]). Throughout this paper, counting will start with 0. The variables i, j, k, m, n, p will always implicitly be taken to range over the nonnegative integers. For instance, the n-tuple $(a_0, \ldots, a_{n-1})$ will be denoted $(a_k \mid k < n)$.

2. Free binary systems do not support idempotent measures

Let $(S, *)$ be a free binary system generated by I, fixed for the remainder of the section. Notice that the binary operation $*$ is in fact an injection from $S \times S \to S \setminus I$; this is in fact equivalent to the freeness of $(S, *)$. Define $\# : (S, *) \to (\mathbf{N}, +)$ to be the homomorphism which maps every element of I to 1. Thus $\#(s)$ is the size of the (unique) nonassociative product of generators used to produce s. In particular, if $s = a * b$, then $\#(a), \#(b) < \#(s) = \#(a) + \#(b)$. Set $S_n := \{s \in S \mid \#(s) \leq n\}$.

Define membership to sets $Z \subseteq S$ and $T_p \subseteq S$ recursively on $\#(\cdot)$ as follows:

- $T_0 = S$ and $T_{p+1} = (S \setminus Z) * T_p$;
- $s \in Z$ if and only if $s = a * b$ where $b \in T_{\#(a)}$.

Observe that $Z = \bigcup_p S_p * T_p$.

Recall that if μ and ν are means on S, then

$$\mu * \nu(f) = \int \left(\int f(s * t) \; d\nu(t) \right) d\mu(s)$$

defines a mean on S. The following two facts are immediate from this definition and the fact that $*$ is injective.

Fact 1. *If μ and ν are means on S and $A, B \subseteq S$, then $\mu * \nu(A * B) = \mu(A)\nu(B)$.*

Fact 2. *If $X \subseteq S$ and for some c and μ-a.e. $s \in S$,*

$$\nu(\{t \in S \mid s * t \in X\}) = c$$

then $\mu * \nu(X) = c$.

Suppose now for contradiction that μ is an idempotent mean on S and set $r := \mu(Z)$. Fact 1 inductively implies that $\mu(T_n) = (1 - r)^n$ (if $n = 0$, then $T_0 = S$ and $\mu(T_0) = 1$). On the other hand, Facts 1 and 2 inductively implies that

$$\mu(S_n) = \mu(\bigcup_{i+j=n} S_i * S_j) = \sum_{i+j=n} \mu(S_i)\mu(S_j) = 0$$

for all n. Moreover, by Fact 1 $\mu(S_n * S) = 0$.

If $r > 0$, then let n be sufficiently large that $(1 - r)^n < r$. Observe that

$$Z = \bigcup_{k<n} S_k * T_k \cup \bigcup_{k=n}^{\infty} S_k * T_k \subseteq \bigcup_{k<n} S_k * T_k \cup \bigcup_{k=n}^{\infty} S_k * T_n.$$

Since $\mu(\bigcup_{k<n} S_k * T_k) = 0$ and since $\mu(\bigcup_{k=n}^{\infty} S_k * T_n) = \mu(T_n) = (1 - r)^n$, we have $r = \mu(Z) \leq (1 - r)^n < r$, contrary to our choice of n.

If $r = 0$, then for every $s \in S$,

$$\mu(\{t \in S \mid s * t \in Z\}) = \mu(\{T_{\#(s)}\}) = (1-0)^{\#(s)} = 1.$$

However by Fact 2, $\mu(Z) = \mu * \mu(Z) = 1$, which is also a contradiction.

3. The nonassociative analog of Hindman's Theorem is false

In this section we will refute Conjecture 1.2 of [5]. For the duration of the section, we will let $(S, *)$ denote the free binary system on one generator $\mathbf{1}$. The sets S_n, T_n, and Z are defined as in the previous section. Note that in the present context S_n is finite for each n. If $\mu \in \bigcup_p M(S_p)$, we will write $\#(\mu)$ to denote the unique p such that $\mu \in M(S_p)$.

Conjecture 1.2 of [5] can now be stated as follows:

> *If $c : S \to [0,1]$ and $\epsilon > 0$, then there is an $r \in [0,1]$ and an increasing sequence $(\mu_i \mid i < \infty)$ of elements of $\bigcup_p M(S_p)$ such that whenever s is in S_n and $(i_k \mid k < n-1)$ is admissible for s:*
>
> $$|c(s(\mu_{i_k} \mid k < n)) - r| < \epsilon$$

Here $s(\mu_{i_k} \mid k < n)$ is the result of taking the unique term used to generate s from $\mathbf{1}$ and $*$ and replacing the k^{th} occurrence of $\mathbf{1}$ with μ_{i_k} and evaluating the resulting expression in $M(S)$. If $\#(\mu_{i_k}) = p_k$, then $\#(s(\mu_{i_k} \mid k < n)) = \sum_{k<n} p_k$.

We will work the following equivalent recursive definition of *admissible*:

- $\mathbf{1}$ is admissible for any sequence of positive integers of length 1;
- if $s \in S_m$ and $t \in S_n$, then $(i_k \mid k < m+n)$ is admissible for $s*t$ if and only if $(i_k \mid k < m)$ is admissible for s and $(i_k - m \mid m \le k < n)$ is admissible for t.

In particular, a sequence $(i_k \mid k < m)$ is admissible for any element of S_m provided that $m \le i_0$.

Returning to Conjecture 1.2 of [5], we will show that the conclusion of the conjecture fails when c is the characteristic function of Z defined in the previous section with ϵ and positive number less than $1/2$; note that $c(\mu) = \mu(Z)$. Toward this end, let $(\mu_i \mid i < \infty)$ be given and let r be any accumulation point of the set $\{\mu_i(Z) \mid i < \infty\}$. We will be finished once we prove the following three claims.

Claim 1. *For every m and $\epsilon > 0$, there is a $s \in S$ and $(i_k \mid k < n)$ such that:*

1. *$(i_k - m \mid k < n)$ is admissible for s;*
2. *$s(\mu_{i_k} \mid l < n)(Z) < \epsilon$.*

Proof. Let m and $\epsilon > 0$ be given. If $r = 0$, then we can take $s = \mathbf{1}$ and choose $i_0 > m$ so that $\mu_{i_0}(Z) < \epsilon$. Therefore suppose that $r > 0$ and let l be sufficiently large that $(1-r)^l < \epsilon$. If $k < l$, define $i_k = m+k+1$ and let $u := \mathbf{1} * (\ldots \mathbf{1} * (\mathbf{1} * \mathbf{1}))$ be the right associated product of l many $\mathbf{1}$'s. We have that $(i_k - m \mid k < l)$ is admissible for u. Set $p := \#(u(\mu_{i_k} \mid k < l))$, $n := l + p + 1$, and let v be the right associated product of $p+1$ many $\mathbf{1}$'s. Since $p \ge l$, it is possible to pick an increasing sequence $(i_k \mid l \le k < n)$ of indices such that $i_l > m + l$ and:

$$\prod_{k=l}^{l+p+1} \mu_{i_k}(S \setminus Z) < \epsilon$$

This implies in particular that $v(\mu_{i_k} \mid l \le k < n)(T_p) < \epsilon$. Set $s := u * v$ and observe that $(i_k - m - l \mid l \le k < n)$ is admissible for v and thus $(i_k - m \mid i < n)$

is admissible for s. Since $u(\mu_{i_k} \mid k < l)(S_p) = 1$ and since $Z \cap (S_p * S) = S_p * T_p$, we have that $s(\mu_{i_k} \mid k < n)(Z) < \epsilon$ as desired. □

Claim 2. *For every m and p and every $\epsilon > 0$, there is an $s \in S$ and a $(i_k \mid k < n)$ such that:*

(1) *$(i_k - m \mid k < n)$ is admissible for s;*
(2) *$s(\mu_{i_k} \mid k < n)(T_p) > 1 - \epsilon$.*

Proof. The proof is by induction on p. The base case is trivial since $T_0 = S$. Suppose the claim holds for a given p and let m and $\epsilon > 0$ be given. Fix a $\delta > 0$ such that $(1-\delta)^2 > 1 - \epsilon$. By Claim 1, there are $u \in S$ and $(i_k \mid k < l)$ such that:

- $u(\mu_{i_k} \mid k < l)(Z) < \delta$ and
- $(i_k - m \mid k < l)$ is admissible for u.

Set $p := \#(u(\mu_{i_k} \mid k < l))$. By our inductive hypothesis, there exist v and $(i_k \mid l \leq k < n)$ such that:

- $v(\mu_{i_k} \mid l \leq k < n)(T_p) > 1 - \delta$ and
- $(i_k - m - l \mid l \leq k < n)$ is admissible for v.

It follows that $s := u * v$ and $(i_k \mid k < n)$ now satisfy the conclusion of the claim. □

Claim 3. *For every $\epsilon > 0$, there is an $s \in S$ and a $(i_k \mid k < n)$ which is admissible for s such that $s(\mu_{i_k} \mid k < n)(Z) > 1 - \epsilon$.*

Proof. Setting $p := \#(\mu_0)$, by Claim 2 there is a sequence $(i_k \mid 0 < k < n)$ and a $t \in S$ such that:

- $(i_k - 1 \mid 0 < k < n)$ is admissible for t and
- $t(\mu_{i_k} \mid 1 \leq k < n)(T_p) > 1 - \epsilon$.

Now $s := \mathbf{1} * t$ and $(i_k \mid k < n)$ satisfies the conclusion of the claim. □

The desired contradiction to Conjecture 1.2 of [5] now follows from Claims 1 and 3 with $m = 0$ by noting that for any $0 \leq r \leq 1$ and $0 < \epsilon < 1/2$, either 0 or 1 is not in the interval $[r - \epsilon, r + \epsilon]$.

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