ARONSZAJN LINES AND THE CLUB FILTER

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Abstract. The purpose of this note is to demonstrate that a weak form of club guessing on ω_1 implies the existence of an Aronszajn line with no Countryman suborders. An immediate consequence is that the existence of a five element basis for the uncountable linear orders does not follow from the forcing axiom for ω-proper forcings.

1. Introduction

In [7], Shelah constructed an uncountable linear order C with the property that C^2 is the union of countably many non decreasing relations. Linear orders with this property are now known as *Countryman lines*. These orders are necessarily *Aronszajn* — they do not contain uncountable scattered or separable suborders.

At the end of his construction, Shelah made the following conjecture: *It is consistent that every Aronszajn line contains a Countryman suborder.* As the understanding of Aronszajn and Countryman lines progressed, it became clear that this conjecture was, assuming the Proper Forcing Axiom (PFA), equivalent to the following stronger statement: *The uncountable linear orders have a five element basis.* Recall that a *basis* for the uncountable linear orders is a collection $\mathscr{B}$ of uncountable linear orders such that any other contains an isomorphic copy of an element of $\mathscr{B}$. It is not difficult to prove that any five element basis for the uncountable linear orders must be of the following form (up to equimorphism of its members): X, ω_1, $-\omega_1$, C, and $-C$ where X is a set of reals of cardinality $\aleph_1$ and C is a Countryman line. Here, if L is a linear ordering, $-L$ denotes the reverse order on L.

In [5] it was demonstrated that PFA does in fact imply that every Aronszajn line contains a Countryman suborder. The proof, however,

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utilized a newly isolated consequence of PFA known as the Mapping Reflection Principle (MRP) which was first considered in [4]. Unlike most consequences of PFA, MRP does not follow from the Forcing Axiom ω-proper forcings (ωPFA). The main purpose of this note is to show that the use of MRP in [5] is, in some sense, unavoidable. This is done by showing that a weak form of club guessing — the axiom $\mho$ — implies the existence of an Aronszajn line with no Countryman suborder.

Theorem 1.1. *($\mho$) There is an Aronszajn line with no Countryman suborder.*

The axiom $\mho$ is a strong failure of MRP at the level of ω_1. It is defined as follows:

> $\mho$: There is a sequence $\langle f_\alpha : \alpha < \omega_1 \rangle$ such that for all $\alpha < \omega_1$, f_α is a continuous map from α into ω and whenever $E \subseteq \omega_1$ is closed and unbounded, there is a δ in E such that f_δ takes all values in ω on $E \cap \delta$.

We will pause for a moment to make a few observations concerning this statement. Notice that if $\alpha < \omega_1$ and $f : \alpha \to \omega$ is continuous, then α can be partitioned into open intervals on which f is constant. In such a situation, there is a cofinal $C \subseteq \alpha$ of ordertype at most ω such that $f(\xi)$ depends only on the size of $\xi \cap C$. From this observation, it should be immediately clear that $\mho$ follows from Ostaszewski's $\clubsuit$ and hence Jensen's $\diamondsuit$. This shows that $\mho$ follows from club guessing on ω_1 (where $\clubsuit$ is modified by replacing uncountable subsets of ω_1 with clubs). Furthermore, $\mho$ is immune to c.c.c. forcing and therefore is consistent with $\mathrm{MA}_{\aleph_1}$. This is because every club $E \subseteq \omega_1$ in a c.c.c. forcing extension contains a club from the ground model. A somewhat more involved argument can be used to show that $\mho$ is preserved by ω-proper forcings.

While the most conspicuous feature of Theorem 1.1 is that it establishes that $\mathrm{FA}(\mathscr{A})$ is not sufficient to imply the existence of a five element basis for the uncountable linear orders, I would argue that this is more than "just another independence result." Most of the well known consequences of the Proper Forcing Axiom — $2^{\aleph_0} = \aleph_2$, the failure of $\square(\kappa)$, the non-existence of S spaces and Kurepa trees, for instance — all follow from ωPFA. However a result such as Theorem 1.1 tells us that *if* a combinatorial statement under consideration is to follow from PFA, methods such as those in [5] are possibly relevant.

On the other hand, $\mho$ is such a weak assumption that once it is successfully used as a hypothesis, it is sometimes possible to refine the construction to obtain a ZFC result. We will see that the construction of Theorem 1.1, with the proper interpretation, also yields a hereditarily Lindelöf non separable T_3 space — an L *space*. This construction was originally circulated in [3] and predates the one in [6] by two months, where the assumption of $\mho$ was removed. Furthermore, at present it seems plausible that $\mho$ may follow from the assumption that $2^{\aleph_0} > \aleph_2$. If this is indeed the case, this construction would give new a method for extracting consequences from the assumption $2^{\aleph_0} \neq \aleph_2$.

The paper is intended to be fairly self contained, though the reader will benefit from some exposure to the material in [10]. Notation is standard and generally follows [2] (see also [1] for background in set theory).

2. Construction of the underlying combinatorial object

In this section we will see how to use $\mho$ to define a C-sequence so that the corresponding function ϱ_0 introduced in [9] realizes some additional properties. First I will recall some definitions from [9]; see [10] for further reading and proofs. A *C-sequence* (on ω_1) is a sequence $\langle C_\alpha : \alpha < \omega_1 \rangle$ such that:

(1) $C_{\alpha+1} = \{\alpha\}$ and $C_0 = \emptyset$.
(2) If $\alpha > 0$ is a limit, then C_α is a cofinal subset of α of ordertype ω which consists of successor ordinals.

Given a C-sequence, define the trace function recursively by

$$\mathrm{Tr}(\alpha, \alpha) = \emptyset$$

$$\mathrm{Tr}(\alpha, \beta) = \{\beta\} \cup \mathrm{Tr}(\alpha, \min(C_\beta \setminus \alpha))$$

for $\alpha < \beta < \omega_1$. The function $\varrho_0(\alpha, \beta)$ is given recursively, by

$$\varrho_0(\alpha, \alpha) = \langle\rangle$$

$$\varrho_0(\alpha, \beta) = |C_\beta \cap \alpha| \,\hat{}\, \varrho_0(\alpha, \min(C_\beta \setminus \alpha))$$

for $\alpha < \beta < \omega_1$. It is easily shown that $\varrho_0(\alpha, \beta)$ is equal to

$$\langle |C_{\beta^i(\alpha)} \cap \alpha| : i < |\,\mathrm{Tr}(\alpha, \beta)| \rangle.$$

The function ϱ_1 is defined by letting $\varrho_1(\alpha, \beta)$ be the maximum value in the sequence $\varrho_0(\alpha, \beta)$ (with $\varrho_1(\alpha, \alpha) = 0$). We will need the following standard facts about these functions.

Fact 2.1. *For every $\delta \leq \beta < \omega_1$ with δ a positive limit ordinal, there is a $\bar{\delta} < \delta$ such that δ is in $\mathrm{Tr}(\alpha, \beta)$ whenever $\bar{\delta} < \alpha < \delta$.*

Fact 2.2. *The following are equivalent for $\alpha < \beta < \gamma < \omega_1$:*

(1) *β is in $\mathrm{Tr}(\alpha, \gamma)$.*

(2) $\mathrm{Tr}(\alpha,\gamma) = \mathrm{Tr}(\alpha,\beta) \cup \mathrm{Tr}(\beta,\gamma)$.
(3) $\varrho_0(\beta,\gamma)$ *is an initial part of* $\varrho_0(\alpha,\gamma)$.
(4) $\varrho_0(\alpha,\gamma) = \varrho_0(\beta,\gamma)\hat{\ }\varrho_0(\alpha,\beta)$.

Fact 2.3. *If $\beta < \omega_1$, then $\varrho_0(\cdot,\beta)$ is a strictly increasing map from β into $\omega^{<\omega}$.*

Fact 2.4. *If $\beta \leq \beta'$ and there is an $\alpha < \beta$ such that $\varrho_0(\alpha,\beta) \neq \varrho_0(\alpha,\beta')$, then the least such α is a successor ordinal.*

Fact 2.5. *For every $n < \omega$ and $\beta \leq \beta'$, the following sets are finite:*

$$\{\alpha < \beta : \varrho_1(\alpha,\beta) \leq n\}$$

$$\{\alpha < \beta : \varrho_1(\alpha,\beta) \neq \varrho_1(\alpha,\beta')\}.$$

This has the following as an immediate consequence.

Fact 2.6. *If $B \subseteq \omega_1$ is uncountable, then*

$$\{\varrho_1(\alpha,\beta) : \beta \in B \setminus \alpha\}$$

is infinite for all but a countable set of $\alpha < \omega_1$.

Now suppose that f_α $(\alpha < \omega_1)$ is a $\mho$-sequence. Fix a function $h : \omega \to \omega$ such that $h^{-1}(n)$ is infinite for all $n < \omega$. It is routine to construct a C-sequence C_α $(\alpha < \omega_1)$ such that for all limit ordinals α we have $f_\alpha(\xi) = h(|C_\alpha \cap \xi|)$ whenever $\xi < \alpha$ is a limit ordinal. For the duration of this paper, I will fix a C-sequence which is derived in this manner and use it to construct the functions Tr, ϱ_0, and ϱ_1 above.

I will now define a function $\varphi : \omega^{<\omega} \to \mathbb{Z}$ and show that the composition of φ and ϱ_0 exhibits strong combinatorial properties if ϱ_0 is derived from a $\mho$-sequence as above. Let η be the composition of h followed by a bijection from ω to the finitely supported functions from ω to $\mathbb{Z}$.

Definition 2.7. If s is a finite sequence of elements of ω, define

$$\varphi(s) = \sum_{i<|s|} \eta(s(i))(\max(s \restriction i)).$$

Here $\max(s)$ denotes the maximum entry of s with $\max(\langle\rangle) = 0$. It will be convenient to let $\varphi(\alpha,\beta)$ denote $\varphi(\varrho_0(\alpha,\beta))$. The following lemma captures the property of $\varphi : [\omega_1]^2 \to \mathbb{Z}$ which we will be interested in using.

Lemma 2.8. *Let $\delta < \omega_1$ be a limit ordinal and let β_i $(i < m)$ be countable ordinals greater than δ such that $\varrho_1(\delta,\beta_i)$ $(i < m)$ are all* *distinct. There is a finitely supported $\sigma : \omega \to \mathbb{Z}$ and a $\bar{\delta} < \delta$ such that whenever $\eta(|C_\delta \cap \alpha|) = \sigma$ and $\bar{\delta} < \alpha < \delta$, it follows that*

$$\varphi(\alpha, \beta_i) = \varphi(\alpha, \beta_0) + i$$

for all $i < m$.

Remark 2.9. This lemma holds for ϱ-functions derived from an arbitrary C-sequence and does not utilize the assumption $\mho$.

Proof. Let $\bar{\delta} < \delta$ satisfy the following conditions:

(1) $\bar{\delta}$ is an upper bound for all $\xi < \delta$ such that $\varrho_1(\xi, \beta_i) \neq \varrho_1(\xi, \beta_{i'})$ for some $i < i' < m$.

(2) If $\bar{\delta} < \alpha < \delta$ and $i < m$, then $\mathrm{Tr}(\delta, \beta_i) \subseteq \mathrm{Tr}(\alpha, \beta_i)$.

Let $\sigma : \omega \to \mathbb{Z}$ be finitely supported such that for each $i < m$

$$\sigma(\varrho_1(\delta, \beta_i)) = i - \varphi(\delta, \beta_i) + \varphi(\delta, \beta_0);$$

such a σ exists since $\varrho_1(\delta, \beta_i) \neq \varrho_1(\delta, \beta_{i'})$ if $i \neq i'$.

It now suffices to check that σ and $\bar{\delta}$ work. Let α be such that $\bar{\delta} < \alpha < \delta$ and $\eta(|C_\delta \cap \alpha|) = \sigma$. Observe that for $i < m$, δ is in $\mathrm{Tr}(\alpha, \beta_i)$ and the following quantities do not depend on i: $\mathrm{Tr}(\alpha, \beta_i) \cap \delta$, and the restriction of $\varrho_1(\cdot, \beta_i)$ to the interval (α, δ). For a given $i < m$,

$$\varphi(\alpha, \beta_i) = \varphi(\delta, \beta_i) + \sigma(\varrho_1(\delta, \beta_i)) + \sum_{\xi \in \mathrm{Tr}(\alpha, \beta_i) \cap \delta} \Big[\eta(|C_\xi \cap \alpha|)\Big](\varrho_1(\xi, \beta_i)).$$

Notice that the last summand does not depend on i by the comments made above and

$$\varphi(\delta, \beta_i) + \sigma(\varrho_1(\delta, \beta_i)) = i + \varphi(\delta, \beta_0)$$

by arrangement. Hence $\varphi(\alpha, \beta_i) = \varphi(\alpha, \beta_0) + i$ for all $i < m$. $\square$

Before proceeding with the proof of Theorem 1.1, I will first demonstrate the following property of φ. It is both of independent interest and contains the main elements of the more involved proof to come.

Proposition 2.10. *($\mho$) If A and B are uncountable subsets of ω_1, then there exist $\alpha \in A$ and $\beta, \beta' \in B \setminus \alpha$ such that $\varphi(\alpha, \beta)$ is even and $\varphi(\alpha, \beta')$ is odd.*

Proof. Let $E \subseteq \omega_1$ be a closed unbounded set with the property that every element of E is a limit point of A and if δ is in E, then

$$\{\varrho_1(\delta, \beta) : \beta \in B \setminus \delta\}$$

is infinite. Since $\langle C_\alpha : \alpha < \omega_1 \rangle$ was derived from a $\mho$-sequence, there is a δ in E such that for every finitely supported $\sigma : \omega \to \mathbb{Z}$ and every $\bar{\delta} < \delta$, there is a ν in $E \cap (\bar{\delta}, \delta)$ such that $\eta(|C_\delta \cap \nu|) = \sigma$.

Now let β_0 and β_1 be two elements of $B \setminus \delta$ such that $\varrho_1(\delta,\beta_0) \neq \varrho_1(\delta,\beta_1)$. By Lemma 2.8, there is a finitely supported $\sigma : \omega \to \mathbb{Z}$ and a $\bar{\delta} < \delta$ such that if $\bar{\delta} < \alpha < \delta$ and $\eta(|C_\delta \cap \alpha|) = \sigma$, then $\varphi(\alpha,\beta_1) = \varphi(\alpha,\beta_0) + 1$. By choice of δ, there is a ν in $E \cap (\bar{\delta},\delta)$ such that $\eta(|C_\delta \cap \nu|) = \sigma$. Since ν is a limit point of A, there is an α in $A \cap (\bar{\delta},\delta)$ such that $\eta(|C_\delta \cap \alpha|) = \sigma$ and hence $\varphi(\alpha,\beta_1) = \varphi(\alpha,\beta_0) + 1$. If i is such that $\varphi(\alpha,\beta_i)$ is even, then let $\beta = \beta_i$ and $\beta' = \beta_{1-i}$. $\square$

By generalizing Lemma 2.6 and adapting the above arguments appropriately, one can prove the following strengthening of Proposition 2.10; we leave the details to the interested reader. This should be compared to Theorem 1.5 of [6].

Theorem 2.11. *($\mho$) If $\mathscr{A} \subseteq [\omega_1]^k$ and $\mathscr{B} \subseteq [\omega_1]^l$ are uncountable families of pairwise disjoint sets, then for every $m < \omega$, there is an a in $\mathscr{A}$ and b_x $(x < m)$ in $\mathscr{B}$ with, for all $i < k$, $j < l$, and $x < m$*

$$a(i) < b_x(j)$$

$$\varphi(a(i), b_x(j)) = \varphi(a(i), b_0(j)) + x.$$

Put in the context of [6, §5], this yields an example of an L space.

3. Using φ to define an Aronszajn line with no Countryman suborder

Todorcevic showed in [9] that $C(\varrho_0)$ – which is the set

$$\{\varrho_0(\cdot,\beta) : \beta < \omega_1\}$$

ordered lexicographically — is a Countryman line. In fact, under mild assumptions, $C(\varrho_0)$ and its reverse form a two element basis for the class of all Countryman lines.

Theorem 3.1. [10, 2.1.13] *($\mathrm{MA}_{\aleph_1}$) Every Countryman order contains an isomorphic copy of either $C(\varrho_0)$ or $-C(\varrho_0)$.*

Furthermore, we have the following lemma which reduces the task of proving Theorem 1.1 to demonstrating the existence of a certain pathological partition of

$$T(\varrho_0) = \{\varrho_0(\cdot,\beta) \restriction \alpha : \alpha \leq \beta < \omega_1\}.$$

Suppose that K is a subset of $T(\varrho_0)$. We can define a linear order $\leq_K$ on $T(\varrho_0)$ so that $\leq_K$ and $\leq_{\mathrm{lex}}$ agree on a pair $\{\varrho_0(\cdot,\alpha), \varrho_0(\cdot,\beta)\}$ iff $\varrho_0(\cdot,\alpha) \wedge \varrho_0(\cdot,\beta)$ is not in K. It is not difficult to show that this defines an Aronszajn order.

Lemma 3.2. [8, 8.7] *($\mathrm{MA}_{\aleph_1}$) Suppose $K \subseteq T$. If for every uncountable $X \subseteq \omega_1$ there are $\alpha, \alpha', \beta, \beta'$ in X with*

$$\varrho_0(\cdot,\alpha) \wedge \varrho_0(\cdot,\beta) \in K$$
$$\varrho_0(\cdot,\alpha') \wedge \varrho_0(\cdot,\beta') \notin K,$$

then $(T(\varrho_0), \leq_K)$ does not contain a Countryman suborder.

I will prove momentarily that φ can be used to define a $K \subseteq T(\varrho_0)$ which will have the property stated in Lemma 3.2, so long as the C-sequence used in the definition of ϱ_0 is derived from a $\mho$-sequence. To see this is sufficient to prove Theorem 1.1, assume V is a model of $\mathrm{ZFC} + \mho$ and go into a c.c.c. forcing extension $V[G]$ which satisfies $\mathrm{MA}_{\aleph_1}$. In $V[G]$, K still satisfies the hypothesis of Lemma 3.2 and hence $(T(\varrho_0), \leq_K)$ does not contain a Countryman suborder by Theorem 3.1 and Lemma 3.2. Since any Countryman suborder of $(T(\varrho_0), \leq_K)$ in V would remain Countryman in $V[G]$, it follows that $(T(\varrho_0), \leq_K)$ did not have a Countryman suborder in V.

Define $K \subseteq T(\varrho_0)$ by putting τ in K iff τ has successor height $\zeta + 1$ and $\varphi(\tau(\zeta))$ is even. We will need the following lemma. Let $\Delta(\alpha, \beta)$ denote the least ξ such that $\varrho_0(\xi, \alpha) \neq \varrho_0(\xi, \beta)$.

Lemma 3.3. *If $X \subseteq \omega_1$ is uncountable, then there is a club $E \subseteq \omega_1$ such that if δ is in E, $\bar{\delta} < \delta$, and β_i $(i < m)$is a finite sequence in $X \setminus \delta$, then there are β_i' $(i < m)$ in $X \cap \delta$ with*

$$\bar{\delta} < \Delta(\beta_0, \beta_0') = \Delta(\beta_i, \beta_i') < \delta$$

whenever $i < m$.

Proof. If $\nu < \omega_1$, choose $f(\nu) < \omega_1$ so that for all countable ordinals δ and β_i $(i < m)$, if:

(1) $\nu < \delta \leq \beta_i$ for $i < m$;
(2) β_i is in X for each $i < m$;
(3) δ is in $\mathrm{Tr}(\nu, \beta_i)$ for each $i < m$,

then there are δ', β_i' $(i < m)$ satisfying the same conditions such that additionally:

(1) each are less than $f(\nu)$;
(2) $\varrho_0(\cdot, \beta_i) \restriction \nu = \varrho_0(\cdot, \beta_i') \restriction \nu$ for each $i < m$;
(3) $\varrho_0(\delta', \beta_i') = \varrho_0(\delta, \beta_i)$ for each $i < m$;
(4) $\varrho_0(\nu, \delta) = \varrho_0(\nu, \delta')$.

Notice that $f(\nu)$ exists since $T(\varrho_0)$ has countable levels and ϱ_0 takes values in a countable set.

It suffices to show that if E is the set of all limit ordinals δ which are closed under f, then E satisfies the conclusion of the lemma. To this end, let δ be in E, $\bar{\delta} < \delta$, and let β_i $(i < m)$ be elements of $X \setminus \delta$. By increasing $\bar{\delta}$, we may assume that δ is in $\mathrm{Tr}(\bar{\delta}, \beta_i)$ for all $i < m$. Let δ' and β_i' $(i < m)$ be ordinals satisfying 1–4 for $\nu = \bar{\delta}$. Clearly $\varrho_0(\delta', \beta_i) \neq \varrho_0(\delta', \beta_i')$ since $\varrho_0(\cdot, \beta_i)$ is one-to-one by Fact 2.3. Hence $\bar{\delta} < \Delta(\beta_i, \beta_i') \leq \delta' < \delta$. Furthermore, by Fact 2.2,

$$\varrho_0(\xi, \beta_i) = \varrho_0(\delta, \beta_i)\,\hat{}\,\varrho_0(\xi, \delta)$$

$$\varrho_0(\xi, \beta_i') = \varrho_0(\delta', \beta_i')\,\hat{}\,\varrho_0(\xi, \delta')$$

whenever $\bar{\delta} < \xi < \delta'$. It follows that $\Delta(\beta_i, \beta_i')$ does not depend on i. □

It is now sufficient to prove that the K defined above satisfies the hypothesis of Lemma 3.2. Let X be given and select a club $E \subseteq \omega_1$ witnessing the conclusion of Lemma 3.3 for this X such that, moreover,

$$\{\varrho_1(\delta, \beta) : \beta \in X \setminus \delta\}$$

is infinite for all δ in E. Using that $\langle C_\alpha : \alpha < \omega_1 \rangle$ was derived from a $\mho$-sequence, it is possible to select a δ in E such that for all finitely supported $\sigma : \omega \to \mathbb{Z}$ and $\bar{\delta} < \delta$, there is a ν in E with $\eta(|C_\delta \cap \nu|) = \sigma$. Let β_0 and β_1 be elements of $X \setminus \delta$ such that $\varrho_1(\delta, \beta_0) \neq \varrho_1(\delta, \beta_1)$. By Lemma 2.8, there is a finitely supported $\sigma : \omega \to \mathbb{Z}$ such that if $\bar{\delta} < \alpha < \delta$ and $\eta(|C_\delta \cap \alpha|) = \sigma$, then $\varphi(\alpha, \beta_1) = \varphi(\alpha, \beta_0) + 1$. Let ν be an element of E with $\bar{\delta} < \nu < \delta$ and $\eta(|C_\delta \cap \nu|) = \sigma$. Let $\bar{\nu} < \nu$ be such that $C_\delta \cap \bar{\nu} = C_\delta \cap \nu$. By choice of E, there are α_0 and α_1 in X such that

$$\bar{\nu} < \Delta(\alpha_0, \beta_0) = \Delta(\alpha_1, \beta_1) < \nu$$

and hence if $\zeta + 1 = \Delta(\alpha_0, \beta_0)$, then $C_\delta \cap \zeta = C_\delta \cap \nu$. Putting this all together, we have that

$$\varphi(\zeta, \alpha_i) = \varphi(\zeta, \beta_i),$$

$$\varphi(\zeta, \alpha_1) = \varphi(\zeta, \alpha_0) + 1$$

and hence $\varrho_0(\cdot, \alpha_0) \wedge \varrho_0(\cdot, \beta_0)$ is in K iff $\varrho_0(\cdot, \alpha_1) \wedge \varrho_0(\cdot, \beta_1)$ is not in K. This finishes the proof.

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