MATH 6870: SET THEORY

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1. Introduction

Why set theory? A bit over a century ago, there was a push to put mathematics on a rigorous, axiomatic foundation. Since the concepts of set and membership were so primitive and since more complex objects such as the real line and Euclidean space can be synthesized using set-theoretic constructions, sets made for a natural domain to carry out the axiomatization.

This is not to say that mathematics is naturally or canonically encoded inside of set theory, just that it *can* be encoded. Nor does set theory typically inform us with intuition as to how to resolve *most* mathematical problems. Category theory is often said to provide a competing foundation for mathematics. This is incorrect. Rather it provides a *complementary* foundation for mathematics. Category theory provides a high level language and methodology that facilitates mathematical thought and intuition. Set theory provides the low level mechanics of mathematics which allows careful and precise analysis when intuition fails to inform or to yield a rigorous justification.

If one were to draw an analogy with computer science, category theory would be object oriented programming whereas set theory would be machine language and hardware. Both are essential aspects of computer science; each complements the other. The same is true of the foundational roles of set theory and category theory. In the physical sciences, category theory would correspond to organic chemistry whereas set theory would correspond to quantum mechanics and (sub)atomic processes. Notice that, aside from the locations of elements on the periodic table, quantum effects to are not typically required to understand chemical reactions.

Set theory is in practice the study of the combinatorics and discrete mathematics of infinite sets. In many cases, the basic axioms of set theory — ZFC — are not sufficient to completely analyze questions of this nature. As a result, there is a need to develop meta-mathematical tools as well — tools to decide when set-theoretic statements are consistent with the standard axioms. Here are some examples of mathematical questions which are fundamentally set-theoretic in nature.

Problem 1. *Is every automorphism of the Boolean ring* $\mathscr{P}(\mathbb{N})/\mathrm{fin}$ *induced by a function from* $\mathbb{N}$ *to* $\mathbb{N}$*?*

Problem 2. *If* A *is an abelian group such that* $\mathrm{Ext}^1(A,\mathbb{Z}) = 0$*, must* A *be a free abelian group? (*$\mathrm{Ext}^1(A,\mathbb{Z}) = 0$ *is equivalent to the assertion that whenever* $h : B \to A$ *is a surjective homomorphism with kernel* $\mathbb{Z}$*, there is a* $g : A \to B$ *such that* $h \circ g$ *is the identity.)*

Problem 3. *Suppose that G is a graph on a separable metric space X such that the adjacency relation is open as a subset of X^2. Must G either be countably chromatic or else contain an uncountable clique?*

Problem 4. *Is there a partition of a square into finitely many Borel pieces such that these pieces can be rearranged through rigid motions of the plane to partition a disc?*

Problem 5. *Suppose $\Gamma \subseteq 2^{\mathbb{N}}$ is a Borel set. If two players alternately play digits of a sequence $a \in 2^{\mathbb{N}}$, does either the first player have a strategy to force the outcome to be in Γ or the second player have a strategy to force the outcome to be in the complement of Γ? What if Γ is the continuous image of a Borel set?*

Problem 6. *Is there a countably additive probability measure μ defined on all subsets of a set X such that $\mu(\{x\}) = 0$ for every $x \in X$? What about the case $X = [0, 1]$?*

Problem 7. *Does every infinite dimensional Banach space have an infinite dimensional quotient with a basis?*

The first three problems are neither provable nor refutable based on the axioms of ZFC. The fourth problem and the first part of the fifth have positive answers. The sixth problem and the second part of the fifth problem can not be proved within ZFC but can be proved using strong forms of the *Axiom of Infinity*. The last problem is open but seems to be essentially set-theoretic in nature.

The goals of this course are:

- Introduce the axioms of set theory and briefly outline how set-theory can be used to synthesize mathematical constructions.
- Introduce basic tools in set theory: the ordinals and cardinals, transfinite induction and recursion, stationary sets and the pressing down lemma.
- Present two axiomatic extensions of ZFC — $\diamondsuit$ and Martin's Axiom — an discuss their mathematical consequences.
- Develop Cohen's method of forcing and use it to establish the independence of CH from ZFC. Forcing will also be used to establish the consistency of $\diamondsuit$ and MA.
- Present Solovay's model of ZF in which all sets of reals are Lebesgue measurable.

Time permitting, we will cover material on Martin's Maximum, a strengthening of Martin's Axiom. While we will not be able to completely solve the problems mentioned above, students will be equipped to understand the solutions most of these problems by the end of the course.

2. The Axioms of Zermelo-Frankel (ZF) Set Theory

We'll now begin the process of formalizing the axioms of set theory. To illustrate why some care is needed, consider the following: does

$$\{x \mid x \notin x\}$$

describe a set S? If it did, then $S \in S$ if and only if $S \notin S$, which is absurd. This is known as Russell's Paradox.

Just as the axioms of group theory describes the properties of a single binary operation $*$ and the axioms of partial orders describes a single binary relation $\leqslant$, the language of set theory describes the properties of a binary relation $\in$.

The first axiom asserts that two sets are equal if they have the same elements.

Axiom 1 (Extensionality). $\forall x \forall y((\forall z((z \in x) \leftrightarrow (z \in y))) \rightarrow (x = y))$

As natural as this may seem, it has the effect of asserting that *everything* under discussion is a set. This is unnatural if we think ahead to our goal of using sets to model all of mathematics: is π a set? is the ordered pair $(2, 3)$? What are their elements? If they don't have elements, why aren't they the emptyset? We'll return to this later.

The next axiom asserts that the emptyset exists.

Axiom 2 (Emptyset).

$$\exists x \forall y(y \notin x)$$

The Axiom of Extensionality implies that the x postulated by this axiom is unique; we will denote it by $\varnothing$.

The next axiom asserts that if x and y are sets, then there is a set, which has exactly x and y as its elements.

Axiom 3 (Pairing).

$$\forall x \forall y \exists z((x \in z) \wedge (y \in z) \wedge \forall u((u \in z) \rightarrow ((u = x) \vee (u = y))))$$

As with $\varnothing$, the z postulated by this axiom for a given x and y is unique; it is denoted $\{x, y\}$. Notice that Extensionality also implies that $\{x, y\} = \{y, x\}$.

It's perhaps worth pausing to point out that the convention of having the language of set theory consist only of binary relation $\in$ for membership is one of economy and not of convenience (or is it the other way around?). There are many set theoretic operations and constants which are definable from $\in$ using the axioms of set theory. In practice we work and write in an enriched language which includes symbols such as $\varnothing$ and the binary function $\{x, y\}$. This ambiguity in the language is analogous the issue of whether the language of group theory includes formal symbols for the identity and inversion. More involved examples in set theory include the Cartesian product $\times$, which is a binary operation, and logical constants for things like π, e, $\mathbb{Z}$ and $\mathbb{R}$. This language can always be converted into the more minimalist language of set theory (though often with great pain!) and when proving results by induction on formulas, it is often useful to appeal to this fact. Typically, however, we will work with the enriched language.

The next axiom asserts that the union of a set x is again a set.

Axiom 4 (Union)**.**

$$\forall x \exists y \forall z ((z \in y) \leftrightarrow \exists u ((u \in x) \wedge (z \in u)))$$

The y postulated by the axiom for a given x is denoted $\bigcup x$ (again, uniqueness is ensured by Extensionality). If x and y are sets, we will use $x \cup y$ to denote $\bigcup\{x, y\}$. That is $\bigcup$ denotes a unary operation on sets and $\cup$ denotes a binary operation.

Observe that the pairing and union axiom can be combined to show that if $x_0, \ldots, x_{n-1}$ is a finite list of sets, then $\{x_0, \ldots, x_{n-1}\}$ is also a set. Also, if x and y are sets, then we define the ordered pair $(x, y) := \{\{x\}, \{x, y\}\}$. It can be shown that $(x, y) = (x', y')$ if and only if $x = x'$ and $y = y'$.

Next we turn to the powerset axiom. It will be useful to define $x \subseteq y$ is an abbreviation for $\forall z ((z \in x) \rightarrow (z \in y))$.

Axiom 5 (Powerset)**.**

$$\forall x \exists y \forall z ((z \subseteq x) \leftrightarrow (z \in y))$$

The set y postulated by this axiom is called the *powerset of* x and is uniquely determined by x. It is denoted $\mathscr{P}(x)$.

The next "axiom" is actually an axiom scheme — an infinite family of axioms parametrized by logical formulas. Throughout this course, if ϕ is a logical formula then we will write $\phi(v_0, \ldots, v_{n-1})$ indicate that the free variables in ϕ are among $v_0, \ldots, v_{n-1}$. We will also write, e.g., $\bar{v}$ as shorthand for $v_0, \ldots, v_{n-1}$, in contexts like $\phi(\bar{v})$ or $\exists \bar{v} \phi(\bar{v})$.

Axiom 6 (Separation Scheme)**.** *If $\phi(y, \bar{w})$ is a formula in the language of set theory, then the following is an axiom*

$$\forall x \forall \bar{w} \exists z \forall y ((y \in z) \leftrightarrow ((y \in x) \wedge \phi(y, \bar{w}))).$$

For a given x and $\bar{w}$, the witnessing z is typically denoted

$$\{y \in x \mid \phi(y, \bar{w})\}.$$

If A and B are sets, then the *Cartesian product* of A and B is defined by

$$A \times B := \{(a,b) \mid (a \in A) \wedge (b \in B)\}.$$

That this set exists based on the axioms will be left as a homework exercise.

The separation scheme can be viewed as saying that subsets of sets exist if they can be described. The next axiom scheme addresses "collecting" the image of a set under a description of a function.

Axiom 7 (Collection Scheme)**.** *If $\phi(x, y, \bar{w})$ is a formula in the language of set theory then the following is an axiom*

$$\forall X \forall \bar{w} (\forall x \in X\ \exists! y \phi(x, y, \bar{w}) \rightarrow \exists Y \forall x \in X\ \exists y \in Y\ \phi(x, y, \bar{w}))$$

When combined with separation, it asserts that

$$\{y \mid \exists x \in X\ \phi(x, y, \bar{w})\}$$

describes a set whenever $\phi(x, y, \bar{w})$ is a formula, X is a set and $\bar{w}$ is a tuple of sets. Moreover, we can apply Separation to conclude that

$$f = \{(x, y) \in X \times Y \mid \phi(x, y, \bar{w})\}$$

forms a set and hence a function. That is, a function on X can be specified by a formula which defines it.

The next axiom asserts a key feature of $\in$: that it is *well founded.* A binary relation is *well founded* if every nonempty subset has a minimal element.

Axiom 8 (Foundation)**.**

$$\forall x((x = \varnothing) \vee \exists y \in x\ (x \cap y = \varnothing))$$

The significance of this axiom is in part that it will afford us an understanding of how models of set theory are structured. We'll wait to state the final ZF axiom — the Axiom of Infinity — until as have introduced some further definitions.

3. Classes, Ordinals, and Transfinite Induction

At this point it is worthwhile to discuss the difference between sets and classes. Formally speaking, a class simply a formula $\phi(v, \bar{a})$ where v is a free variable and $\bar{a}$ is a finite sequence of sets. We think of $\phi(v, \bar{a})$ as describing the class $\{x \mid \phi(x, \bar{a})\}$ of all sets x such that $\phi(x, \bar{a})$ is true. This collection may or may not be a set, as we have seen with Russell's Paradox.

If x is a set, then the formula $v \in x$ describes x itself. In particular, every set is a class. On the other hand, the formula $v = v$ describes the class of all sets. Since the Axiom of Foundation implies that no set is an element of itself, this class does not correspond to a set.

Theorem 3.1. *For all x, $x \notin x$.*

Proof. Let x be given and apply the Axiom of Foundation to $y = \{x\}$ to obtain a $z \in y$ such that $z \cap y = \varnothing$. Since the only element of y is x, this translates to $x \cap \{x\} = \varnothing$ or $x \notin x$. □

Often capital letters in boldface are used to denote classes. For instance $\mathbf{V}$ denotes the class of all sets. This notation is meant to suggest that classes should be thought of as collections while alerting the reader to the possibility that they need not be sets. We will sometimes use "class" as an adjective — e.g. "class relation", "class function" — to alert the reader that something under discussion may not be a set. For example, a class function is a class $\mathbf{F}$ such that the elements of $\mathbf{F}$ are ordered pairs and for each set x there is at most one y such that $(x, y) \in \mathbf{F}$. The domain of a class function $\mathbf{F}$ is the class of all x such that for some y $(x, y) \in \mathbf{F}$. If x is in the domain of $\mathbf{F}$, we write $\mathbf{F}(x)$ to denote the unique y such that $(x, y) \in \mathbf{F}$.

Unlike sets, the axioms of ZF do not directly address the properties of classes. The Separation and Collection schemes can be viewed as indirectly making assertions about the relationship between classes and sets. Specifically, the Separation scheme can be interpreted as asserting that the intersection of a set and a class is a set. The Collection scheme is equivalent to saying that a class function defined on a set domain is a function.

A set x is *transitive* if whenever $y \in x$ and $z \in y$, $z \in x$. Phrased in another way, x is transitive if each of its elements is also a subset of x. An *ordinal* is a transitive set α which is *well ordered* by $\in$ — every nonempty subset of α has an $\in$-minimum element. We will write $\mathbf{ON}$ to denote the class of all ordinals. Informally, an ordinal is the set of ordinals which are smaller than it. The following theorems bear this intuition out.

Theorem 3.2. *An element of an ordinal is an ordinal.*

Proof. Suppose α is an ordinal and $\beta \in \alpha$. Since α is transitive, $\beta \subseteq \alpha$ and in particular, β is well ordered by $\in$. If $\gamma \in \beta$ and $\delta \in \gamma$, then since $(\alpha, \in)$ is transitive, $\delta \in \beta$. Thus β is also transitive and hence an ordinal. $\square$

Theorem 3.3. *If α and β are ordinals, then one of the following must be true: $\alpha \in \beta$, $\alpha = \beta$, or $\beta \in \alpha$.*

Proof. Suppose for contradiction that the theorem is false and let (α, β) be a counterexample. Using the fact that both α and β are well ordered by $\in$, we may assume that additionally if $\alpha' \in \alpha$ then (α', β) satisfies the conclusion of the theorem and if $\beta' \in \beta$ then (α, β') satisfies the conclusion of the theorem. We'll refer to such a pair (α, β) as a *minimal counterexample.*

Suppose that $\gamma \in \alpha$. By our assumption, we know that either $\gamma \in \beta$, $\gamma = \beta$, or $\beta \in \gamma$. Since $\beta \notin \alpha$, we know that $\gamma \neq \beta$. Also, if $\beta \in \gamma$, then since α is transitive, we would have that $\beta \in \alpha$, which we have also assumed is not possible. Thus $\alpha \subseteq \beta$. Note however that if (α, β) is an minimal counterexample, then so is (β, α). Thus we also have $\beta \subseteq \alpha$ and hence, by Extensionality, $\alpha = \beta$. $\square$

If α and β are ordinals, we will write $\alpha < \beta$ to mean $\alpha \in \beta$ and $\alpha \leqslant \beta$ to mean $\alpha \in \beta$ or $\alpha = \beta$. The previous theorem asserts that $\leqslant$ is a class linear order on **ON**. If α is an ordinal, define $\alpha + 1 = \alpha \cup \{\alpha\}$. Notice that $\alpha < \alpha + 1$ and that if β is an ordinal greater than α, then it must be that $\alpha + 1 \leqslant \beta$. An ordinal of the form $\alpha + 1$ is called a *successor ordinal.* All other nonzero ordinals are called *limit ordinals.*

We will now pause to state the final axiom of ZF.

Axiom 9 (Infinity)**.** *There is a limit ordinal.*

Given that there is a limit ordinal, there is a least limit ordinal. This ordinal is denoted ω. We also note the following corollary to Theorem 3.3. It is known as the Burali-Forti Paradox.

Corollary 3.4. *The class of all ordinals is not a set.*

Proof. If the class of all ordinals was a set α, then Theorem 3.3 implies that it would be an ordinal. But then $\alpha \in \alpha$ by Theorem 3.1. $\square$

The next two "theorems" formalize transfinite induction and recursion. In fact, each is a family of theorems, one for each class.

Theorem 3.5 (Transfinite Induction)**.** *If* $\mathbf{C} \subseteq \mathbf{ON}$ *is nonempty, then* $\mathbf{C}$ *has a least element.*

Proof. Let $\alpha \in \mathbf{C}$. If $\alpha \cap \mathbf{C}$ is empty, then Theorem 3.3 implies that $\alpha \leqslant \beta$ whenever $\beta \in \mathbf{C}$. If $\alpha \cap \mathbf{C}$ is nonempty, then it is a nonempty subset of α. Since α is well ordered by $\in$, $\alpha \cap \mathbf{C}$ has a least element γ. If $\beta \in \mathbf{C}$, then Theorem 3.3 implies that either $\gamma \leqslant \beta$ or else $\beta \in \gamma$. If $\beta \in \gamma$, then since α is a transitive set, $\beta \in \alpha \cap \mathbf{C}$, which contradicts that γ was minimal. It follows that γ is the least element of $\mathbf{C}$. $\square$

4. Well Founded Relations and Transfinite Recursion

We will now set up a very general framework for making recursive definitions. This is a cornerstone to set theory and will be used throughout the course, often in subtle ways.

Suppose that $\mathbf{R}$ is a binary class relation. We say that $\mathbf{R}$ is *set-like* if for every y, $\mathbf{R}^y := \{x \mid (x, y) \in \mathbf{R}\}$ is a set. Notice that $\in$ is a set-like relation. A relation $\mathbf{R}$ is *well founded* if for every nonempty set x, there is a $y \in x$ such that for all $z \in x$, (z, y) is not in $\mathbf{R}$ — i.e. every nonempty class has an $\mathbf{R}$-minimal element. Note that the Axiom of Foundation simply asserts that $\in$ is well-founded.

Theorem 4.1 (Transfinite Recursion)**.** *Suppose $\mathbf{A}$ is a class and $\mathbf{R}$ is a set-like well-founded relation on $\mathbf{A}$. For every class function $\mathbf{F} : \mathbf{A} \times \mathbf{V} \to \mathbf{V}$ there is a unique class function $\mathbf{G} : \mathbf{A} \to \mathbf{V}$ such that for all $a \in \mathbf{A}$, $\mathbf{G}(a) = \mathbf{F}(a, \mathbf{G} \restriction \mathbf{R}^a)$.*

Notice that since we do not require that $\mathbf{R}$ is extensional in this theorem, $\mathbf{R}^a$ does not uniquely determine a and thus allowing a as an input to $\mathbf{F}$ adds some additional generality. Before we prove this theorem, we will need to prove two lemmas concerning set-like relations.

Lemma 4.2. *If $\mathbf{R}$ is a set-like relation and x is a set, then there is a set X such that $x \in X$ and whenever y is in X, $\mathbf{R}^y \subseteq X$.*

Proof. Let $\phi(n, x, y)$ be the assertion $n \in \omega$ and there exists a $k \leqslant n$ and a function s with domain n such that $s(0) = y$, $s(n-1) = x$ and for all $i \in n-1$, $(s(i), s(i+1)) \in \mathbf{R}$. We claim that for all $n \in \omega$ there is a unique set X_n such that $y \in X_n$ if and only if $\phi(n, x, y)$. This is proved by induction on n. If $n = 0$, then $X_0 = \{x\}$. Also $X_{n+1} = \bigcup\{\mathbf{R}^y \mid y \in X_n\}$, which is a set by our inductive assumption, the union axiom, and the assumption that $\mathbf{R}$ is set-like. By Replacement, $\{X_n \mid n \in \omega\}$ is a set and by Union, $X := \bigcup\{X_n \mid n \in \omega\}$ is a set. □

Also observe that, by taking an intersection, there is always a $\subseteq$-minimum set X satisfying the conclusion of this lemma (in fact this is the set constructed in the proof). This will be called the *transitive $\mathbf{R}$-closure* of x. If $\mathbf{R}$ is the membership relation, we will simply refer to it as the *transitive closure of* x and denote it $\mathrm{tc}(x)$. It is the $\subseteq$-minimum transitive set which has x as an element.

Lemma 4.3. *Suppose $\mathbf{R}$ is a well-founded set-like class relation. Every nonempty class has an $\mathbf{R}$-minimal element.*

Proof. Suppose that $\mathbf{X}$ is a nonempty class and let $x \in \mathbf{X}$. Define Y to be the transitive $\mathbf{R}$-closure of x. By Separation, $\mathbf{X} \cap Y$ is a set. Since

$x \in \mathbf{X} \cap Y$, $\mathbf{X} \cap Y$ has an $\mathbf{R}$-minimal element y. Since $\mathbf{R}^y \subseteq Y$, it must be that y is an $\mathbf{R}$-minimal element of $\mathbf{X}$. $\square$

Proof of Theorem 4.1. Fix $\mathbf{F} : \mathbf{A} \times \mathbf{V} \to \mathbf{V}$ and $\mathbf{R}$ as in the statement of the theorem. An *approximation* is a function g such that:

- the domain of g is contained in $\mathbf{A}$;
- if x is in the domain of g, then $\mathbf{R}^x$ is contained in the domain of g and $g(x) = \mathbf{F}(x, g \restriction \mathbf{R}^x)$.

We will first show that if g and h are approximations and x is in the domain of both, then $g(x) = h(x)$. Suppose for contradiction that this is not the case and let x be an $\mathbf{R}$-minimal counterexample. Notice though that this implies that $g \restriction \mathbf{R}^x = h \restriction \mathbf{R}^x$ and hence

$$g(x) = \mathbf{F}(x, g \restriction \mathbf{R}^x) = \mathbf{F}(x, h \restriction \mathbf{R}^x) = h(x),$$

which is a contradiction.

Define $\mathbf{G}$ to be all (a, b) which are in some approximation. By the previous observation, $\mathbf{G}$ is a class function defined on a subclass of $\mathbf{A}$. It suffices to show that the domain of $\mathbf{G}$ is all of $\mathbf{A}$. Again suppose for contradiction that this is not the case and let $a \in \mathbf{A}$ be $\mathbf{R}$-minimal with respect to not being in the domain of an approximation. Let A be the transitive $\mathbf{R}$-closure of a. By minimality of a, $A \backslash \{a\}$ is contained in the domain of $\mathbf{G}$. Moreover, $\mathbf{G} \restriction (A \backslash \{a\})$ is an approximation — that this is a function follows from Collection. Define g to be the extension of $\mathbf{G} \restriction (A \backslash \{a\})$ to A defined by $g(a) = \mathbf{F}(a, \mathbf{G} \restriction \mathbf{R}^a)$, noting that $\mathbf{G} \restriction \mathbf{R}^a$ is a restriction of $\mathbf{G} \restriction (A \backslash \{a\})$. Since g is an approximation which is defined at a, we have a contradiction. It must therefore be that the domain of $\mathbf{G}$ is all of $\mathbf{A}$. $\square$

The special class of the Transfinite Recursion Theorem in which $\mathbf{A}$ is $\mathbf{ON}$ and $\mathbf{R}$ is $\in$ is important and can be stated as follows.

Theorem 4.4. *For every class function $\mathbf{F} : \mathbf{V} \to \mathbf{V}$ there is a unique class function $\mathbf{G} : \mathbf{ON} \to \mathbf{V}$ such that for all $\alpha \in \mathbf{ON}$, $\mathbf{G}(\alpha) = \mathbf{F}(\mathbf{G} \restriction \alpha)$.*

A *sequence* is a function whose domain is an ordinal. The domain of a sequence is called its *length*. Observe that, in the statement of this theorem, it is only relevant what the values of $\mathbf{F}$ are on sequences. The function $\mathbf{F}$ can be thought of as specifying a recursive rule which described a class length sequence $\mathbf{G}$: it tells you how to compute $\mathbf{G}(\alpha)$ given $\mathbf{G} \restriction \alpha$.

5. Ordinal arithmetic

We will now turn to ordinal arithmetic. Note that if A is a set of ordinals, then $\bigcup A$ is the least upper bound of A. We will denote this by $\sup A$. For a fixed α, define the class function $\beta \mapsto \alpha+\beta$ recursively by:

$$\alpha + \beta := \begin{cases} \alpha & \text{if } \beta = 0 \\ (\alpha+\gamma)+1 & \text{if } \beta = \gamma+1 \\ \sup\{\alpha+\gamma \mid \gamma \in \beta\} & \text{if } \beta \text{ is a limit ordinal} \end{cases}$$

It is easily checked that $+$ agrees with (or rather formalizes) ordinary addition on the finite ordinals. Note, however, that $+$ is not commutative. For example,

$$1+\omega = \sup\{1+n \mid n \in \omega\} = \omega < \omega+1.$$

Observe that if $\alpha \leqslant \beta$, then there is a unique γ such that $\alpha+\gamma=\beta$: γ is the greatest ordinal such that $\alpha+\gamma \leqslant \beta$.

Similarly, one can recursively define multiplication and exponentiation:

$$\alpha \cdot \beta := \begin{cases} 0 & \text{if } \beta = 0 \\ (\alpha\cdot\gamma)+\alpha & \text{if } \beta = \gamma+1 \\ \sup\{\alpha\cdot\gamma \mid \gamma \in \beta\} & \text{if } \beta \text{ is a limit ordinal} \end{cases}$$

$$\alpha^\beta := \begin{cases} 1 & \text{if } \beta = 0 \\ (\alpha^\gamma)\cdot\alpha & \text{if } \beta = \gamma+1 \\ \sup\{\alpha^\gamma \mid \gamma \in \beta\} & \text{if } \beta \text{ is a limit ordinal} \end{cases}$$

Proposition 5.1. *The operations $+$ and $\cdot$ are associative on* **ON**.

Proof. As the arguments are similar, a proof will only be given that $+$ is associative. We will prove by induction on γ that if α and β are ordinals then $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$. If $\gamma=0$ there is nothing to show. If $\gamma=\delta+1$, then

$$\begin{aligned}(\alpha+\beta)+(\delta+1) &= ((\alpha+\beta)+\delta)+1 = (\alpha+(\beta+\delta))+1 \\ &= \alpha+((\beta+\delta)+1) = \alpha+(\beta+(\delta+1)) \\ &= \alpha+(\beta+\gamma).\end{aligned}$$

If γ is a limit ordinal, then

$$\begin{aligned}(\alpha+\beta)+\gamma &= \sup\{(\alpha+\beta)+\delta \mid \delta\in\gamma\} = \sup\{\alpha+(\beta+\delta) \mid \delta \in \gamma\} \\ &= \sup\{\alpha+\eta \mid \eta \in \beta+\gamma\} = \alpha+(\beta+\gamma).\end{aligned}$$

The third inequality is justified by the fact that $\{\beta + \delta \mid \delta \in \gamma\}$ is a cofinal subset of $\beta + \gamma$ combined with the fact that $\xi \mapsto \alpha + \xi$ is strictly increasing. $\square$

The following proposition is left as a homework exercise.

Proposition 5.2. *For all $\alpha, \beta, \gamma \in \mathbf{ON}$ the following identities hold:*

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma, \quad \alpha^{\beta+\gamma} = \alpha^{\beta} \cdot \alpha^{\gamma}, \quad (\alpha^{\beta})^{\gamma} = \alpha^{\beta \cdot \gamma}.$$

The next theorem shows that ordinals can be written uniquely in any ordinal base β with $\beta > 1$. If $\langle \xi_i \mid i < n \rangle$ is a finite sequence of ordinals, we define $\sum_{i<n} \xi_i := \xi_0 + \ldots + \xi_{n-1}$ with the convention that the sum is 0 if $n = 0$.

Theorem 5.3 (Cantor Normal Form)**.** *Suppose $\beta > 1$ is an ordinal. For any ordinal α there exist a unique pair of finite sequences of ordinals $\langle \gamma_i \mid i < n \rangle$ and $\langle \delta_i \mid i < n \rangle$ such that $\langle \gamma_i \mid i < n \rangle$ is strictly decreasing, $0 < \delta_i < \beta$ for all $i < n$, and*

$$\alpha = \sum_{i<n} \beta^{\gamma_i} \cdot \delta_i.$$

The special case $\beta = \omega$ is usually what is meant by "Cantor normal form." The case $\beta = 2$ is also of interest — in this case $\delta_i = 1$ and we simply have $\alpha = \sum_{i<n} 2^{\gamma_i}$ for a unique strictly decreasing sequence $\langle \gamma_i \mid i < n \rangle$. Before proving this theorem, we will establish the following lemma.

Lemma 5.4. *For any ordinal $\beta > 1$, any nonnull finite strictly decreasing sequence $\langle \gamma_i \mid i < n \rangle$ of ordinals, and any sequence $\langle \delta_i \mid i < n \rangle$ of ordinals less than β we have:*

$$\sum_{i<n} \beta^{\gamma_i} \cdot \delta_i < \beta^{\gamma_0 + 1}$$

Proof. The proof is by induction on n. If $n = 1$, then $\beta^{\gamma_0} \cdot \delta_0 < \beta^{\gamma_0} \cdot \beta = \beta^{\gamma_0 + 1}$. If $n > 1$, then by induction

$$\sum_{i=1}^{n-1} \beta^{\gamma_i} \cdot \delta_i < \beta^{\gamma_1 + 1}$$

and therefore we have

$$\sum_{i<n} \beta^{\gamma_i} \cdot \delta_i < \beta^{\gamma_0} \cdot \delta_0 + \beta^{\gamma_1 + 1} \leqslant \beta^{\gamma_0} \cdot \delta_0 + \beta^{\gamma_0} = \beta^{\gamma_0} \cdot (\delta_0 + 1) \leqslant \beta^{\gamma_0 + 1}$$

as desired. $\square$

Proof of Theorem 5.3. We will prove existence by induction on α for a given $\beta > 1$. If $\alpha = 0$, then we take $n = 0$ and the null sequences witness the conclusion. Suppose now that $\alpha > 0$. Let γ_0 be the greatest ordinal such that $\beta^{\gamma_0} \leqslant \alpha$; such an ordinal exists since $\beta > 1$. Let δ_0 be the greatest ordinal such that $\beta^{\gamma_0} \cdot \delta_0 \leqslant \alpha$. Since

$$\beta^{\gamma_0} \cdot 1 \leqslant \alpha < \beta^{\gamma_0+1} = \beta^{\gamma_0} \cdot \beta,$$

it must be that $1 \leqslant \delta_0 < \beta$. Let ρ be the unique ordinal such that $\alpha = \beta^{\gamma_0} \cdot \delta_0 + \rho$. Since $\rho < \alpha$, we can apply our induction hypothesis to find sequences $\langle \gamma_i \mid i < n \rangle$ and $\langle \delta_i \mid i < n \rangle$ such that $\gamma_1 > \cdots > \gamma_{n-1}$, $0 < \delta_i < \beta$, and

$$\rho = \sum_{i=1}^{n-1} \beta^{\gamma_i} \cdot \delta_i.$$

Since $\rho < \beta^{\gamma_0}$, it follows that $\gamma_0 > \gamma_1$. Thus $\alpha = \sum_{i<n} \beta^{\gamma_i} \cdot \delta_i$.

Uniqueness is established by induction on α using the observation that if $\alpha = \sum_{i<n} \beta^{\gamma_i} \cdot \delta_i$, then

$$\gamma_0 = \max\{\gamma \mid \beta^{\gamma} \leqslant \alpha\}$$
$$\delta_0 = \max\{\delta \mid \beta^{\gamma_0} \cdot \delta \leqslant \alpha\}$$

and $\rho = \sum_{i=1}^{n-1} \beta^{\gamma_i} \cdot \delta_i$ is the unique ordinal such that $\alpha = \beta^{\gamma_0} \cdot \delta_0 + \rho$. □

Theorem 5.5. *Suppose that $\beta > 1$. For every positive ordinal α, there exist unique ordinals γ and δ and $0 < \rho < \beta$ such that $\alpha = \beta^{\gamma} \cdot (\beta \cdot \delta + \rho)$. In particular, for every $\alpha > 0$, there are unique ordinals β and γ such that $\alpha = 2^{\beta} \cdot (2 \cdot \gamma + 1)$.*

Proof. Let $\alpha = \sum_{i \leqslant n} \beta^{\gamma_i} \cdot \delta$ be the Cantor normal form for α in base β. Set $\gamma = \gamma_n$ for each $i < n$ let ξ_i be such that $\gamma_i = \gamma + 1 + \xi_i$. If we define $\delta = \sum_{i<n} \beta^{\xi_i}$ and $\rho = \delta_n$, then by Proposition 5.2, $\alpha = \beta^{\gamma} \cdot (\beta \cdot \delta + \rho)$. Uniqueness follows from the observation that if $\alpha = \beta^{\gamma} \cdot (\beta \cdot \delta + \rho)$ then the Cantor normal form of δ and ρ in base β can be manipulated using Proposition 5.2 into a Cantor normal form for α in base β. Since α has a unique Cantor normal form in base β, it must be that γ, δ and ρ are unique. □

6. Applications of transfinite recursion

Theorem 6.1 (Mostowski collapse)**.** *Suppose that* $\mathbf{A}$ *is a class and* $\mathbf{R}$ *is a well founded set-like relation on* $\mathbf{A}$*. There is a unique transitive class* $\mathbf{M}$ *and a unique class surjection* $\pi : \mathbf{A} \to \mathbf{M}$ *such that for all* $x, y \in \mathbf{A}$*,* $(x, y) \in \mathbf{R}$ *if and only if* $\pi(x) \in \pi(y)$*. Moreover, if* $\mathbf{R}$ *is extensional,* π *is injective.*

Proof. Define $\mathbf{F}(g)$ to be the range of g if g is a function and $\mathbf{F}(g) = \varnothing$ otherwise. Observe that if $\pi : \mathbf{A} \to \mathbf{V}$ is given by Theorem 4.1 and $\mathbf{M}$ is the range of π, then

$$\pi(y) = \{\pi(x) \mid (x, y) \in \mathbf{R}\} = \mathrm{range}(\pi \restriction \mathbf{R}^y).$$

To see that $\mathbf{M}$ is transitive, observe that if $x \in \mathbf{M}$, then $x = \pi(a)$ for some a. If $y \in \pi(a)$, then $y \in \mathrm{range}(\pi \restriction \mathbf{R}^a) \subseteq \mathrm{range}(\pi) = \mathbf{M}$. □

Corollary 6.2. *If* $(W, \lhd)$ *is a well ordered set, then* $(W, \lhd)$ *is isomorphic to* $(\alpha, \in)$*. Moreover* α *and the isomorphism are unique.*

Proof. Take the Mostowski collapse of $(W, \lhd)$ to obtain a unique transitive set α and isomorphism $\pi : (W, \lhd) \cong (\alpha, \in)$. Since $\lhd$ is a linear order, $(\alpha, \in)$ must be as well and hence α is an ordinal. □

If $(W, \lhd)$ is a well ordering, then the unique α such that $(W, \lhd) \cong (\alpha, \in)$ is called the *ordertype* of $(W, \lhd)$ and denoted $\mathrm{otp}(W, \lhd)$. If W is a set of ordinals and $\lhd$ is $\in$, then we write $\mathrm{otp}(W)$ for $\mathrm{otp}(W, \in)$.

Corollary 6.3. *Given any two well orders, one is uniquely isomorphic to an initial segment of the other.*

Proof. By the previous corollary, we may assume that the two well orders are ordinals α and β. Since either $\alpha \in \beta$, $\alpha = \beta$, or $\beta \in \alpha$, we have that α is an initial part of β or vice versa. □

Corollary 6.4. *If* $\mathbf{R}$ *is any well-founded set-like relation, then*

$$\rho(y) = \sup\{\rho(x) + 1 \mid (x, y) \in \mathbf{R}\}$$

defines an $\mathbf{R}$*-increasing class function into the ordinals.*

The class function ρ is called the *rank function* for $\mathbf{R}$. If $\mathbf{R}$ is the membership relation, then $\rho(x)$ is called the *rank of* x. Observe that if α is an ordinal, then the rank of α is α.

For $\alpha \in \mathbf{ON}$, define V_α recursively by

$$V_\alpha = \begin{cases} \varnothing & \text{if } \alpha = 0 \\ \mathscr{P}(V_\beta) & \text{if } \alpha = \beta + 1 \\ \bigcup\{V_\beta \mid \beta \in \alpha\} & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

This sequence of sets is known as *the cumulative hierarchy* and was first defined by John von Neumann. The next theorem summarizes its key properties.

Theorem 6.5. *The following are true:*

(a) For every ordinal α and every set x, x is in V_α if and only if the rank of x is less than α. In particular the class of all sets of rank less than α is a set.
(b) For every ordinal α, V_α is a transitive set and $V_\alpha \cap \mathbf{ON} = \alpha$.
(c) For each ordinal α, $(V_\alpha, \in)$ satisfies the axioms of Extensionality, Foundation, Union, and Separation.
(d) If $\alpha > 0$, $(V_\alpha, \in)$ satisfies Emptyset and if $\alpha > \omega$, then $(V_\alpha, \in)$ satisfies Infinity.
(e) For every limit ordinal α, $(V_\alpha, \in)$ satisfies the axioms of Pairing and Powerset.

In particular, $(V_{\omega\cdot 2}, \in)$ satisfies all of the axioms of ZF except for the Collection Scheme.

Proof. That V_α consists exactly of the sets of rank less than α is proved by induction on α. Suppose that this is false and let α be the least ordinal witnessing this. Observe that α must be a successor; let $\alpha = \beta + 1$. To see that V_α only consists of sets of rank less than α, suppose that $x \in V_\alpha$. In this case $\rho(x) = \sup_{y \in x} \rho(y) + 1$. Since $x \subseteq V_\beta$, each $y \in x$ is in V_β. Thus $\rho(y) < \beta$ and $\rho(y) + 1 < \alpha$. To see that V_α contains every set x of rank less than α, observe that the elements of x must each have rank less than β. Thus $x \subseteq V_\beta$ and therefore $x \in \mathscr{P}(V_\beta) = V_\alpha$.

Since the class of sets of rank less than α is transitive, V_α is transitive for each α. Since the rank function is the identity on the ordinals, $V_\alpha \cap \mathbf{ON} = \alpha$. Observe that if X is any transitive set, then $(X, \in)$ satisfies the axioms of Extensionality and Foundation. If x is a set, then

$$\rho(\bigcup x) = \sup_{y \in x} \sup_{z \in y} \rho(z) + 1 \leqslant \sup_{y \in x} (\sup_{z \in y} \rho(z) + 1) + 1 = \rho(x)$$

and thus if x is in V_α, so is $\bigcup x$. Finally, if x has rank less than α and $y \subseteq x$, then y has rank less than α. Hence if $x \in V_\alpha$ and $y \subseteq x$, then $y \in V_\alpha$.

Since Emptyset and Infinity just postulate the existence of 0 and ω, respectively, (d) follows. Finally if x and y are in V_α, then $\mathscr{P}(x)$ is in $V_{\alpha+1}$ and $\{x, y\}$ is in $V_{\alpha+2}$. In particular, if α is a limit ordinal, V_α satisfies Powerset and Pairing. $\square$

The next theorem is another application of the Mostowski collapse.

Theorem 6.6 (Hartog ordinal)**.** *For each set x, there is a least ordinal γ such that there is no injection from γ into x.*

Proof. Given x, define

$$W := \{(y, \lhd) \in \mathscr{P}(x) \times \mathscr{P}(x^2) \mid (y, \lhd) \text{ is a well order}\}.$$

Since W is a set, otp $\restriction W$ is a function whose range is contained in the ordinals. Let γ be the least strict upper bound for the range of W. If there were an injection $f : \gamma \to x$, then its range y would be a subset of x which could be equipped with a well order isomorphic to γ: define $f(\alpha) \lhd f(\beta)$ if and only if $\alpha \in \beta$. Since this is contrary to our choice of γ, there is no injection from γ into x. $\square$

7. Cardinals and Cardinality

Recall that if X and Y are sets, then the *cardinality of X is at most that of Y* if there is an injection from X into Y. This is denoted by $|X| \leqslant |Y|$. If there is a bijection between X and Y, this is denoted $|X| = |Y|$. In the presence of well orderings, the existence of surjective maps have implications for cardinality inequalities.

Proposition 7.1. *If A is a set and β is an ordinal and there is a surjection $f : \beta \to A$, then $|A| \leqslant |\beta|$.*

Proof. $a \mapsto \min f^{-1}(a)$ defines an injection from A to β and hence $|A| \leqslant |\beta|$. □

We note the following two well known theorems on cardinality.

Theorem 7.2 (Cantor-Schröder-Bernstein)**.** *If $|X| \leqslant |Y|$ and $|Y| \leqslant |X|$, then $|X| = |Y|$.*

Proof. Suppose that $f : X \to Y$ and $g : Y \to X$ are injections. Define

$$\begin{aligned} X_0 &:= X \backslash g[Y] & \qquad Y_0 &:= Y \backslash f[X] \\ X_{n+1} &:= g[Y_n] & \qquad Y_{n+1} &:= f[X_n] \end{aligned}$$

Set $X_\omega := X \backslash \bigcup_{n\in\omega} X_n$ and $Y_\omega := Y \backslash \bigcup_{n\in\omega} Y_n$. Observe that the composition $g^{-1} \circ f$ defines a bijection between X_n and X_{n+1} for all n. Since X_0 is disjoint from the range of $g^{-1} \circ f$ and since $g^{-1} \circ f$ translates $X_0, \ldots, X_{n-1}$ to $X_1, \ldots, X_n$ for each n, it follows by induction on n that $\{X_n \mid n \in \omega + 1\}$ is pairwise disjoint. Similarly $\{Y_n \mid n \in \omega + 1\}$ is pairwise disjoint. Furthermore $f \restriction X_{2n}$ is a bijection between X_{2n} and Y_{1+2n} if $n \leqslant \omega$. Similarly $g^{-1} \restriction X_{1+2n}$ is a bijection between X_{1+2n} and Y_{2n}. Define $h : X \to Y$ by

$$h(x) := \begin{cases} g^{-1}(x) & \text{if } x \in \bigcup_{n\in\omega} X_{2n+1} \\ f(x) & \text{otherwise} \end{cases}$$

It follows that h is a bijection. □

Theorem 7.3 (Cantor)**.** *For any set x, there is no surjection from x to $\mathscr{P}(x)$.*

Proof. If $f : x \to \mathscr{P}(x)$, then $\{y \in x \mid y \notin f(y)\}$ is not in range(f). □

The Hartog function allows us to give a short proof of the following remarkable fact.

Corollary 7.4. *If for every pair of sets x and y, $|x| \leqslant |y|$ or $|y| \leqslant |x|$, then every set can be well ordered.*

Proof. Let x be given and let γ be an ordinal such that $|\gamma| \nleqslant |x|$. If $|x| \leqslant |\gamma|$, then there is an injection f from x into γ. If we define $\lhd$ on x by $a \lhd b$ if and only if $f(a) \in f(b)$, then $\lhd$ is a well ordering on x. □

The conclusion of this corollary is known as the *Well Ordering Principle*. Notice that a set X can be well ordered precisely when $|X| \leqslant |\alpha|$ for some ordinal α.

Corollary 7.5. *If for every infinite X, $|X| = |X^2|$, then the Well Ordering Principle holds.*

Proof. Let X be a given infinite set and let γ be an ordinal such that $|\gamma| \nleqslant |X|$ and set $Y = X \cup \gamma$. If $|Y| = |Y^2|$, let $f : Y \to Y^2$ be a bijection and define

$$g(x) = \min\{\xi \in \gamma \mid \exists y \; f(\xi) = (x, y)\}.$$

Notice that $g(x)$ is always defined since otherwise $\eta \mapsto f^{-1}(x, \eta)$ would define an injection from γ into X. Since g is clearly one-to-one, we've showing that $|X| \leqslant |\gamma|$ and hence that X can be well ordered. □

An ordinal κ is a *cardinal* if whenever $\alpha \in \kappa$, $|\alpha| < |\kappa|$. The Well Ordering Principle is equivalent to the assertion that every set has the same cardinality as a cardinal. If $|x| = |\kappa|$ for some cardinal κ it is customary to adopt the convention that $|x| := \kappa$. Notice that the finite cardinals are exactly the elements of ω. On the other hand, ordinals such as $\omega + 1$, $\omega \cdot \omega$, $\omega^{\omega+1}$, etc. are not cardinals — they all have the same cardinality as ω. If $|x| \leqslant \omega$, we will say that x is *countable*.

The infinite cardinals have a canonical enumeration which can be described as follows. If α is an ordinal, define α^+ to be the least cardinal β greater than α. A cardinal of the form α^+ is called a *successor cardinal*. All other infinite cardinals are called *limit cardinals*. Define the hierarchy of infinite cardinals recursively by

$$\omega_\alpha := \begin{cases} \omega & \text{if } \alpha = 0 \\ (\omega_\beta)^+ & \text{if } \alpha = \beta + 1 \\ \sup\{\omega_\beta \mid \beta \in \alpha\} & \text{if } \alpha \text{ is a limit ordinal} \end{cases}$$

Observe that every infinite cardinal is of the form ω_α for some ordinal α (otherwise there would be a least counterexample and this is easily shown to be impossible). Cardinals both play of role of well orderings and as representative cardinalities of sets. If we wish to emphasize that ω_α is a cardinal, we will instead write $\aleph_\alpha$. If, on the other hand, we want to signal that ordinal arithmetic is involved or that the underlying $\in$-ordering is important, we will write ω_α. Generally one writes ω instead of ω_0 but writes $\aleph_0$ and not $\aleph$.

Theorem 7.6. *The following are true for every infinite cardinal κ:*

(a) κ is closed under addition, multiplication, and exponentiation.
(b) $|\kappa^{<\omega}| = \kappa$.

Proof. We will first show that, for a given κ that (a) implies (b). To see this, define $\mathbf{F} : \mathbf{ON} \to \mathbf{ON}^{<\omega}$ and $\mathbf{G} : \mathbf{ON}^{<\omega} \to \mathbf{ON}$ by

$$\mathbf{F}(\alpha) := \begin{cases} \varepsilon & \text{if } \alpha = 0 \\ \langle \beta \rangle^\frown \mathbf{F}(\gamma) & \text{if } \alpha = 2^\beta(2 \cdot \gamma + 1). \end{cases}$$

$$\mathbf{G}(s) := \begin{cases} 0 & \text{if } s = \varepsilon \\ 2^\xi \cdot (2 \cdot \mathbf{G}(t) + 1) & \text{if } s = \langle \xi \rangle^\frown t \end{cases}$$

(Here ε denotes the null sequence and $s^\frown t$ is the concatenation of s and t.) Theorem 5.5 implies that $\mathbf{G} \circ \mathbf{F} = \mathbf{F} \circ \mathbf{G}$ is the identity function and hence that both $\mathbf{F}$ and $\mathbf{G}$ are class bijections. If κ is closed under the arithmetic operations, then $\mathbf{F} \restriction \kappa$ is a bijection between κ and $\kappa^{<\omega}$.

We will now prove (a) by induction on κ. Define

$$\mathbf{I} = \{\xi \in \mathbf{ON} \mid 2^\xi = \xi\}$$

and observe that the elements of $\mathbf{I}$ are exactly those ordinals which are closed under addition, multiplication, and exponentiation. Observe that 0 and ω are in $\mathbf{I}$ and if $A \subseteq \mathbf{I}$ is a set, then $\sup(A) \in \mathbf{I}$. In order to show that every infinite cardinal is in $\mathbf{I}$, it suffices to show that if $\mu < \nu$ are consecutive elements of $\mathbf{I}$, then $|\mu| = |\nu|$. To this end, let $\mu < \nu$ be given and define $h : \mu \to \nu$ by

$$h(\alpha) := \begin{cases} \alpha & \text{if } \alpha \in \mathbf{I} \\ \sum_{i<n} 2^{h(\beta_i)} & \text{if } \alpha \notin \mathbf{I} \text{ and } \mathbf{F}(\alpha) = \langle \beta_i \mid i < n \rangle \text{ is decreasing} \\ \mu & \text{otherwise} \end{cases}$$

Notice that this is a recursive definition since if $\mathbf{F}(\alpha) = \langle \beta_i \mid i < n \rangle$, and $\alpha \leqslant \beta_i$ for some $i < n$, then $i = 0$, $n = 1$ and $\alpha = \beta_0$ is in $\mathbf{I}$. By Proposition 7.1, it suffices to show that if $\gamma \in \nu$, γ is in the range of h. We will prove this by induction. If $\gamma \in \mathbf{I} \cap \mu$, there is nothing to show. Also, since $\mathbf{F}(3) = \langle 0, 1 \rangle$, $h(3) = \mu$. If $\gamma \in \nu$ is not in $\mathbf{I}$, then $\gamma = \sum_{i<n} 2^{\gamma_i}$ for some strictly decreasing sequence $\langle \gamma_i \mid i < n \rangle$ with $\gamma_0 < \gamma$. Applying our inductive assumption, let β_i be the least ordinal such that $h(\beta_i) = \gamma_i$ and let α be such that $\mathbf{F}(\alpha) = \langle \beta_i \mid i < n \rangle$. It follows that $h(\alpha) = \gamma$. $\square$

8. The Axiom of Choice

Up to this point, all of our analysis has been carried out just using the ZF axioms. We will now introduce the final axiom of ZFC, the Axiom of Choice.

Axiom 10 (Choice)**.** *If X is a set and every element of X is nonempty, then there is a function $f : X \to \bigcup X$ such that $f(x) \in x$ for all $x \in X$.*

Frequently the Axiom of Choice is used through one of its many equivalent forms. Recall that Zorn's Lemma asserts that whenever $(P, \leqslant)$ is a partially ordered set in which every totally ordered subset has an upper bound, P has a maximal element.

Theorem 8.1. *Assuming the axioms of ZF, the following are equivalent:*

(a) The Axiom of Choice.
(b) Zorn's Lemma.
(c) The Well Ordering Principle.

Proof. We will show (a)⇒(b)⇒(c)⇒(a). To see (a) implies (b), let $(P, \leqslant)$ be a partially ordered set satisfying the hypothesis of Zorn's Lemma. Define

$$\mathcal{C} := \{C \in \mathscr{P}(P) \mid (C, \leqslant) \text{ is a total order}\}.$$

If $C \in \mathcal{C}$, define

$$B(C) := \{p \in P \mid p \text{ is a strict upper bound for } C\}$$

and set $\mathcal{B} := \{B(C) \mid C \in \mathcal{C}\}$. If $B(C)$ is empty for some $C \in \mathcal{C}$, then C has an upper bound p but no strict upper bound. This means that p is a maximal element of P. Suppose for contradiction that no element of $\mathcal{B}$ is empty. By the Axiom of Choice, there is a function $f : \mathcal{B} \to \bigcup \mathcal{B}$ such that $f(B) \in B$ for all $B \in \mathcal{B}$. If $s : \alpha \to P$ is a strictly increasing sequence, define $\mathbf{F}(s) = f(\text{range}(s))$. By the Transfinite Recursion Theorem, there is a $\mathbf{G} : \mathbf{ON} \to \mathbf{V}$ such that $\mathbf{G}(\alpha) = f(\text{range}(\mathbf{G} \restriction \alpha))$ for all α. But then $\mathbf{G}$ is a strictly increasing — and in particular injective — class function from $\mathbf{ON}$ into P, which is absurd.

To see (b) implies (c), define Q to be the class of all injections from an ordinal into X. Observe that if γ is the Hartog ordinal for X, then $Q \subseteq \mathscr{P}(\gamma \times X)$ and hence Q is a set. Order Q by $p \leqslant q$ if p is a restriction of q. If $C \subseteq Q$ is a chain, then $\bigcup C$ is also in Q and is an upper bound for C. By Zorn's Lemma, Q has a maximal element $p : \alpha \to X$. If p is not a surjection, then $q \cup \{(\alpha, x)\} : \alpha + 1 \to X$ is an injection where $x \in X$ is not in the range of p. Since such a q would be strictly above p, it must be that p is a surjection.

To see that (c) implies (a), let X be given and let $\lhd$ be a well ordering of $\bigcup X$. Define $f : X \to \bigcup X$ so that $f(x)$ is the $\lhd$-least element of x. $\square$

We've also shown the theorem.

Theorem 8.2. *The following are each equivalent to the Well Ordering Principle:*

(a) If X and Y are sets, then either $|X| \leqslant |Y|$ or $|Y| \leqslant |X|$.
(b) If X is an infinite set, then $|X^{<\omega}| = |X|$.
(c) If X is an infinite set, then $|X \times X| = |X|$.

From this point forward, all proofs will be carried out in ZFC unless explicitly stated otherwise.

If α is a ordinal limit, then the *cofinality* of α, denoted $\mathrm{cof}(\alpha)$, is the minimum cardinality of a subset $A \subseteq \alpha$ such that $\sup(A) = \alpha$. If κ is an infinite cardinal and $\mathrm{cof}(\kappa) = \kappa$, we say that κ is a *regular cardinal*; if an infinite cardinal is not regular it is *singular*. The cardinals $\aleph_n$ for $n \in \omega$ are all regular cardinals whereas $\aleph_\omega = \sup\{\aleph_n \mid n \in \omega\}$ has cofinality ω and hence is singular.

Theorem 8.3. *Every infinite successor cardinal is regular.*

Proof. Let κ be an infinite cardinal and suppose for contradiction that there is an $A \subseteq \kappa^+$ such that $\sup(A) = \kappa^+$ and $|A| \leqslant \kappa$. Fix a surjection $f : \kappa \to A \times \kappa$. For each $\alpha \in A$, define E_α to be the collection of all surjections from κ to α. Since $\{E_\alpha \mid \alpha \in A\}$ exists by Collection and since each E_α is nonempty by hypothesis, there exists a function e with domain A such that $e(\alpha) \in E_\alpha$ for all $\alpha \in A$. Writing e_α for $e(\alpha)$ define $g : \kappa \to \kappa^+$ by $g(\alpha) = e_\beta(\gamma)$ where $f(\alpha) = (\beta, \gamma)$. Clearly g is a surjection: if $\xi \in \kappa^+$ then there is a $\beta \in A$ with $\xi \in \beta$ and a $\gamma \in \kappa$ such that $e_\beta(\gamma) = \xi$. If $f(\alpha) = (\beta, \gamma)$, then $g(\alpha) = e_\beta(\gamma) = \xi$. $\square$

9. Inaccessible cardinals and the reflection theorem

A cardinal κ is a *strong limit cardinal* if whenever $\alpha < \kappa$, $|\mathscr{P}(\alpha)| < \kappa$. A cardinal is *weakly inaccessible* if it is an uncountable regular limit cardinal and *strongly inaccessible* if it is a uncountable regular strong limit cardinal. Notice that ω is both regular and a strong limit cardinal.

Theorem 9.1. *Assume ZF. If κ is a strongly inaccessible cardinal, then $(V_\kappa, \in)$ satisfies ZF.*

Remark 9.2. If α is a limit ordinal and $X \in V_\alpha$, then any well ordering of X is in V_α. Thus if the Well Ordering Principle holds, it is satisfied by $(V_\alpha, \in)$ if α is a limit ordinal.

Proof. If κ is strongly inaccessible, then suppose that $X \in V_\kappa$, $\phi(u, v, \bar{w})$ is a formula, and $\bar{a} \in V_\kappa^{<\omega}$ are such that

$$(V_\kappa, \in) \models \forall x \in X \ \exists! y \ \phi(x, y, \bar{a}).$$

Let $\psi(x, y, \bar{w})$ be such that for any sets $x, y, \bar{w}$, $\psi(x, y, \bar{w})$ if and only if $(V_\kappa, \in) \models \phi(x, y, \bar{w})$. By Separation applied to $X \times V_\kappa$ and ψ, there is a function $f : X \to V_\kappa$ such that if $x \in X$, $(V_\kappa, \in) \models \phi(x, f(x), \bar{a})$. Let $A := \{\rho(f(x)) \mid x \in X\} \subseteq \kappa$.

Claim 9.3. *For each $\alpha \in \kappa$, $\theta_\alpha := |V_\alpha| \in \kappa$.*

Proof. The proof is by induction on α. If $\alpha = \beta + 1$, then $\theta_\alpha = |\mathscr{P}(\theta_\beta)| < \kappa$ since $\theta_\alpha < \kappa$ and κ is a strong limit. If α is a limit ordinal, then $\{\theta_\beta \mid \beta \in \alpha\}$ is a subset of κ of cardinality $|\alpha| < \kappa$. Since κ is regular,

$$\theta_\alpha := \sup\{\theta_\beta \mid \beta \in \alpha\}$$

is less than κ. □

Let α be minimal such that $X \in V_\alpha$. Since V_α is transitive, $X \subseteq V_\alpha$ and $|X| < \kappa$ by the claim. Since $\rho \circ f : X \to A$ is a surjection, A is bounded in κ. Let $\beta < \kappa$ be a strict upper bound for A. Thus $f \in V_\beta \subseteq V_\kappa$ and therefore $(V_\kappa, \in)$ satisfies the corresponding instance of Collection. □

Theorem 9.4. *If $(V_\kappa, \in)$ satisfies ZFC, then κ is a strong limit cardinal.*

Proof. Now suppose that $(V_\kappa, \in)$ satisfies Collection for some uncountable limit ordinal κ. As we have noted already, this means that $(V_\kappa, \in)$ satisfies ZF. Let $\alpha \in \kappa$ be arbitrary and observe that $\mathscr{P}(\alpha) \in V_\kappa$. Since the Axiom of Choice holds, there is a well ordering $\lhd$ of $\mathscr{P}(\alpha)$ which is therefore also in V_κ. Since $(V_\kappa, \in)$ satisfies Collection, there is an ordinal $\beta \in V_\kappa$ which is isomorphic to $(\mathscr{P}(\alpha), \lhd)$. Since $|\mathscr{P}(\alpha)| = |\beta|$

and since κ is a cardinal, $|\mathscr{P}(\alpha)| < \kappa$. Since $\alpha < \kappa$ was arbitrary, κ is a strong limit cardinal. ☐

It will be useful throughout the course to connect truth in an ambient model of set theory to truth in transitive sets. Given a formula $\phi(\bar{v})$ and transitive classes $\mathbf{M} \subseteq \mathbf{N}$, we say that $\phi(\bar{v})$ is *absolute for* $\mathbf{M}$ *and* $\mathbf{N}$ if whenever $\bar{a}$ is a tuple from $\mathbf{M}$, $(\mathbf{M}, \in) \models \phi(\bar{a})$ is true if and only if $(\mathbf{N}, \in) \models \phi(\bar{a})$. If $\mathbf{N}$ is $\mathbf{V}$, we will just say that ϕ is absolute for $\mathbf{M}$.

Certain formulas are always absolute for transitive models. A Σ_0-*formula* is a formula $\phi(\bar{v})$ in the language of set theory such that all quantification in ϕ is bounded. Namely all atomic formulas are Σ_0-formulas, the Σ_0-formulas are closed under conjunctions, disjunctions, and negation, and if $\phi(u, \bar{v})$ is a Σ_0-formula, so are $\exists u \in w\ \phi(u, \bar{v})$ and $\forall u \in w\ \phi(u, \bar{v})$. The following proposition is very useful; its proof is a routine induction on formulas and is left as an exercise.

Proposition 9.5. *If M is a transitive set, then any Σ_0-formula is absolute for M.*

Now we turn to the statement of the Reflection Theorem. A *stratified transitive class* is a class length sequence $\langle M_\alpha \mid \alpha \in \mathbf{ON} \rangle$ such that:

- for all $\alpha \in \mathbf{ON}$, M_α is a transitive set;
- if $\alpha \in \beta \in \mathbf{ON}$, then $M_\alpha \subseteq M_\beta$;
- if $\alpha \in \mathbf{ON}$ is a limit ordinal, $M_\alpha = \bigcup_{\beta \in \alpha} M_\beta$.

We will let $\mathbf{M}$ denote $\bigcup \{M_\alpha \mid \alpha \in \mathbf{ON}\}$.

The Reflection Theorem is actually a scheme of theorems one for each stratified transitive class $\mathbf{M}$ and one for each formula $\phi(\bar{v})$.

Theorem 9.6. *Let $\langle M_\alpha \mid \alpha \in \mathbf{ON} \rangle$ be a stratified class and ϕ be a formula. The class*

$$\{\alpha \in \mathbf{ON} \mid \phi \textit{ is absolute for } M_\alpha \textit{ and } \mathbf{M}\}$$

contains a closed and unbounded class $\mathbf{E} \subseteq \mathbf{ON}$.

Proof. If ϕ is an atomic formula, then ϕ is absolute for M_α for all $\alpha \in \mathbf{ON}$. Observe that the set of formulas for which the Reflection Theorem holds is closed under taking conjunctions, disjunctions, and negations. Suppose now that the conclusion of the reflection theorem holds for every proper subformula of ϕ. We may assume without loss of generality that ϕ is of the form $\exists u \psi(u, \bar{v})$. Let $\mathbf{E}$ be a closed and unbounded class contained in

$$\{\alpha \in \mathbf{ON} \mid \psi(u, \bar{v}) \text{ is absolute for } M_\alpha\}.$$

Define $\mathbf{F} : \mathbf{E} \to \mathbf{E}$ by $\mathbf{F}(\alpha)$ is the least element β of $\mathbf{E}$ such that $\alpha \leqslant \beta$ and for all $\bar{a} \in M_\alpha^n$, if $(\mathbf{M}, \in) \models \exists u \psi(u, \bar{a})$ then $(M_\beta, \in) \models \exists u \phi(u, \bar{a})$.

Notice that $\mathbf{F}$ is defined on all of $\mathbf{E}$ by our assumption that $\mathbf{E}$ is closed and unbounded and by Collection. Observe that if $\mathbf{F}(\alpha) = \alpha$, then ϕ is absolute for M_α and $\mathbf{M}$. Also observe that if $\alpha = \sup(\mathbf{E} \cap \alpha)$, then

$$\mathbf{F}(\alpha) = \sup\{\mathbf{F}(\beta) \mid \beta \in \mathbf{E} \cap \alpha\}.$$

In particular, $\{\alpha \in \mathbf{E} \mid \mathbf{F}(\alpha) = \alpha\}$ is closed under taking supremums. It suffices to show that for each α in $\mathbf{E}$ there is a $\beta \geqslant \alpha$ in $\mathbf{E}$ such that $\mathbf{F}(\beta) = \beta$. Define $\alpha_0 := \alpha$, $\alpha_{n+1} := \mathbf{F}(\alpha_n)$ and $\beta := \sup\{\alpha_n \mid n \in \omega\}$. $\langle \alpha_n \mid n \in \omega \rangle$ exists by the recursion theorem and hence β exists. It follows that

$$\mathbf{F}(\beta) = \sup\{\mathbf{F}(\alpha_n) \mid n \in \omega\} = \beta.$$

□

Corollary 9.7. *ZF is not finitely axiomatizable.*

Proof. Suppose that this is true and observe that then there is a single sentence ϕ such that ϕ proves all of the axioms of ZF. Then ϕ proves that there is a least α such that $(V_\alpha, \in) \models \phi$. But now

$$(V_\alpha, \in) \models \exists \beta \in \mathbf{ON}\ ((V_\beta, \in) \models \phi).$$

This is impossible since this implies $(V_\beta, \in) \models \phi$, contradicting the minimality of ϕ. □

10. Truth, definability, and the Löwenheim-Skolem theorem

Suppose that we are working in an ambient model of ZF and, within this model $\mathcal{L}$ is the signature of a language. If $\mathcal{S}$ is the set of all logical symbols (including a countable set of variables), then the collection of $\mathcal{L}$-formulas constitutes a set by Separation — it is contained within $(\mathcal{L} \cup \mathcal{S})^{<\omega}$. Moreover, the operation which returns the sequence of variables which are free in a given formula is in fact a function. For any $\mathcal{L}$-formula $\phi(\bar{v})$, there is an $\mathcal{L} \cup \{\in\}$-formula $\psi(u, \bar{v})$ such that for an $\mathcal{M}$ and $\bar{a}$ with the same length as $\bar{v}$, $\psi(\mathcal{M}, \bar{a})$ is true if and only if $\mathcal{M}$ is an $\mathcal{L}$-structure, $\bar{a}$ is a sequence from $\mathcal{M}$ and $\mathcal{M} \models \phi(\bar{a})$. Contrast this with the following theorem of Tarski (formulated here for ZF).

Theorem 10.1 (Undefinability of Truth)**.** *ZF does not prove the following statement: there is a $\{\in\}$-formula $\phi(v)$ such that for all formulas ψ, $\phi(\psi) \leftrightarrow \psi$. Furthermore,*

$$\{(\phi(v_0, \ldots, v_{n-1}), (a_0, \ldots, a_{n-1})) \mid \phi(\bar{v}) \textit{ is a formula and } \phi(\bar{a}) \textit{ is true}\}$$

is not a class — it can not be formalized.

In particular, while for each finite set F of formulas in the language of set theory, the Reflection Theorem implies that

$$\mathbf{E}_F := \{\delta \in \mathbf{ON} \mid F \text{ is absolute for } V_\delta\}$$

is a closed unbounded class, it is meaningless to talk about the intersection of these classes over the countable set of formulas. Note, however, that if κ is an ordinal, then

$$\{\mathbf{E}_F \cap \kappa \mid F \text{ is a finite set of formulas}\}$$

is a set. This gives the following proposition.

Proposition 10.2. *Suppose that κ is a strongly inaccessible cardinal. There is a closed and unbounded subset $E \subseteq \kappa$ such that if $\delta \in E$, then every formula in the language of set theory is absolute between V_δ and V_κ. In particular, $(V_\delta, \in)$ is a model of ZFC if $\delta \in E$.*

Proof. If F is a finite set of formulas in the language of set theory, let E_F denote the set of all $\delta \in \kappa$ such that all formulas in F are absolute between V_δ and V_κ. By the Reflection Theorem, applied in V_κ, $E_F \subseteq \kappa$ is closed and unbounded. Let E be the intersection of the E_F's. Clearly E is closed. That E is unbounded will be verified momentarily as part of a more general phenomenon. □

If X is a set and $M \subseteq X$, we say that *M is an elementary submodel of X* if every formula in the language of set theory is absolute between

M and X. We will write $M \prec X$ to denote that M is an elementary submodel of X. Our next goal will be to prove the (Downward) Löwenheim-Skolem Theorem which ensures the existence of many elementary submodels when X is uncountable.

Theorem 10.3. *For any X and infinite cardinal $\kappa \leqslant |X|$, if $X_0 \in [X]^\kappa$, there is an elementary submodel M of X of cardinality κ which contains X_0 as a subset.*

This theorem will be a consequence of two propositions. In order to state these propositions, we need to develop some terminology. If X is a set and $f : X^{<\omega} \to X$, define

$$C_f := \{M \in \mathscr{P}(X) \mid f[M^{<\omega}] \subseteq M\}.$$

A set of this form is said to be *club* in $\mathscr{P}(X)$.

A set S is *stationary* if S intersects every club in $\mathscr{P}(\bigcup S)$. If S is stationary and $C \subseteq S$, then we say that C is club in S if $C = S \cap C_f$ for some $f : X^{<\omega} \to X$. If X is a set and κ is a cardinal, define $[X]^\kappa$ to be the collection of all subsets of X of cardinality κ.

Proposition 10.4. *For any X, $\{M \in \mathscr{P}(X) \mid M \prec X\}$ contains a club.*

Proof. First recall the Tarski-Vaught criterion for elementarity.

Lemma 10.5. *$M \prec X$ if whenever $\phi(\bar{v}, w)$ is a formula in the language of set theory, $\bar{a}$ is a tuple from M, and $(X, \in) \models \phi(\bar{a}, b)$ for some b in X, then b can be found in M.*

Observe that if X is finite, then $X \prec X$ and there is a function $f : X^{<\omega} \to X$ such that X is the only f-closed set. Let $\langle x_n \mid n \in \omega\rangle$ be a list of elements of X without repetitions and let $\lhd$ be a well ordering of X. Let ϕ_n $(n \in \omega)$ list the formulas in the language of set theory in such a way that there are at most n free variables in ϕ_n. Define $f : X^{<\omega} \to X$ by

$$f(\bar{a}) := \begin{cases} x_0 & \text{if } \bar{a} = \varepsilon \\ x_{n+1} & \text{if } \bar{a} = \langle x_n\rangle \\ \lhd\text{-min}\{b \in X \mid (X, \in) \models \phi_i(\bar{a}\restriction i, b)\} & \text{if } \ell h(\bar{a}) = 2^{i+2}(2j+1) \end{cases}$$

Observe that if M is f-closed, then $M \prec X$ by the Tarski-Vaught criterion. $\square$

Proposition 10.6. *If X is any infinite set, $\kappa \leqslant |X|$ is any infinite cardinal, then and $X_0 \in [X]^\kappa$, then*

$$\{M \in [X]^\kappa \mid X_0 \subseteq M\}$$

is stationary.

Proof. Let X, κ, and X_0 be given. Define $X_{n+1} := X_n \cup f[X_n^{<\omega}]$ and set $Y := \bigcup\{X_n \mid n \in \omega\}$. Notice that $Y^{<\omega} = \bigcup\{X_n^{<\omega} \mid n \in \omega\}$. Since it follows that

$$f[Y^{<\omega}] = \bigcup\{f[X_n^{<\omega}] \mid n \in \omega\} \subseteq \bigcup\{X_{n+1} \mid n \in \omega\} = Y.$$

□

Lemma 10.7. *Suppose that X is a nonempty set and $\langle f_k \mid k \in \omega\rangle$ is such that $f_k : X^{<\omega} \to X$ is a function for each $k \in \omega$. There is single $g : X^{<\omega} \to X$ such that if $M \subseteq X$ is g-closed, then M is f_k-closed for all k. In particular, a countable intersection of clubs contains a club.*

Proof. If X is finite, then it is possible to find a g such that the only g-closed subset of X is X. Thus we may assume that X is infinite. Let $\langle a_k \mid k \in \omega\rangle$ be sequence of distinct elements of X and define

$$g(\bar{x}) := \begin{cases} a_0 & \text{if } \bar{x} = \varepsilon \\ a_{k+1} & \text{if } \bar{x} = \langle a_k\rangle \\ f_k(\bar{y}) & \text{if } \bar{x} = \langle a_k\rangle^\frown\bar{y}^\frown\langle a_0\rangle \end{cases}$$

Clearly any g-closed set is f_k-closed for each k. □

Lemma 10.8. *Suppose that $X \subseteq Y$ are nonempty. For every $f : Y^{<\omega} \to Y$, there is a $g : X^{<\omega} \to X$ such that if $M \subseteq Y$ is f-closed, then $M \cap X$ is g-closed.*

Proof. For each $\langle x_i \mid i < n\rangle \in X^{<\omega}$, the g-closure of $\{x_i \mid i < n\}$ is a countable set. Choose functions $f_k : X^{<\omega} \to X$ such that $\{f_k(\bar{x}) \mid k \in \omega\}$ is the intersection of X with the g-closure of $\{x_i \mid i < n\}$. By Lemma 10.7, there is a function $f : X^{<\omega} \to X$ such that any f-closed set is f_k-closed for all $k \in \omega$. If $M \subseteq Y$ is g-closed, then $M \cap X$ is f_k-closed for all k and hence f-closed. □

This lemma has the following useful consequence.

Lemma 10.9. *If $X \subseteq Y$ are uncountable sets and E is club in $[Y]^\omega$, then $\{M \cap X \mid M \in E\}$ contains a club in $[X]^\omega$. In particular if θ is a regular uncountable cardinal, then for any countable $X \subseteq H_\theta$,*

$$\{M \cap \omega_1 \mid (M \in [H_\theta]^\omega) \wedge (M \prec H_\theta)\}$$

contains a closed unbounded subset of ω_1.

11. Intersecting clubs and the Pressing Down Lemma

An important family of examples of stationary sets is provided by the uncountable regular cardinals.

Proposition 11.1. *If κ is a regular uncountable cardinal, κ is stationary. Moreover, if $\lambda < \kappa$ is a regular cardinal, $\{\alpha \in \kappa \mid \operatorname{cof}(\alpha) = \lambda\}$ is stationary.*

Remark 11.2. Notice that if κ is a regular uncountable cardinal, then $E \subseteq \kappa$ is club in κ if and only if $E \subseteq \kappa$ is closed an unbounded in κ.

Proof. Noting that $\bigcup \kappa = \kappa$, suppose that $f : \kappa^{<\omega} \to \kappa$ is a function. Define $g : \kappa \to \kappa$ by

$$g(\alpha) := \max(\alpha + 1, \sup\{f(s) \mid s \in \alpha^{<\omega}\}).$$

Notice that $g(\alpha) \in \kappa$ since κ is a regular cardinal. Define $\langle \alpha_\xi \mid \xi \in \kappa \rangle$ by recursion:

$$\alpha_\xi := \begin{cases} 0 & \text{if } \xi = 0 \\ f(\alpha_\eta) & \text{if } \xi = \eta + 1 \\ \sup\{\alpha_\eta \mid \eta \in \xi\} & \text{if } \xi \text{ is a limit ordinal} \end{cases}$$

Observe that this sequence is strictly increasing and if ξ is a limit ordinal, then $f[\alpha_\xi^{<\omega}] = \alpha_\xi$ and $\operatorname{cof}(\alpha_\xi) = \operatorname{cof}(\xi)$. In particular, if $\lambda < \kappa$ is a regular cardinal, $\operatorname{cof}(\alpha_\lambda) = \lambda$. □

Proposition 11.3. *If κ is an uncountable regular cardinal and $\mathcal{E}$ is a family of fewer than κ club subsets of κ, then $\bigcup \mathcal{E}$ is club. In particular every club in κ is stationary and a partition of a stationary subset of κ into fewer than κ pieces has at least one stationary piece.*

We will derive this proposition from a more powerful lemma known as the *Pressing Down Lemma*.

Lemma 11.4. *Suppose that S is a stationary set and r is a function defined on S such that $r(M) \in M$ for all $M \in S$. There is an $x \in \bigcup S$ such that $\{M \in S \mid r(M) = x\}$ is stationary.*

Proof. Suppose for contradiction that that the lemma is false for some S and r and set $X := \bigcup S$. For each x in X, let $f_x : X^{<\omega} \to X$ be such that if $M \subseteq X$ is f-closed, then $r(M) \neq x$. Define $f : X^{<\omega} \to X$ so that $f(\langle x \rangle^\frown \bar{y}) = f_x(\bar{y})$. By assumption, there is an M in S such that M is f-closed. Observe, however, that if $x \in M$, then M is f_x-closed and hence $r(M) \neq x$. But this contradicts our assumption that $r(M) \in M$. □

The pressing down lemma is equivalent to the following assertion.

Lemma 11.5. *Suppose that X is a nonempty set and $f_x : X^{<\omega} \to X$ is a function for each $x \in X$. The following set is a club:*

$$\{M \in \mathscr{P}(X)\backslash\{\varnothing\} \mid \forall x \in M \ (M \in C_{f_x})\}$$

The club in the conclusion of the previous theorem is called the *diagonal intersection* of the clubs $\{C_{f_x} \mid x \in X\}$ and is denoted $\Delta_{x\in X} C_{f_x}$. In particular, the previous lemma shows that any two clubs intersect and hence any club is stationary.

The following corollary captures the content of the Pressing Down Lemma for stationary sets of ordinals.

Corollary 11.6. *Suppose that κ is a regular cardinal and $S \subseteq \kappa$ is stationary. If $r : S \to \kappa$ satisfies $r(\alpha) < \alpha$ for all $\alpha \in S$, then r is constant on a stationary set.*

To see how to prove Proposition 11.3, suppose that $\mathcal{E} = \{E_\xi \mid \xi \in \lambda\}$ is a list of club subsets of κ for some $\lambda < \kappa$. Let S be the set of ordinals in κ which are greater than λ. If $\bigcap \mathcal{E}$ is not closed and unbounded, then $S := \kappa\backslash(\lambda \cup \bigcap \mathcal{E})$ is stationary. Define $r : S \to \lambda$ by

$$r(\alpha) = \min\{\xi \in \lambda \mid \alpha \notin E_\xi\}.$$

Clearly r is regressive since $\lambda < \min S$. Let $\xi \in \lambda$ be such that $T := \{\alpha \in S \mid r(\alpha) = \xi\}$ is stationary. But T is disjoint from E_ξ, which is a contradiction.

We've seen that any two closed unbounded sets must intersect. What about stationary sets? This is addressed by the following result of Ulam.

Proposition 11.7 (Ulam)**.** *Suppose that κ is an infinite cardinal. If $S \subseteq \kappa^+$, then S can be partitioned into κ^+ many stationary sets.*

Remark 11.8. In fact any stationary subset of a regular uncountable cardinal λ can be partitioned into λ stationary sets, but we will not prove this.

Proof. Using the Axiom of Choice, fix a sequence $\langle e_\beta \mid \beta \in \kappa^+\rangle$ such that for each $\beta \in \kappa^+$, $e_\beta : \beta \to \kappa$ is an injection. For $\alpha \in \kappa$ and $\beta \in \kappa^+$, define

$$S_{\alpha,\beta} := \{\gamma \in S \mid e_\gamma(\beta) = \alpha\}.$$

Observe that for each $\alpha < \kappa$, $\{S_{\alpha,\beta} \mid \beta \in \kappa^+\}$ is pairwise disjoint. On the other hand, for each $\beta \in \kappa^+$, $S = \bigcup\{S_{\alpha,\beta} \mid \alpha \in \kappa\}$. By Proposition 11.3, for each $\beta \in \kappa^+$ there is an $\alpha_\beta \in \kappa$ such that $S_{\alpha_\beta,\beta}$ is stationary. Since κ^+ is regular, there must be a single $\alpha \in \kappa$ such that $B := \{\beta \in \kappa^+ \mid \alpha = \alpha_\beta\}$ has cardinality κ^+. We now have that $\{S_{\alpha,\beta} \mid \beta \in B\}$ is a pairwise disjoint family of stationary sets. $\square$

12. The constructible universe $\mathbf{L}$

We will now turn to the task of proving the following result of Kurt Gödel.

Theorem 12.1. *There is a class* $\mathbf{L}$ *with the following properties:*

- *(a)* $\mathbf{L}$ *is transitive;*
- *(b) ZF proves* $(\mathbf{L}, \in) \models$ *ZFC;*
- *(c) if* $\mathbf{M}$ *is any transitive class such that* $\mathbf{M} \models$ *ZF and* $\mathbf{ON} \subseteq \mathbf{M}$*, then* $\mathbf{L} \subseteq \mathbf{M}$.
- *(d) if* $\mathbf{M}$ *is a transitive class such that* $\mathbf{M} \models$ *ZF, then* $\mathbf{M} \models x \in \mathbf{L}$ *if and only if* $x \in \mathbf{L}$.
- *(e)* $(\mathbf{L}, \in)$ *satisfies that for every infinite set* X, $|\mathscr{P}(X)| = |X|^+$.

The basic idea behind this theorem is to revisit the construction of the cumulative hierarchy so as to only add sets to $\mathbf{L}$ when it is required by the Separation Scheme. First we will need to develop a variant of the powerset operation, which we will denote $\mathscr{D}$. Let $\langle \phi_i \mid i \in \omega \rangle$ be a recursive listing of the formulas in the language of set theory so that ϕ_i has at most i free variables. If A is a set, define $\mathscr{D}(A)$ to be the collection of all sets of the form

$$\{b \in A \mid (A, \in) \models \phi_i(\bar{a}, b)\}$$

where $i \in \omega$ and $\bar{a} \in A^i$.

Proposition 12.2. *For any set* A*, the following are true:*

- *(a) every finite subset of* A *is in* $\mathscr{D}(A) \subseteq \mathscr{P}(A)$ *and* $\mathscr{D}(A)$ *is a Boolean algebra;*
- *(b) if* A *is transitive, then* $A \subseteq \mathscr{D}(A)$*;*
- *(c) if* A *can be well ordered, then so can* $\mathscr{D}(A)$ *and moreover* $|A| \leqslant |\mathscr{D}(A)| \leqslant |A| + \aleph_0$*;*
- *(d) there is a* Σ_0*-formula* $\phi(u, v, w)$ *such that* $\phi(A, B, \omega)$ *is equivalent to* $B \in \mathscr{D}(A)$*;*
- *(e) if* $\mathbf{M}$ *is a transitive class,* $(\mathbf{M}, \in) \models$ ZF*, and* $A \in \mathbf{M}$*, then* $\mathscr{D}(A) \subseteq \mathbf{M}$*;*

Proof. Items (a) and (b) are immediate from the definition of $\mathscr{D}(A)$. To see the second, suppose that $a \in A$. Since A is transitive, $a \subseteq A$. Thus $a = \{b \in A \mid (A, \in) \models b \in a\}$ is in $\mathscr{D}(A)$.

To see (c), suppose that A can be well ordered. If A is empty then $\mathscr{D}(A) = \{\varnothing\}$ and there is nothing so show. Suppose now that $f : \alpha \to A$ be a surjection for some infinite ordinal α. Define $g : \alpha^{<\omega} \to \mathscr{D}(A)$

by

$$g(\xi_0, \dots, \xi_k) = \begin{cases} \{b \in A \mid (A, \in) \models \phi_i(f(\xi_1), \dots, f(\xi_k))\} & \text{if } \xi_0 = i \leqslant k \\ g(0) & \text{otherwise} \end{cases}$$

Clearly g is a surjection. Since we've seen that $|\alpha^{<\omega}| = |\alpha|$, this completes the proof.

To see (e), suppose $\mathbf{M}$ is a transitive class and $(\mathbf{M}, \in) \models \mathbf{ZF}$. Let $A \in \mathbf{M}$ and suppose that $B \in \mathscr{D}(A)$. Fix an i and $\bar{a} \in A^i$ such that

$$B = \{b \in A \mid (A, \in) \models \phi_i(\bar{a}, b)\}$$

Observe that $(t, \in) \models \phi_u(\bar{v}, w)$ is expressible by a Σ_0-formula. By Separation applied in $\mathbf{M}$ and Extensionality, B must be in $\mathbf{M}$. $\square$

For each $\alpha \in \mathbf{ON}$, define L_α recursively as follows:

$$L_\alpha := \begin{cases} \emptyset & \text{if } \alpha = 0 \\ \mathscr{D}(L_\beta) & \text{if } \alpha = \beta + 1 \\ \bigcup\{L_\beta \mid \beta \in \alpha\} & \text{if } \alpha \text{ is a limit ordinal} \end{cases}$$

Similarly, if X is a set, define $L_\alpha(X)$ recursively so that:

$$L_\alpha(X) := \begin{cases} \mathrm{tc}(\{X\}) & \text{if } \alpha = 0 \\ \mathscr{D}(L_\beta(X)) & \text{if } \alpha = \beta + 1 \\ \bigcup\{L_\beta(X) \mid \beta \in \alpha\} & \text{if } \alpha \text{ is a limit ordinal} \end{cases}$$

Set $\mathbf{L} := \bigcup\{L_\alpha \mid \alpha \in \mathbf{ON}\}$ and $\mathbf{L}(X) := \bigcup\{L_\alpha(X) \mid \alpha \in \mathbf{ON}\}$.

Theorem 12.3. *The following are true:*

(a) *for each $\alpha \in \mathbf{ON}$ and set X, L_α and $L_\alpha(X)$ are transitive sets and $\mathbf{L}$ and $\mathbf{L}(X)$ are transitive classes;*
(b) *if $\alpha \in \beta \in \mathbf{ON}$, then $L_\alpha \in L_\beta$;*
(c) *$(\mathbf{L}, \in)$ and $(\mathbf{L}(X), \in)$ both satisfy ZF;*
(d) *if $\mathbf{M}$ is a transitive class and $(\mathbf{M}, \in) \models$ ZF, then $\mathbf{L} \subseteq \mathbf{M}$.*
(e) *if $\mathbf{M}$ is a transitive class with $X \in \mathbf{M}$ and $(\mathbf{M}, \in) \models$ ZF, then $\mathbf{L}(X) \subseteq \mathbf{M}$.*

Proof. We will only give the proofs for $\mathbf{L}$; the arguments for $\mathbf{L}(X)$ are similar. That each L_α is transitive is proved by induction on α. If $\alpha = 0$, this is trivial. If $\alpha = \beta + 1$ and L_β is assumed to be transitive, then every element of L_α is a subset of $L_\beta \subseteq \mathscr{D}(L_\beta) = L_\alpha$. If α is a limit ordinal, then L_α is (inductively) a union of transitive sets and hence is transitive. Since clearly $L_\alpha \in \mathscr{D}(L_\alpha)$, it follows by induction on β that $L_\alpha \in L_\beta$ whenever $\alpha \in \beta$.

To see that $(\mathbf{L}, \in)$ satisfies the axioms of ZFC, first observe that Extensionality and Foundation are satisfied by virtue of $\mathbf{L}$ being a transitive class. Also, Emptyset and Infinity hold since 0 and ω are in $\mathbf{L}$, respectively. Pairing holds in $\mathbf{L}$ since if $x, y \in A$, $\{x, y\} \in \mathscr{D}(A)$. Similarly, $\mathbf{L}$ satisfies Union since if A is transitive and $B \in A$, then $\bigcup B \in \mathscr{D}(A)$.

In order to verify the Separation Scheme, suppose that $A, \bar{x} \in \mathbf{L}$ and $\phi(u, \bar{v})$ is a formula. By the Reflection Theorem, there is an ordinal δ such that $A, \bar{x} \in L_\delta$ and ϕ is absolute between L_δ and $\mathbf{L}$. It follows that

$$\{a \in A \mid (L_\delta, \in) \models \phi(a, \bar{x})\} = \{a \in A \mid (\mathbf{L}, \in) \models \phi(a, \bar{a})\}.$$

The former set is in $L_{\delta+1}$ while the latter collection is the subset of A postulated to exist by Separation for ϕ, $A, \bar{x}$.

The Collection Scheme is handled in a similar way: if $A, \bar{x} \in \mathbf{L}$ and $\phi(u, v, \bar{w})$ is a formula such that

$$(\mathbf{L}, \in) \models \forall u \in A \ \exists! v \phi(a, v, \bar{x})$$

then find a δ such that $\forall u \in s \ \exists! v \phi(u, v, \bar{w})$ is absolute between L_δ and $\mathbf{L}$.

Finally, to see that $\mathbf{L}$ satisfies the Powerset Axiom, it is sufficient to show that if $A \in \mathbf{L}$, then there is a δ such that $\mathscr{P}(A) \cap \mathbf{L} \subseteq L_\delta$. Define $\rho : \mathscr{P}(A) \to \mathbf{ON}$ by $\rho(B) := \min\{\alpha \in \mathbf{ON} \mid B \in L_\alpha\}$ if $B \in \mathbf{L}$ and $\rho(B) := 0$ if B is not in $\mathbf{L}$. By Collection, the range of ρ is a set; let $\delta \in \mathbf{ON}$ be any strict upper bound for the range of ρ. $\square$

13. The Axiom of Choice and the Generalized Continuum Hypothesis in L

We've already seen the basic mechanism for why $\mathbf{L}$ satisfies the Axiom of Choice: if A can be well ordered, then so can $\mathcal{D}(A)$. With a little more care, we can prove the following result.

Theorem 13.1. *There is a formula $\phi(u, v)$ in the language of set theory such that all quantification in ϕ is restricted to $\mathbf{L}$ and such that the following are theorems of ZF:*

(a) $\forall x \forall y(\phi(x, y) \to ((x \in \mathbf{L}) \wedge (x \in \mathbf{L}) \wedge (x \neq y)))$
(b) $\forall x \in \mathbf{L}\ \forall y \in \mathbf{L}\ (\phi(x, y) \vee \phi(y, x) \vee (x = y))$
(c) $\forall X \in \mathbf{L}\ (X = \varnothing \vee \exists x \in X\ \forall y \in X\ ((x = y) \vee \phi(x, y)))$

It is customary to write $x <_{\mathbf{L}} y$ to denote $\phi(x, y)$. The theorem asserts that ZF proves $<_{\mathbf{L}}$ is a class well-ordering of $\mathbf{L}$. In particular, $(\mathbf{L}, \in)$ satisfies the Axiom of Choice. The ordering $<_{\mathbf{L}}$ is defined recursively: given $<_{\mathbf{L}} \restriction L_\alpha$, we define $<_{\mathbf{L}} \restriction L_{\alpha+1}$. If $x \in L_\alpha$ and $y \in L_{\alpha+1} \backslash L_\alpha$, then $x <_{\mathbf{L}} y$. If $x \in L_{\alpha+1} \backslash L_\alpha$, then there is a tuple $(i, \bar{a})$ such that $i \in \omega$, $\bar{a} \in L_\alpha^i$, and

$$x = \{z \in L_\alpha \mid (L_\alpha, \in) \models \phi_i(x, \bar{a})\}$$

Let p_x denote the lexicographically least such tuple with respect to $<_{\mathbf{L}}$. If $x, y \in L_{\alpha+1} \backslash L_\alpha$, define $x <_{\mathbf{L}} y$ if and only if $p_x <_{\text{lex}} p_y$. Thus if γ is the ordertype of $(L_\alpha, <_{\mathbf{L}})$, then the ordertype of $(L_{\alpha+1}, <_{\mathbf{L}})$ is (at most) $\sup_n \sum_{i<n} \gamma^i = \gamma^\omega$.

Now we turn to the task of proving that $\mathbf{L}$ satisfies the Generalized Continuum Hypothesis (GCH): for every infinite cardinal κ, $|\mathscr{P}(\kappa)| = \kappa^+$. Let us begin by noting that for each infinite α, $|L_\alpha| = |\alpha|$. Thus it suffices to prove the following theorem.

Theorem 13.2. *For every infinite cardinal κ, $\mathscr{P}(\kappa) \cap \mathbf{L} \subseteq L_{\kappa^+}$.*

The proof that GCH holds in $\mathbf{L}$ makes use of a general phenomenon known as *condensation* which is both powerful and characteristic of $\mathbf{L}$. In order to prove this, it will be helpful to give an alternate, more elementary description of $\mathscr{D}(A)$.

Proposition 13.3. *For every set A, there is a unique sequence $\langle \mathrm{Def}(A, n) \mid n \in \omega \rangle$ of sets such that:*

(a) $\mathrm{Def}(A, n) \subseteq \mathscr{P}(A^n)$*;*
(b) for each n and $i, j < n$ the following sets are in $\mathrm{Def}(A, n)$*: $\{\bar{a} \in A^n \mid a_i \in a_j\}$ and $\{\bar{a} \in A^n \mid a_i \in a_j\}$;*
(c) $\mathrm{Def}(A, n)$ *is closed under taking intersections and complements;*

(d) *if* $R \in \mathrm{Def}(A, n+1)$*, then* $\{\bar{a} \in A^n \mid \exists b \in A\ (\bar{a}^\frown\langle b\rangle)\}$ *is in* $\mathrm{Def}(A, n)$*;*

(e) *if* $\langle \mathrm{Def}'(A, n) \mid n \in \omega\rangle$ *is any other sequence satisfying the above conditions,* $\mathrm{Def}(A, n) \subseteq \mathrm{Def}'(A, n)$ *for all* $n \in \omega$*.*

Moreover for every A*,* $\mathscr{D}(A)$ *equals*

$$\{B \subseteq A \mid \exists n \exists R \in \mathrm{Def}(A, n+1)\ \exists \bar{a} \in A^n\ (B = \{b \in A \mid \bar{a}^\frown\langle b\rangle \in R\})\}.$$

The proof is left as an exercise. Let Λ be the conjunction of:

- the axioms of Pairing, Foundation, Union;
- "There is a least infinite ordinal ω";
- "For every A there is a unique sequence $\langle \mathrm{Def}(A, n) \mid n \in \omega\rangle$ satisfying the main conclusion of Proposition 13.3";
- "for every A, $\mathscr{D}(A)$ exists," taking the definition of $\mathscr{D}(A)$ from the conclusion of Proposition 13.3;
- "for every ordinal α, L_α exists."
- "for every x, there is an ordinal α such that $x \in L_\alpha$."

Notice that assertions such as "L_α exists" are really shorthand for an assertion like "There is a sequence $\langle L_\xi \mid \xi \in \alpha + 1\rangle$ which satisfies the recursive definition of the **L**-hierarchy." The key feature of Λ is captured by the following proposition, whose proof is self-evident.

Proposition 13.4. *If* (M, E) *is a set equipped with a well founded relation, then* $(M, E) \models \Lambda$ *if and only if there is a limit ordinal* $\nu > \omega$ *such that* $(M, E) \cong (L_\nu, \in)$*.*

Proof of Theorem 13.2. Let $X \subseteq \kappa$ be in $\mathbf{L}$ and fix a limit ordinal $\nu \geqslant \kappa$ such that X is in L_ν. Let $M \prec L_\nu$ such that $\kappa \subseteq M$, $X \in M$ and $|M| = \kappa$. Observe that $(L_\nu, \in)$ satisfies Λ and therefore $(M, \in)$ satisfies Λ. Let $\pi : M \cong L_\alpha$ be the transitive collapse of $(M, \in)$. Since $|M| = \kappa$, $\kappa \leqslant \alpha < \kappa^+$. It therefore suffices to show that $\pi(X) = X$, since then $X \in L_\alpha \subseteq L_{\kappa^+}$. This followed from the following general fact.

Proposition 13.5. *Suppose that* M *is a set and* $A \in M$ *is transitive and a subset of* M*. If* $\pi : M \to N$ *is the transitive collapse, then* $\pi(B) = B$ *for every* $B \subseteq A$ *in* M*.*

Proof. Suppose that the proposition is false and let A be an $\in$-minimal counterexample. By minimality, we have that $\pi \restriction A$ is the identity. Thus if $B \subseteq A$, then $\pi(B) = \{\pi(b) \mid b \in B\} = B$. But this contradicts that A was a counterexample. $\square$

$\square$

14. The constructions of $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$

Now we'll turn to showing that familiar mathematical constructions can be carried out in a model of set theory. The main difficulty is to give a set theoretic definition of $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ — more involved constructions such as rings of polynomials, function spaces, manifolds, and tangent bundles are themselves usually defined set-theoretically in terms of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ via Cartesian products.

Let $\mathbb{N}$ denote $\omega\backslash\{0\}$. In order to define $\mathbb{Z}$ and $\mathbb{Q}$, we need to work with equivalence relations and their quotients. We've already seen how to formalize the Cartesian product. The quotient of a set by an equivalence relation is also a fundamental construction which we will need. Recall that an equivalence relation E on a set X is a reflexive, symmetric, transitive binary relation. We define X/E to be the collection of all E-equivalence classes. That this is a set is a consequence of the Powerset Axiom and the Separation Scheme:

$$X/E := \{A \in \mathscr{P}(X) \mid A \text{ is an } E\text{-equivalence class}\}$$

where A *is an* E*-equivalence class* abbreviates

$$(A \neq \varnothing) \wedge (\forall a \in A\ \forall x \in X\ ((a,x) \in E \leftrightarrow x \in A))$$

The integers are defined as formal differences, up to an appropriate equivalence. Define $\sim$ on ω^2 by $(m,n) \sim (m',n')$ if $m + n' = m' + n$. Here $+$ refers to ordinal arithmetic on ω. Intuitively a pair (m,n) is thought of as representing a formal difference $m - n$. Addition, inversion, and multiplication on $\omega^2/\sim$ are defined by

$$[(m,n)]_\sim + [(m',n')]_\sim = [(m+m', n+n')]_\sim$$
$$-[(m,n)]_\sim = [(n,m)]_\sim$$
$$[(m,n)]_\sim \cdot [(m',n')]_\sim = [(m\cdot m' + n\cdot n', m\cdot n' + m'\cdot n)]_\sim.$$

It is left to the reader to check that this is well defined. Next observe that each $\sim$-class contains a unique representative (m,n) in which $\min(m,n) = 0$. We define $\mathbb{Z}$ to be the set of all such representative pairs which the operations $+$, $\cdot$, and $-$ induced by those on $\omega^2/\sim$. The advantage of defining $\mathbb{Z}$ formally in terms of canonical representatives of $\sim$-classes instead of $\sim$-classes themselves is that then $\mathbb{Z} \subseteq V_\omega$ as opposed to $\mathbb{Z} \subseteq V_{\omega+1}$. It is common to abuse notation and write n for $(n,0)$ and $-n$ for $(0,n)$. Notice that this embedding of ω inside $\mathbb{Z}$ respects the operations $+$ and $\cdot$.

Similarly, one defines $\mathbb{Q}$ to be the set of representatives of equivalence classes of pairs $(m,n) \in \mathbb{Z} \times \mathbb{N}$. Specifically (m,n) is equivalent to (m',n') if $m \cdot n' = m' \cdot n$, noting that each equivalence class contains a unique element (m,n) where there is no $k > 1$ which divides into both

m and n. The advantage of working with representatives becomes even more apparent here: with this definition $\mathbb{Q} \subseteq V_\omega$ whereas if we defined both $\mathbb{Q}$ and $\mathbb{Z}$ in terms of equivalence classes, $\mathbb{Q}$ would only be contained in $V_{\omega+4}$.

Define $\mathbb{R} \subseteq \mathscr{P}(\mathbb{Q})$ to consist of all Dedekind cuts: all $r \subseteq \mathbb{Q}$ such that that $r \neq \emptyset$, $r \neq \mathbb{Q}$, r has no last element, and r is an initial interval in $\mathbb{Q}$. The order on $\mathbb{R}$ is simply containment. We view $\mathbb{Q}$ as a subset of $\mathbb{R}$ via the map $q \mapsto \{s \in \mathbb{Q} \mid s < q\}$. It is left to the reader to check that the operations of $+$, $\cdot$, and $-$ extend continuously to $\mathbb{R}$ where $\mathbb{Q}$ and $\mathbb{R}$ are given the order topology.

15. Some nonmeasurable sets of reals

Suppose now that $(G, \cdot)$ is a locally compact topological group (i.e. the group operation and the inversion operation are both continuous). Recall that the Borel subsets of G are the smallest σ-algebra which contains the open sets.

Theorem 15.1. *For any locally compact topological group G, there is a function μ defined on a subset $\mathscr{M}$ of $\mathscr{P}(G)$ such that the following are true:*

- *$\mathscr{M}$ is a σ-algebra including the Borel subsets of G and μ takes values in $[0, \infty]$;*
- *if $\{A_i \mid i \in \omega\}$ is a countable collection of elements of $\mathscr{M}$, then $\mu(\bigcup_{i=0}^{\infty} A_i) = \sum_{i=0}^{\infty} \mu(A_i)$;*
- *if U is an open set with compact closure, then $0 < \mu(U) < \infty$;*
- *if $A \in \mathscr{M}$ and $g \in G$, then $\mu(gA) = \mu(A)$;*
- *if $X \subseteq G$, then X is in $\mathscr{M}$ if and only if there are Borel sets $B, E \subseteq G$ such that $X \triangle B \subseteq E$ and $\mu(E) = 0$.*

Moreover, if μ_0 and μ_1 satisfy the above conditions, then their domains coincide and for some $0 < C < \infty$, $\mu_1(A) = C\mu_0(A)$ for all A in their common domain.

The measure μ in the previous theorem is called Haar measure. If G is compact, then generally μ is chosen so that $\mu(G) = 1$. In other cases, there is typically a natural choice of an open set with compact closure which is chosen to be measure 1.

Some important examples of Haar measure are given by $\mathbb{R}^d$ with coordinatewise addition. The unique invariant measure on $\mathbb{R}^d$ assigning measure 1 to $(0, 1)^d$ is Lebesgue measure. If we regard 2^ω as a compact group with coordinatewise addition modulo 2, then the normalized Haar measure is the same as the product measure where $\{0, 1\}$ is given the uniform measure. We also note the following general fact, which is a form of Kolmogorov's 0-1 law.

Theorem 15.2. *Suppose that G is a locally compact metric group and $H \leqslant G$ is a dense subgroup. If $B \subseteq G$ is Haar measurable and $\mu(B \triangle hB) = 0$ for all $h \in H$, then $\mu(B) = 0$ or $\mu(G \backslash B) = 0$.*

It is natural to ask whether $\mathscr{M}$ is all of $\mathscr{P}(G)$. In the presence of the Axiom of Choice, this is not the case. We'll consider three different examples.

The first is the classic construction of a nonmeasurable subset of $\mathbb{R}$ known as a *Vitali set*. Define $\sim$ on $[0, 1]$ by $x \sim y$ if $y - x$ is in $\mathbb{Q}$. By the Axiom of Choice, there is a function $f : [0, 1]/\sim \to [0, 1]$ such that $f(a) \in a$ whenever $a \in [0,1]/\sim$. Let X be the range of f and observe that $X \subseteq [0,1]$ meets each $\sim$-class in exactly one point. We claim that X must be nonmeasurable. Suppose that this is not the case. Then $\{2^{-n} + X \mid n \in \omega\}$ is an infinite pairwise disjoint family: if $z \in 2^{-m} + X \cap 2^{-n} + X$, then $z = 2^{-m} + x = 2^{-n} + y$ for $x, y \in X$ which are necessarily distinct which would contradict that X meets each $\sim$-class in a unique point. Since $\cup\{2^{-n} + X \mid n \in omega\} \subseteq [0,2]$, it must be that $\lambda(X) = 0$. On the other hand, $[0,1] \subseteq \bigcup\{q + X \mid q \in \mathbb{Q}\}$ which violates countable additivity.

Next suppose that $\{E_\alpha \mid \alpha \in \beta\}$ is a collection of measure 0 subsets of $[0,1]$ whose union is not measure 0 such that if $\alpha \in \beta$, then $E_\alpha \subseteq E_\beta$. Notice that such a collection exists (assuming only ZF) if there is a well orderable collection of measure 0 sets whose union does not have measure 0. We claim that either $X := \bigcup\{E_\alpha \mid \alpha \in \beta\}$ is nonmeasurable or else $R := \bigcup\{E_\alpha \times (X \backslash E_\alpha) \mid \alpha \in \beta\}$ is a nonmeasurable subset of $[0,1] \times [0,1]$. Suppose that X is measurable. Observe that for each $x \in X$ if $x \in E_\alpha$, $\chi_R(x,y) = 1$ if $y \in X \backslash E_\gamma$ and 0 otherwise. Thus $\int_0^1 \int_0^1 \chi_R(x,y)\ dy\ dx = \int_0^1 \lambda(X)\chi_X\ dx = \lambda(X)^2$. On the other hand, for each $y \in X$ if $y \in E_\alpha$, then $\chi_R(x,y) = 0$ unless $x \in E_\alpha$. Thus $\int_0^1 \int_0^1 \chi_R(x,y)\ dx\ dy = \int_0^1 0\ dy = 0$. Since we assumed $\lambda(X) > 0$, this violates Fubini's Theorem.

Finally, suppose that $\mathscr{U}$ is a *nonprincipal ultrafilter* on ω: $\mathscr{U} \subseteq \mathscr{P}(\omega)$ is closed under taking supersets and finite intersections, contains X or $\omega \backslash X$ for every $X \subseteq \omega$, and does not contain any singletons. Notice that $\mathscr{P}(\omega)$ is also a compact group when given the operation of symmetric difference (it is in fact isomorphic to 2^ω equipped with coordinatewise addition mod 2). We claim that any ultrafilter is nonmeasurable with respect to the Haar measure. This follows from two observations. First, the map $X \mapsto \omega \triangle X$ preserves Haar measure and maps $\mathscr{U}$ to its complement and vice versa. Thus if $\mathscr{U}$ were measurable, it would have to be that

$$2\mu(\mathscr{U}) = \mu(\mathscr{U}) + \mu(\mathscr{P}(\omega) \backslash \mathscr{U}) = 1$$

and hence that $\mu(\mathscr{U}) = 1/2$. On the other hand consider the collection $\mathscr{F}$ of finite subsets of ω. This is a dense subgroup of $\mathscr{P}(\omega)$. It follows from the definition of an ultrafilter that if $F \in \mathscr{F}$, then $F \triangle \mathscr{U} = \mathscr{U}$. Thus $\mu(\mathscr{U}) = 0$ or $\mu(\mathscr{U}) = 1$, both of which contradict our previous observation. This argument shows, in particular, that if $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is a *filter* — it is closed under finite intersections and supersets — and it contains the complement of every finite subset of ω, then $\mathscr{F}$ is measure 0 provided it is measurable.

16. Souslin's problem

Recall that a linear order is a set L equipped with a binary relation $\leqslant$ which is transitive, reflexive, antisymmetric, and satisfies that for all x and y in L, either $x \leqslant y$ or $y \leqslant x$. A linear order is *dense* if it has no first or last element and for every $x < y$ in L, there is a z in L such that $x < z < y$. A subset D of a linear order is *dense* if it intersects every nonempty open interval. A linear order is separable if it has a countable dense subset. A linear order is *complete* if every bounded subset has a supremum. Cantor proved the following result.

Theorem 16.1. *Any countable dense linear order is isomorphic to* $(\mathbb{Q}, \leqslant)$ *and any separable complete linear order is isomorphic to* $(\mathbb{R}, \leqslant)$.

Observe that if L is separable, then every collection of pairwise disjoint open intervals must be countable. Such a linear order is said to satisfy the countable chain condition (c.c.c.). Souslin asked whether separability can be relaxed to the c.c.c. in Cantor's characterization. This is become known as *Souslin's Problem*; a positive answer is known as *Souslin's Hypothesis* and a counterexample is a *Souslin continuum.*

It turns out that while completeness plays a crucial role in Cantor's theorem, it is quite irrelevant to Souslin's problem. Specifically, we say that an uncountable linear order is a *Souslin line* if it is c.c.c. but the closure of every countable subset is countable. We will first establish some properties of c.c.c. linear orders.

Proposition 16.2. *If L is a c.c.c. linear order, L does not contain any uncountable chain of intervals which is well ordered by either $\subseteq$ or $\supseteq$. In particular, L does not contain an uncountable well order or the reverse of an uncountable well order.*

Proof. Suppose that $\langle I_\xi \mid \xi < \omega_1 \rangle$ is a sequence of intervals which is either strictly increasing or strictly decreasing with respect to $\subseteq$. Let J_ξ be the symmetric difference of I_ξ and $I_{\xi+1}$. It follows that $\{J_\xi \mid \xi < \omega_1\}$ is an uncountable family of pairwise disjoint intervals. □

Proposition 16.3. *If L is a c.c.c. linear order and $\mathscr{U}$ is a collection of open subsets of L, then there is a countable subcollection $\mathscr{U}_0 \subseteq \mathscr{U}$ which has the same union.*

Proof. Since every nonempty open set is a union of basic intervals, we may assume that $\mathscr{U}$ consists of open intervals. Since L has the c.c.c., there is a countable $\mathscr{V} \subseteq \mathscr{U}$ such that every element of $\mathscr{U}$ intersects some element of $\mathscr{V}$. Otherwise, we could recursively construct an uncountable pairwise disjoint family of elements of $\mathscr{U}$. Now let $V \in \mathscr{V}$. It suffices to show that there is a countable subset of

$\{U \in \mathscr{U} \mid U \cap V \neq \emptyset\}$ which has the same union. Suppose not. Recursively construct $\langle U_\alpha \mid \alpha \in \omega_1 \rangle$ consisting of elements of $\mathscr{U}$ which intersect V such that $I_\xi := \bigcup\{U_\alpha \mid \alpha < \xi\}$ is a proper subset of $\bigcup\{U_\alpha \mid \alpha < \xi + 1\}$. Since the union of two intersecting intervals is an interval and since an increasing union of intervals is an interval, it follows that $\langle I_\xi \mid \xi < \omega_1 \rangle$ is a strictly increasing sequence of intervals, which contradicts Proposition 16.2. □

Proposition 16.4. *If L is a c.c.c. linear order and $D \subseteq X \subseteq L$, then the closure of D with respect to the order topology on X differs from the closure with respect to the subspace topology by a countable set.*

Proof. Homework. □

Theorem 16.5. *The following are equivalent:*

(a) There is a c.c.c. nonseparable linear order.
(b) There is a Souslin continuum.
(c) There is a Souslin line.

Proof. Trivially (c) implies (a). To see that (a implies (b), suppose that L is a c.c.c. and nonseparable. Let $\mathscr{U}$ be the collection of all separable open intervals in L. By Proposition 16.3, there is a countable subset of $\mathscr{U}$ with the same union. It follows that $\bigcup \mathscr{U}$ is separable and that $L' := L \setminus \bigcup \mathscr{U}$ is nonempty and has the property that every every nonempty interval in L' is nonseparable. Let $L_0 \subseteq L'$ consist of all x in L' which do not have an immediate predecessor in L' and which are not the greatest of least element of L'. If $x < y$ are in L_0, then $(x, y) \cap L'$ is nonempty. Furthermore, observe that the least element of L' — if it exists — does not have an immediate successor in L' since otherwise L' would have a nonempty finite and hence separable open interval. It follows that L_0 has no least elements and, by an analogous argument, no greatest elements. Let K be the collection of Dedekind cuts of L_0 ordered by $\subseteq$. Since L_0 is dense, c.c.c., and nonseparable and since K contains a dense suborder isomorphic to L_0, K is a Souslin continua.

To see that (b) implies (c), suppose that K is a Souslin continuum and construct a sequence of points $\{x_\alpha \mid \alpha \in \omega_1\}$ in K by transfinite recursion. Given $\{x_\alpha \mid \alpha \in \beta\}$, let x_β be any element of K not in the closure of $\{x_\alpha \mid \alpha \in \beta\}$. This is always possible since K is not separable. Let $L = \{x_\alpha \mid \alpha \in \omega_1\}$ with the order inherited from K. Clearly L is uncountable and it inherits the countable chain condition from K. Since the closure of any countable subset of L in the subspace topology is countable, Proposition 16.4 implies that the closure of any countable subset of L in the interval topology is countable. □

17. Trees and linear orders

Often it is useful to translate questions about linear orders into questions about *trees*. A *tree* is a partially ordered set $(T, \leqslant)$ such that for all $t \in T$, $\{s \in T \mid s < t\}$ is well ordered by $\leqslant$. The ordertype of $(\{s \in T \mid s < t\}, \leqslant)$ is called the *height* of t. The set of all elements of T of height α is called the α^{th} level of T and is denoted T_α. The height of T is the least α such that T_α is empty.

An example of a tree is $\sigma\mathbb{Q}$, which consists of all subsets s of $\mathbb{Q}$ which are well ordered by the usual order on $\mathbb{Q}$. $\sigma\mathbb{Q}$ is ordered by $s \leqslant t$ if s is an initial part of t. This is isomorphic to the collection of all strictly increasing sequences of rationals. The α^{th}-level of $\sigma\mathbb{Q}$ consists of those $s \in \sigma\mathbb{Q}$ which have order type α. The tree $\sigma\mathbb{Q}$ has no uncountable chains — the union of such a chain would be an uncountable well ordered subset of $\mathbb{Q}$, which is absurd. We have seen previously in the homework that $\sigma\mathbb{Q}$ does not admit a strictly increasing map into $\mathbb{Q}$. Trees which admit a strictly increasing map into $\mathbb{Q}$ are precisely those which are countable unions *antichains* — pairwise incomparable subsets of the tree.

A typical example of a tree is a set of sequences, equipped with the order of extension: $s \leqslant t$ if t extends s as a function. In fact this is a completely general example of a tree. To see this, suppose that $(T, \leqslant)$ is any tree. If t is in T and α is at most the height of T, there is a unique $t' \in T_\alpha$ such that $t' \leqslant t$; this is the projection of t to level α and is denoted $t'|_\alpha$. Define a function σ on T so that $\sigma(t)$ is a sequence of length $\mathrm{ht}(t) + 1$ where $\sigma(t)(\xi) = t|_\xi$ if $\xi \leqslant \mathrm{ht}(t)$.

Proposition 17.1. *If L is a linear order, then L is isomorphic to a set of binary sequences equipped with the lexicographic order.*

Proof. Fix a well ordering $\lhd$ of L and let θ be the ordertype of $(L, \lhd)$. Let x_ξ denote the ξ^{th} element of L with respect to $\lhd$ and define $f_\xi : \theta \to 2$ by $f_\xi(\eta) = 1$ if $x_\eta \leqslant x_\xi$ and $f_\xi(\eta) = 0$ otherwise. If fact for any $\xi, \eta, \alpha < \theta$, if $f_\xi(\alpha) = 0 < f_\eta(\alpha)$, then $x_\xi < x_\alpha \leqslant x_\eta$ and so in particular $x_\xi <_{\text{lex}} x_\eta$. Since $x_\xi <_{\text{lex}} x_\eta$ implies $f_\xi(\eta) = 0 < f_\eta(\eta) = 0$, this implies that $x_\xi \mapsto f_\xi$ is an order preserving map from $(L, \leqslant)$ to $(\{f_\xi \mid \xi < \theta\}, \leqslant_{\text{lex}})$. □

Now suppose that $(T, \leqslant)$ is a tree. Let $\mathscr{C}(T)$ denote the collection of all maximal chains in T. Observe that if x is in $\mathscr{C}(T)$ and $t \in x$, then maximality of x implies that $\{s \in T \mid s < t\} \subseteq s$. If $x \cap T_\xi$ is nonempty, it contains a unique element which we will denote x_ξ. Observe that the set of all ξ such that $x \cap T_\xi$ is nonempty is an ordinal. Also, if $x \neq y$ are in $\mathscr{C}(T)$, then there is a ξ such that $x_\xi \neq y_\xi$ and both are defined.

Define $\Delta(x,y)$ to be the least such ξ. Now fix a linear order $\trianglelefteq$ of T. Define $x <_{\mathrm{lex}} y$ if $x_\delta \lhd y_\delta$.

Lemma 17.2. *If (a,b) is a nonempty open interval in $\mathscr{C}(T)$, then it contains a set of the form $\{x \in \mathscr{C}(T) \mid t \in x\}$. Moreover for every $t \in T$, $\{x \in \mathscr{C}(T) \mid t \in x\}$ is a nonempty interval in $\mathscr{C}(T)$.*

Proof. Suppose that $a <_{\mathrm{lex}} c <_{\mathrm{lex}} b$ are in $\mathscr{C}(T)$. If $\xi := \max(\Delta(a,c), \Delta(c,b))$, then $t := c_\xi$ is defined. If $x \in \mathscr{C}(T)$ has $t \in x$, then $x_\xi = t$ and hence $a <_{\mathrm{lex}} x <_{\mathrm{lex}} b$. To see that $\{x \in \mathscr{C}(T) \mid t \in x\}$ is a nonempty interval in $\mathscr{C}(T)$, first observe that Zorn's lemma implies that any element of T is contained in a maximal chain. Now suppose $x <_{\mathrm{lex}} y$ both have t as an element. If $t \notin z$ for $z \in \mathscr{C}(T)$, then

$$\Delta(x,y) > \mathrm{ht}(t) \geqslant \max(\Delta(x,z), \Delta(y,z))$$

which implies that $x <_{\mathrm{lex}} z$ if and only if $y <_{\mathrm{lex}} z$. □

Now suppose that $L \subseteq 2^\theta$ for some ordinal θ and L is equipped with the $<_{\mathrm{lex}}$ ordering. Define $T(L)$ to be the set of all sequences t such that there exist $x \neq y$ in L with t an initial part of both. Elements of L naturally correspond to elements of $\mathscr{C}(T(L))$: for every f in L, there is a unique x in $\mathscr{C}(T)$ such that $\bigcup x \subseteq f$. Moreover if we define $s \trianglelefteq t$ if $s \leqslant t$ or s and t are incomparable and $s <_{\mathrm{lex}} t$, then $\trianglelefteq$ induced the order on L. Thus for any L, L embeds into $\mathscr{C}(T(L))$.

Observe that if s and t are in a tree T, then s and t are comparable in the tree order precisely when they have a common upper bound. An *antichain* in a tree T is a subset A which is pairwise incomparable. A tree T is a *Souslin tree* if T is uncountable but has no uncountable chains and no uncountable antichains. Observe that any level in a Souslin tree is at most countable and hence the elements of a Souslin tree of height α is countable whenever $\alpha < \omega_1$. A tree T is an *Aronszajn tree* if every level and every chain of T is countable. ZFC proves that Aronszajn trees exist where as we will see it is not sufficient to prove that Souslin trees exist.

Theorem 17.3 (Aronszajn, Kurepa)**.** *Aronszajn lines exists.*

Proof. This construction is guided by the homework set. □

Proposition 17.4. *There is a Souslin tree if and only if there is a Souslin line.*

Proof. We have already seen that every c.c.c. nonseparable linear order contains a Souslin line. Thus for the forward implication, it is sufficient so show that if T is a Souslin tree, then $(\mathscr{C}(T), \leqslant_{\mathrm{lex}})$ is c.c.c. and nonseparable. Suppose that $\mathscr{I}$ is an uncountable collection of intervals in $\mathscr{C}(T)$. For each I in $\mathscr{I}$, Lemma 17.2 implies there is a $u_I \in T$ such that $\{x \in \mathscr{C}(T) \mid u_I \in x\} \subseteq I$. Since T is c.c.c., there are $I \neq J$ in $\mathscr{I}$ such that $u_I \leqslant u_J$. Then $\{x \in \mathscr{C}(T) \mid u_J \in x\} \subseteq I \cap J$ and in particular $\mathscr{I}$ is not pairwise disjoint. To see that $\mathscr{C}(T)$ is nonseparable suppose that $X \subseteq \mathscr{C}(T)$ is countable. Let $\alpha < \omega_1$ be such that if $x \neq y$ are in X, then $\Delta(x, y) < \alpha$. Since T is uncountable and the set of elements of height less than α is countable, there is a $t \in T_\alpha$ such that $\{u \in T \mid t \leqslant u\}$ is uncountable. Since each element of $\mathscr{C}(T)$ is countable, there are uncountably many elements of $\mathscr{C}(T)$ having t as an element. Let $a <_{\text{lex}} c <_{\text{lex}} b$ be three such elements. It follows from Lemma 17.2 that (a, b) is a nonempty interval disjoint from X.

To see the reverse implication, suppose that L is a Souslin line and observe that we may take L to have cardinality ω_1 and moreover to have the form $L = \{f_\xi \mid \xi < \omega_1\} \subseteq 2^{\omega_1}$ with the ordering being the lexicographic ordering. If $t \in T(L)$, then $I_t := \{f \in L \mid t \in f\}$ is an interval in L. If $I_s \cap I_t$ are nonempty, then s and t are comparable. Thus $T(L)$ has no uncountable antichains. Also observe that if $s < t$, then $[t]$ is properly contained in $[s]$. If $\langle t_\xi \mid \xi < \omega_1\rangle$ were an uncountable chain in T, then $\langle [t_\xi] \mid \xi < \omega_1\rangle$ would be an uncountable strictly decreasing sequence of intervals in L, contradicting Proposition 16.2. Finally, to see that $T(L)$ is uncountable, suppose not and let α be an upper bound on the heights of elements of $T(L)$. This means that for every $g, g' \in L$, $g = g'$ provided that $\{\xi < \alpha \mid f_\xi < g\} = \{\xi < \alpha \mid f_\xi < g'\}$. This implies $\{f_\xi \mid \xi < \alpha\}$ is countable and dense, contrary to our assumption. □

Finally, we note the following theorem of Kurepa, which demonstrates a striking property of Souslin lines.

Theorem 17.5 (Kurepa)**.** *If L is a nonseparable linear order, then $L \times L$ contains an uncountable family of pairwise disjoint rectangles.*

Proof. If L isn't c.c.c., then this is trivially true so suppose that L is c.c.c.. By removing countably many points if necessary, we may assume that every interval in L is uncountable. Recursively construct intervals I_α, J_α, and K_α for $\alpha \in \omega_1$ such that if $\alpha < \beta$, no endpoint of J_α is in I_β and J_β and K_β are two disjoint intervals contained in I_β. It suffices to show that $\{J_\alpha \times K_\alpha \mid \alpha \in \omega_1\}$ is pairwise disjoint. Suppose that $\alpha < \beta$. If $J_\alpha \cap J_\beta \neq \emptyset$, then $J_\alpha \cap I_\beta \neq \emptyset$. Since I_β does not contain an endpoint of J_α, it must be contained in J_α. But then $K_\beta \subseteq J_\alpha$, which is disjoint from K_α. Thus

$$(J_\alpha \times K_\alpha) \cap (J_\beta \times K_\beta) = \emptyset.$$

□

18. Jensen's $\diamondsuit$

Thomas Jech and Stanley Tennenbaum independently established that ZFC is consistent with the existence of a Souslin tree by adapting Cohen's method of forcing (Jech in 1967, Tennenbaum in 1968). We will soon see that even the assertion that there are c.c.c. topological spaces whose product is not c.c.c. is something not provable in ZFC — in particular ZFC does not prove or refute Souslin's Hypothesis. After Jech and Tennenbaum's works, Jensen proved that $\mathbf{L}$ satisfies that there is a Souslin tree — and therefore a Souslin continuum. Jensen's construction of a Souslin tree under the assumption $\mathbf{V} = \mathbf{L}$ proceeds by a combinatorial consequence of $\mathbf{V} = \mathbf{L}$ which itself is extremely important in set theory: $\diamondsuit$ is the assertion that there is a sequence $\langle A_\alpha \mid \alpha \in \omega_1 \rangle$ such that for every $X \subseteq \omega_1$

$$\{\alpha \in \omega_1 \mid X \cap \alpha = A_\alpha\}$$

is stationary.

Proposition 18.1. *The following are equivalent:*

(a) $\diamondsuit$

(b) *There is a sequence $\langle \mathscr{A}_\alpha \mid \alpha \in \omega_1 \rangle$ such that for each $\alpha \in \omega_1$, $\mathscr{A}_\alpha \subseteq \mathscr{P}(\alpha)$ is countable and for every $X \subseteq \omega_1$, there is an infinite α such that $X \cap \alpha \in \mathscr{A}_\alpha$.*

Proof. This will be part of the next homework set. □

Theorem 18.2 (Jensen). $\mathbf{L}$ *satisfies* $\diamondsuit$.

Proof. Assume $\mathbf{V} = \mathbf{L}$. Define $h : \omega_1 \to \omega_1$ so that $h(\alpha)$ is the least ordinal β such that L_β contains an injection from α into ω. Define $\mathscr{A}_\alpha := \mathscr{P}(\alpha) \cap L_{h(\alpha)}$, noting that $\mathscr{A}_\alpha$ is countable. By Proposition 18.1, it suffices to show that for every $X \subseteq \omega_1$ in $\mathbf{L}$, there is an α such that $X \cap \alpha \in \mathscr{A}_\alpha$. Let $M \prec L_{\omega_2}$ be countable such that $X \in M$. By condensation, $(M, \in)$ is isomorphic to $(L_\gamma, \in)$ for some $\gamma \in \omega_1$. Let π denote the isomorphism and set $\alpha := M \cap \omega_1$, noting that $\pi(\omega_1) = \alpha$. In particular, $\gamma < h(\beta)$ and hence $L_\gamma \cap \mathscr{P}(\alpha) \subseteq \mathscr{A}_\alpha$. Since $\pi(X) \subseteq \alpha$, it follows that $\pi(X) = X \cap \alpha \in \mathscr{A}_\alpha$. □

Theorem 18.3 (Jensen). *Assume $\diamondsuit$. There is a Souslin tree.*

Proof. Let $\langle A_\alpha \mid \alpha \in \omega_1 \rangle$ be a $\diamondsuit$-sequence. We will construct a tree ordering $\leqslant_T$ on the successor ordinals below ω_1 by recursion. If $m, n \in \omega$, define $m <_T n$ if $m < n$ and there is a k such that $m < 2^k$ and $n - m$ is divisible by 2^{k+1}. Notice that the height of m with respect to $<_T$ is the number of occurrences of `10` in the binary expansion of m. The minimal elements of $(\omega, \leqslant_T)$ are those elements of ω of the form

$2^n - 1$. In particular $(\omega, \leqslant_T)$ has infinitely many elements of height 0 and every m has infinitely many immediate successors.

Our recursive construction will satisfy the following conditions:

(1) for every limit ordinal α if $\beta, \bar{\alpha} < \alpha$, there is a $\gamma < \alpha$ such that $\beta <_T \gamma$ and $\bar{\alpha} \leqslant \gamma$;

(2) for any $\xi \in \omega_1$ and $m, n \in \omega$, $m <_T n$ if and only if $\omega \cdot \xi + m <_T \omega \cdot \xi + n$;

(3) for any $\xi \in \omega_1$,
$$\{\mathrm{ht}(\alpha) \mid \omega \cdot \xi \leqslant \alpha < \omega \cdot \xi + \omega\} = [\omega \cdot \xi, \omega \cdot \xi + \omega)$$

(4) if α is a limit ordinal and A_α is a maximal antichain in $(\alpha, \leqslant_T)$, then A_α is a maximal antichain in $(\alpha + \omega, \leqslant_T)$.

Observe that these conditions hold for $\leqslant_T \restriction \omega$. Now suppose that we have defined $\leqslant_T \restriction \alpha$ for some limit ordinal α so that the above conditions are satisfied. Let B_α denote the collection of all chains in $(\alpha, \leqslant_T)$ which meet every level. By condition (1), the union of B_α is all of α. Moreover if A is a maximal antichain in $(\alpha, \leqslant_T)$, then the union of those elements of B_α which contain some element of A is also all of α. Let $\langle b_n \mid n \in \omega \rangle$ be a sequence of distinct elements of B_α whose union is α such that, if A_α is a maximal antichain in α, each b_n contains an element of A_α. There is now a unique definition of $\leqslant_T \restriction (\alpha + \omega)$ such that $\alpha + (2^n - 1)$ is an upper bound for b_n and such that condition (2) is satisfied. Notice that the height of $\alpha + (2^n - 1)$ is α — the predecessors of this element are the set b_n, which has ordertype α with respect to $\leqslant_T$. Thus the height of $\alpha + n$ is $\alpha + \mathrm{ht}(n)$. Conditions (1), (3), and (4) therefore hold for $(\alpha + \omega, \leqslant_T)$. This completes the recursive construction.

Observe that since every element of $(\omega_1, \leqslant_T)$ has more than one immediate successor, if $(\omega_1, \leqslant_T)$ contains an uncountable chain, it must contain an uncountable antichain. Suppose that $A \subseteq \omega_1$ is a maximal antichain. Define $f : \omega_1 \to \omega_1$ by $f(\alpha)$ the least β such that there is an element of $A \cap \beta$ which is $\leqslant_T$-comparable with α. Observe that δ is f-closed if and only if $A \cap \delta$ is a maximal antichain in $(\delta, \leqslant_T)$. Since $\langle A_\alpha \mid \alpha \in \omega_1 \rangle$ is a $\diamondsuit$-sequence, there is a δ which is f-closed such that $A \cap \delta = A_\delta$. It follows that $A \cap \delta$ is a maximal antichain in $(\delta + \omega, \leqslant_T)$. If there were an $\alpha \in A$ which is greater than $\delta + \omega$, there would be an $\alpha' \leqslant_T \alpha$ of height δ. This α' would satisfy $\delta \leqslant \alpha' < \delta + \omega$ and also would be incomparable with every element of $A \cap \delta$. This would contradict that $A \cap \delta$ is a maximal antichain in $(\delta + \omega, \leqslant_T)$. Thus $A \subseteq \delta$ and hence A is countable. $\square$

19. Martin's Axiom

In 1971, Robert Solovay and Stanley Tennenbaum developed the technique of *iterated forcing* in order to prove the consistency of Souslin's Hypothesis with ZFC. Tony Martin observed that their technique could be used to establish the consistency of a more general principle, now known as $\mathrm{MA}_{\aleph_1}$, which itself is sufficient to prove Souslin's Hypothesis.

In order to state the principle, we need to develop some terminology. Let $(P, \leqslant)$ be a partial order. In the definitions which follow, it is useful to think of P as being the collection of nonempty open sets in a topological space, ordered by $\subseteq$. Two elements of P are *compatible* if they have a common lower bound in P; otherwise they are *incompatible*. It is common to write $p \parallel q$ to mean that p and q are compatible and $p \perp q$ to denote that p and q are incompatible. In the context of set theory, an *antichain* in a partial order is pairwise incompatible subset. Note that this is stronger than being pairwise incomparable. We say that P has the c.c.c. if every antichain in P is countable.

A subset of P is a *filter* if it is nonempty, upward closed, and downward directed. A subset D of a partial order P is *dense* in P if for every p in P there is a q in D such that $q \leqslant p$. If $\mathscr{D}$ is a collection of dense subsets of P, then a filter $G \subseteq P$ is *$\mathscr{D}$-generic* if $G \cap D \neq \varnothing$ for every $D \in \mathscr{D}$.

We can now state Martin's Axiom for θ many dense sets (MA_θ): whenever Q is c.c.c. and $\mathscr{D}$ is a collection of at most θ dense subsets of Q, there is a filter $G \subseteq Q$ which is $\mathscr{D}$-generic. Historically *Martin's Axiom* is the assertion that MA_θ holds for every cardinal θ less than $2^{\aleph_0}$. That said, $\mathrm{MA}_{\aleph_1}$ is become much more important as a hypothesis than Martin's Axiom. Many papers in the 1970s and 1980s in set theory state results under the hypothesis MA + $\neg$CH but really only invoke $\mathrm{MA}_{\aleph_1}$.

MA_θ is actually an assertion about Baire category. Recall that if K is a topological space, a subset E of K is *nowhere dense* if the closure of E has empty interior. This is equivalent to the assertion that for every nonempty open set U of K, there is an nonempty open set $V \subseteq U$ such that $E \cap V$ is empty.

Theorem 19.1. *For any infinite cardinal θ,* MA_θ *is equivalent to the assertion that a c.c.c. locally compact space cannot be covered by θ nowhere dense sets.*

Before proving the theorem, we'll establish the following lemma, which will later be an important part of our proof that MA_θ is consistent with ZFC.

Lemma 19.2. *For any cardinal θ, if the conclusion of* MA_θ *holds for all c.c.c. posets of cardinality at most θ, then* MA_θ *is true.*

Proof. Suppose that Q is any c.c.c. partial order and $\mathscr{D}$ is any collection of dense sets of cardinality at most θ. Let λ be a regular cardinal such that Q and $\mathscr{D}$ are in H_λ and let M be an elementary submodel of H_λ of cardinality θ such that $Q \in M$ and $\mathscr{D} \subseteq M$. Let $Q_0 = Q \cap M$. By elementarity, any two elements of Q_0 have a common lower bound in Q_0 if and only if they have a common lower bound in Q. Thus Q_0 is also c.c.c.. Also by elementarity, if $D \in \mathscr{D}$, then $D \cap Q_0$ is dense in Q_0. By our hypothesis, there is a filter $G \subseteq Q_0$ which meets each element of $\mathscr{D}$. It follows that $\{p \in Q \mid \exists q \in G\ (q \leqslant p)\}$ is a $\mathscr{D}$-generic filter. $\square$

Proof of Theorem 19.1. For the forward implication, let K be given and $\mathscr{E}$ be a collection of nowhere dense subsets of K of cardinality at most θ. Define Q to be the collection of all nonempty open subsets U of K. Since K is c.c.c., so is Q. If E is in $\mathscr{E}$, define D_E to be the collection of all $U \in Q$ such that the closure of U is disjoint from E. Since E is nowhere dense, D_U is dense. Let $G \subseteq Q$ be a filter which intersects D_E for each $E \in \mathscr{E}$ and let $x \in \bigcap\{\overline{U} \mid U \in G\}$. Since G is a filter and K is compact, this intersection is nonempty. Since G contains an element whose closure is disjoint from E, x is not in $\bigcup \mathscr{E}$.

For the reverse implication, let Q be a c.c.c. poset. By Lemma 19.2 we may assume that Q has cardinality at most θ. A subset G of Q is *centered* if every finite subset of G has a common lower bound in Q. Let K be the collection of all $X \subseteq Q$ such that X is upwards closed and centered. It is easily checked that K is a closed subset of the compact space $\mathscr{P}(Q) \approx 2^Q$. If q is in Q, set $U_q := \{X \in K \mid q \in X\}$. It is readily checked that each U_q is a nonempty open set and that every nonempty open subset of Q contains U_q for some q. Furthermore, p and q are compatible if and only if $U_p \cap U_q$ is nonempty. It follows that since Q is c.c.c., so is K. For each $p \in G$, define

$$W_{p,q} := \bigcup\{U_r \mid (r \leqslant p, q) \vee (p \perp r) \vee (q \perp r)\}$$

Observe that for every $p, q, r \in Q$, $W_{p,q} \cap U_r \neq \varnothing$. In particular, W_p is a dense open set for every $p \in Q$. Next observe that if $D \subseteq Q$ is dense, then $\bigcup\{U_p \mid p \in D\}$ is dense open in K. If $\mathscr{D}$ is a collection of at most θ dense subsets of Q, let $G \in K$ be such that for all $D \in \mathscr{D}$, G is in $\bigcup\{U_p \mid p \in D\}$ and for all $p \in Q$, G is in W_p. We first claim that since $G \in \bigcap_{p \in Q} W_p$, G is a filter. To see this, suppose that $p, q \in G$. Since G is in $W_{p,q} \cap U_p \cap U_q$, there is an $r \in G$ such that $r \leqslant p, q$. It follows that G is a filter. Since G is in $\bigcup\{U_p \mid p \in D\}$ if and only if $G \cap D$ is nonempty, it follows that G is $\mathscr{D}$-generic. $\square$

20. Consequences of MA_θ

We'll now give some consequences of MA_θ. First we'll show that $\mathrm{MA}_{\aleph_1}$ implies Souslin's hypothesis. If K is a complete linear order, then K is locally compact and hence homeomorphic to a dense open set in a compact space. Also observe that if K is a Souslin continuum, then there is a closed $P \subseteq K$ such that $K \backslash P$ is separable and every separable subspace of P is nowhere dense in P. In particular, the next theorem implies that $\mathrm{MA}_{\aleph_1}$ implies Souslin's Hypothesis.

Theorem 20.1. *If K is a Souslin continuum such that separable subspaces are nowhere dense, K can be covered by $\aleph_1$ many nowhere dense sets.*

Proof. Recursively construct a $\subseteq$-increasing sequence of countable sets $\langle D_\alpha \mid \alpha \in \omega_1 \rangle$ such that $D_{\alpha+1}$ intersects every maximal open interval of K which is disjoint from D_α. We claim that $K = \bigcup\{\overline{D_\alpha} \mid \alpha \in \omega_1\}$. This is sufficient since our hypothesis implies that each $\overline{D_\alpha}$ is nowhere dense. Suppose for contradiction that some $x \in K$ be outside $\bigcup\{\overline{D_\alpha} \mid \alpha \in \omega_1\}$. For each α, let I_α be the maximal open interval containing x as an element and disjoint from D_α. By construction $\langle I_\alpha \mid \alpha \in \omega_1 \rangle$ is strictly $\subseteq$-decreasing, contradicting Proposition 16.2. □

We will now turn to a striking consequence that MA_θ has for cardinal arithmetic. A I is a countable set, a collection A of infinite subsets of I is *almost disjoint* if every pair of elements have finite intersection. If $r \in 2^\omega$, define $a_r := \{r \restriction n \mid n \in \omega\}$. It follows that $\{a_r \mid r \in 2^\omega\}$ is an almost disjoint family of subsets of $2^{<\omega}$ of cardinality $2^{\aleph_0}$. Since $|2^{<\omega}| = \aleph_0$, there is an almost disjoint family of infinite subsets of ω of cardinality $2^{\aleph_0}$.

Theorem 20.2 (Solovay's almost disjoint coding)**.** *Assume* MA_θ. *If $A \subseteq \mathscr{P}(\omega)$ is an almost disjoint family of cardinality θ, then for every $B \subseteq A$, there is an $x \subseteq \omega$ such that $a \cap x$ is infinite if and only if $a \in B$. In particular, $2^\theta = 2^{\aleph_0}$.*

Proof. Assume MA_θ and let $B \subseteq A$ be given as in the statement of the theorem. Define Q to be the collection of all pairs $q = (x_q, A_q)$ such that:

- x_q is a finite subset of ω;
- A_q is a finite subset of $A \backslash B$.

If $p, q \in Q$, define $q \leqslant p$ to mean:

- x_p is an initial part of x_q and $A_p \subseteq A_q$;
- if $a \in A_p$, then a is disjoint from $x_q \backslash x_p$.

Observe that if $x \subseteq \omega$ is finite then $Q_x = \{q \in Q \mid x_q = x\}$ is centered: if $F \subseteq Q_x$ is finite, then $(x, \bigcup_{p \in F} A_p)$ is a lower bound for F. Thus Q is σ-centered and hence c.c.c.. Next notice that if $a \in A \backslash B$, then $D_a := \{q \in Q \mid a \in A_q\}$ is dense.

Claim 20.3. *For each $b \in B$ and $n \in \omega$,*

$$D_{b,n} := \{q \in Q \mid |b \cap x_q| \geqslant n\}$$

is dense.

Proof. Let $p \in Q$ be arbitrary. Since A is almost disjoint and A_p is finite, $\bigcup\{b \cap a \mid a \in A_p\}$ is finite. Let k be a strict upper bound for this set and let $m > k$ be such that $|b \cap [k, m]| \geqslant n$. Define $q := (x_p \cup (b \cap [k, m]), A_p)$. Since $x_q \backslash x_p \subseteq b \backslash k$, $x_q \backslash x_p$ is disjoint from a for every $a \in A_p$. It follows that $q \leqslant p$ is in $D_{b,n}$. □

Now let $G \subseteq Q$ be a filter which intersects D_a for each $a \in A \backslash B$ and $D_{b,n}$ for each $b \in B$ and $n \in \omega$. Define $x := \bigcup\{x_q \mid q \in G\}$. We will show that for $a \in A$, $x \cap a$ is infinite if and only if $a \in B$.

Suppose that $a \in A \backslash B$ and let $p \in G \cap D_a$. We claim $x \backslash x_p$ is disjoint from a. If $k \in x \backslash x_p$, let $q \in G$ be such that $k \in x_q$ and let $r \in G$ such that $r \leqslant p, q$. It follows that $k \in x_q \backslash x_p \subseteq x_r \backslash x_p$, which is disjoint from a. Now suppose that $b \in B$ and let $n \in \omega$ be arbitrary. Let $p \in G \cap D_{b,n}$. Then $x_p \cap b \subseteq x \cap b$ and $|x_p \cap b| \geqslant n$. It follows that $x \cap b$ is infinite. □

21. Chain conditions

We'll now turn to the study of a family of assumptions that the compatibility relation of a given partial order might satisfy. For historical reasons, these types of assumptions are known as *chain conditions* even though they really concern antichains more than chains. A subset of a partial order P is *linked* if every two elements have a common lower bound, *n-linked* if every n elements have a common lower bound and *centered* if every finite subset has a common lower bound. The modifier "σ-" means "is a countable union of" so, e.g. P is σ-centered means that P is a countable union of centered sets. Also, a partial order P satisfies *Knaster's condition* (or has *property K*) if every uncountable subset contains an uncountable linked subset. Trivially every σ-centered poset is σ-linked, every σ-linked poset has property K, and every property K poset is c.c.c.. Observe that if K is a compact space, the partial order of nonempty open subsets of K ordered by containment is σ-centered if and only if K is separable.

Theorem 21.1. *Assume* MA_θ *for σ-centered posets. Suppose that $\mathscr{F} \subset \mathscr{P}(\omega)$ has the property that every finite subset of $\mathscr{F}$ has infinite intersection. Then there is an infinite $x \subseteq \omega$ such that $x \subseteq^* y$ for every $y \in \mathscr{F}$.*

Remark 21.2. The minimum cardinality of a family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ for which the conclusion of the theorem fails is commonly denoted $\mathfrak{p}$. Murray Bell proved that $\mathfrak{p}$ is the minimum cardinality of a cover of a separable compact space by nowhere dense sets or, equivalently, the minimal cardinal θ for which MA_θ fails for a σ-centered poset.

Proof. We may assume without loss of generality that $\mathscr{F}$ is closed under finite intersections. Define Q to be the set of all $q = (x_q, F_q)$ such that $x_q \subseteq \omega$ is finite and $F_q \in \mathscr{F}$. Define $q \leqslant p$ if x_p is an initial part of x_q, $F_q \subseteq F_p$, and $x_q \backslash x_p \subseteq F_p$. If x is a finite subset of ω, then $Q_x := \{q \in Q \mid x_q = x\}$ is centered and therefore Q is σ-centered. If $F \in \mathscr{F}$ and $n \in \omega$, define

$$D_{F,n} := \{q \in Q \mid (|x_q| \geqslant n) \wedge (F_q \subseteq F)\}.$$

To see that each $D_{F,n}$ is dense, let $p \in Q$ be given. Let $y \subseteq F_p \cap F$ be such that $|y| \geqslant n$ and $\max(x) < \min(y)$. Then $q := (x_p \cup y, F_p \cap F)$ satisfies $q \leqslant p$ and $q \in D_{F,n}$. Now let $G \subseteq Q$ be a filter which intersects $D_{F,n}$ for each $F \in \mathscr{F}$ and $n \in \omega$. Define $x := \bigcup\{x_p \mid p \in G\}$. Suppose that $F \in \mathscr{F}$ and $n \in \omega$ are arbitrary. Let $p \in G \cap D_{F,n}$. As in the proof of Theorem 20.2, x_p is an initial part of x and $x \backslash x_p \subseteq F$. Since n was arbitrary, it follows that x is infinite and $x \subseteq^* F$. $\square$

Theorem 21.3. *Assume* MA_θ*. If* $\mathscr{E}$ *is a collection of Lebesgue measure* 0 *sets and* $|\mathscr{E}| \leqslant \theta$*, then* $\bigcup \mathscr{E}$ *has measure* 0*.*

Remark 21.4. If $\mathscr{I}$ is an ideal of set (i.e. closed under subsets and finite unions), then $\mathrm{add}(\mathscr{I})$ is defined to the minimum cardinality of a subset of $\mathscr{I}$ whose union is not in $\mathscr{I}$. The previous theorem asserts that MA_θ implies $\mathrm{add}(\mathscr{N}) \geqslant \theta$ where $\mathscr{N}$ is the ideal of measure 0 subsets of $\mathbb{R}$. Tomek Bartoszynski has shown that ZFC implies $\mathrm{add}(\mathscr{M}) \geqslant \mathrm{add}(\mathscr{N})$, where $\mathscr{M}$ is the ideal of first category subsets of $\mathbb{R}$.

Proof. Observe that $E \subseteq \mathbb{R}$ has measure 0 if and only if its intersection with $[a, b]$ has measure 0 for each $a < b$. It therefore suffices to consider collections $\mathscr{E}$ of subsets of $[0, 1]$. We will need the following claim.

Claim 21.5. *The following are equivalent for a subset* E *of* $[0, 1]$*.*

(a) E *has measure* 0*;*
(b) for every compact $K \subseteq [0, 1]$ *of positive measure and every* $\varepsilon > 0$*, there is a* $K' \subseteq K$ *of measure at least* $\lambda(K) - \varepsilon$ *which is disjoint from* E*;*
(c) for every compact $K \subseteq [0, 1]$ *of positive measure and there is a* $K' \subseteq K$ *of positive measure which is disjoint from* E*.*

Proof. To see (a) implies (b), suppose that E has measure 0 and let $K \subseteq [0, 1]$ have positive measure and $\varepsilon > 0$. Since $\lambda(K \backslash E) = \lambda(E)$, there is an increasing sequence $\langle K_n \mid n \in \omega \rangle$ of compact subset of $K \backslash E$ such that $\lim_n \lambda(K_n) = \lambda(K)$. Let n be sufficiently large that $\lambda(K_n) > \lambda(K) - \varepsilon$. Trivially (b) implies (c). To see that (c) implies (a), let $\mathscr{A}$ be a maximal pairwise disjoint collection of compact subsets of $[0, 1] \backslash E$ of positive measure. Notice that $\mathscr{A}$ is countable. If $\lambda(\bigcup \mathscr{A}) = 1$, then $\lambda(E) = 0$ and we're done. Suppose for contradiction that $\lambda(\bigcup \mathscr{A}) < 1$ and let $K \subseteq [0, 1] \backslash \bigcup \mathscr{A}$ be a compact set of positive measure. By hypothesis, there is a compact $K' \subseteq K$ of positive measure which is disjoint from E. This contradicts the maximality of $\mathscr{A}$. □

Returning to the main proof, by the claim, it suffices to show that if $K \subseteq [0, 1]$ is a compact set of positive measure, there is a compact $K' \subseteq K$ of positive measure which is disjoint from $\cup \mathscr{E}$. Let K be given and define Q to be the collection of compact subsets q of K such that the measure of q is greater than $r := \lambda(K)/2$.

Claim 21.6. Q *is* σ*-linked.*

Proof. Let $\mathscr{U}$ be the collection of all $U \subseteq \mathbb{R}$ such that U is a finite union of rational open intervals and $\lambda(U \cap K) > r$. For each $U \in \mathscr{U}$, define Q_U to be all $q \in Q$ such that $q \subseteq U$ and

$$\lambda(U \backslash q) < \frac{1}{2}(\lambda(U) - r).$$

Observe that $\mathscr{U}$ is countable and that $Q = \bigcup\{Q_U \mid U \in \mathscr{U}\}$. If $p, q \in \mathscr{U}$, then

$$\lambda(U \backslash p \cap q) \leqslant \lambda(U \backslash p) + \lambda(U \backslash q) < \lambda(U) - r$$

and thus $\lambda(p \cap q) > r$. In particular, Q_U is linked for each U. □

If E has measure 0, define D_E to be the collection of all $q \in Q$ such that q is disjoint from E. By Claim 21.5, D_E is dense. My $\mathrm{MA}_\theta(\sigma - \text{linked})$, there is a $G \subseteq Q$ be a filter which intersects D_E for each $E \in \mathscr{E}$. Observe that since G consists of closed sets, there is a countable filter $G_0 \subseteq G$ such that $\bigcap G_0 = \bigcap G$. Since every element of G_0 has measure greater than r, the measure of $K' := \bigcap G_0$ is at least $r > 0$. It follows that $\bigcup \mathscr{E}$ is disjoint from K' and hence is measure 0. □

Next we turn to question of whether the c.c.c. is preserved by taking products. If P and Q are partial orders, then their product has $P \times Q$ as its underlying set, with $(p_0, q_0) \leqslant (p_1, q_1)$ if and only if $p_0 \leqslant p_1$ and $q_0 \leqslant q_1$. Observe that a poset P is c.c.c. if whenever $\langle p_\xi \mid \xi \in \omega_1 \rangle$ are elements of P (possibly not all distinct), there exist $\xi \neq \eta$ such that p_ξ and p_η are compatible. Similarly, P has property K if whenever $\langle p_\xi \mid \xi \in \omega_1 \rangle$ are elements of P, there is an uncountable $X \subseteq \omega_1$ such that if $\xi \neq \eta$ are in X, then p_ξ and p_η are compatible.

Proposition 21.7. *Suppose that P satisfies the c.c.c. and Q has property K. Then $P \times Q$ has the c.c.c..*

Proof. Let $\langle (p_\xi, q_\xi) \mid \xi \in \omega_1 \rangle$ be a sequence of elements of $P \times Q$. Since Q has property K, there is an uncountable $X \subseteq \omega_1$ such that if $\xi \neq \eta$ are in X, then q_ξ and q_η are compatible. Since P is c.c.c., there are $\xi \neq \eta$ in X such that p_ξ and p_η are compatible. Thus (p_ξ, q_ξ) and (p_η, q_η) are compatible and hence $P \times Q$ has the c.c.c.. □

Proposition 21.8. *Assume* $\mathrm{MA}_{\aleph_1}$. *Every c.c.c. partial order has property K.*

Proof. Let Q be a c.c.c. partial order and suppose that $\langle p_\xi \mid \xi \in \omega_1 \rangle$ is a sequence of elements of Q. We will first show that, for some $p \in Q$, if $q \leqslant p$ then there are uncountably many ξ such that p_ξ is compatible with q. If not, let $q_\xi \leqslant p_\xi$ be such that for all but countably many η, q_ξ is incompatible with p_η. It follows that for all $\xi \in \omega_1$, q_ξ is compatible with q_η for only countably many η. It follows that for some uncountable $X \subseteq \omega_1$, $\{q_\xi \mid \xi \in X\}$ is an uncountable antichain, which is a contradiction.

Let Q_0 be the partial order of all $q \in Q$ such that $q \leqslant p$ and order Q_0 with the order inherited from Q. Clearly Q_0 is c.c.c.. For each $\xi \in \omega_1$, define

$$D_\xi := \{q \in Q_0 \mid \exists \eta > \xi\ (q \leqslant p_\eta)\}.$$

To see that D_ξ is dense, let $q \leqslant p$ and ξ be given. By assumption, there is an $\eta > \xi$ such that q is compatible with p_η. If r is a lower bound for q and p_η, then $r \in D_\xi$ and $r \leqslant q$. Let $G_0 \subseteq Q_0$ be a filter which intersects D_ξ for each $\xi \in \omega_1$ and let G be the upwards closure of G_0 in Q. It follows that G is linked and contains p_ξ for uncountably many $\xi \in \omega_1$. □

Corollary 21.9. $\mathrm{MA}_{\aleph_1}$ *implies that the product of c.c.c. partial orders is c.c.c..*

Remark 21.10. The proof of Proposition 21.8 actually shows that if Q is a c.c.c. partial order and $X \subseteq Q$ is uncountable, then X contains an uncountable centered family. Stevo Todorcevic and Boban Veličković have shown that in fact $\mathrm{MA}_{\aleph_1}$ is equivalent to this assertion. It is an open problem whether $\mathrm{MA}_{\aleph_1}$ is equivalent to the assertion that every c.c.c. partial order has property K. It is also unknown whether $\mathrm{MA}_{\aleph_1}$ is equivalent to the assertion that the product of two c.c.c. partial orders is c.c.c..

22. Specializing trees

Given a tree T, it is very natural to ask when it has an uncountable chain. An obstruction to a tree T containing an uncountable chain is the existence of a cover of T by countably many antichains. Trees with such a cover are said to be *special*. It was shown in the exercises that a tree T is special precisely when there is a function $f : T \to \mathbb{Q}$ such that $f(s) < f(t)$ whenever $s < t$. Many examples of nonspecial trees without uncountable chains. Perhaps the easiest to describe is $\sigma\mathbb{Q}$, which consists of all strictly increasing sequences of elements of $\mathbb{Q}$, ordered by extension. We will see, however, that MA_θ implies that every tree of cardinality at most θ is special unless it contains an uncountable chain. We will begin by the following general consequence of MA_θ.

Theorem 22.1. *Assume* MA_θ. *If Q is a c.c.c. poset and $|Q| \leqslant \theta$, then Q is σ-centered.*

Proof. If $\theta = \aleph_0$, this is trivially true, so we may assume that $\theta \geqslant \aleph_1$. In particular Q^n is c.c.c. for all $n \in \omega$. Let $Q^{<\omega}$ be the partial order of all finite sequences of elements of Q ordered by $q \leqslant p$ if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and whenever $i \in \mathrm{dom}(p)$, $q(i) \leqslant p(i)$. Notice that since each Q^n is c.c.c., so is $Q^{<\omega}$. For each $p \in Q$, define $D_p \subseteq Q^{<\omega}$ to consist of all $q \in Q^{<\omega}$ such that for some n, $q(n) \leqslant p$. Clearly D_p is dense for all p: if $q \in Q^{<\omega}$ then $q^\frown\langle p\rangle$ is in D_p and is below q. By MA_θ, there is a filter $G \subseteq Q^{<\omega}$ such that $G \cap D_p \neq \varnothing$ for all $p \in Q$. If $G_n = \{q(n) \mid q \in G\}$, then G_n is centered and $Q = \bigcup\{G_n \mid n \in \omega\}$. $\square$

Lemma 22.2. *Suppose that T is a tree with no uncountable chains. If $A, B \subseteq T$ are uncountable, there are uncountable $A' \subseteq A$ and $B' \subseteq B$ such that every element of A' is incomparable with every element of B'.*

Proof. Let U be the set of all $u \in T$ such that there are uncountably many $s \in A$ with $u < s$ and let V be the corresponding set for B. First suppose that $U \neq V$. Without loss of generality, we may assume that there is a $u \in U \backslash V$. Define

$$A' := \{s \in A \mid u < s\} \qquad B' := \{t \in B \mid (u \nleqslant t) \wedge (t \nleqslant u)\}.$$

Clearly A' is uncountable and every element of A' is incompatible with every element of B'. To see that B' is uncountable, observe that if $t \in B \backslash B'$, then either $t \geqslant u$ or $t \leqslant u$. Since the predecessors of u form a chain, there are only countably many elements of B below u. Since $u \notin V$, there are only countably many $t \in B$ with $t \geqslant u$. Thus $B \backslash B'$ is countable hence B' is uncountable.

Now suppose that $U = V$. If there exist $u, v \in U$ which are incomparable, then define $A' := \{s \in A \mid u \leqslant s\}$ and $B' := \{t \in B \mid v \leqslant t\}$ and observe that A' and B' are as desired. The alternative is that $U = V$ is a chain, which therefore is countable by our assumption. If U has a greatest element u, then let A_0 and B_0 be the set of immediate successors of u which have elements of A and B respectively above them. Observe that since u is a maximal element of both U and V, A_0 and B_0 are uncountable. Let $A_0' \subseteq A_0$ and $B_0' \subseteq B_0$ be uncountable disjoint sets and define A' be the set of elements of A above some element of A_0' and B' be the set of elements of B above some element of B'. Since A_0' and B_0' are disjoint subsets of a single level of T, it follows that every element of A' is incomparable with every element of B' (in fact $A' \cup B'$ is an antichain). Finally suppose that $U = V$ is a chain but has no greatest element. If $u \in U$, define

$$A_u := \{s \in A \mid u < s\} \qquad B_u := \{t \in B \mid (u \not\leqslant t) \wedge (t \not\leqslant u)\}$$

noting that every element of A_u is incomparable with every element of B_u. By definition of U, each A_u is uncountable. Since U has no last element $B = \bigcup\{B_u \mid u \in U\}$. Since U is countable, some B_u is uncountable and for this u, $A_u \subseteq A$ and $B_u \subseteq B$ satisfy the conclusion of the lemma. □

The following lemma is useful in establishing the c.c.c..

Lemma 22.3. *Suppose that $K \subseteq [\omega_1]^2$ has the property that whenever $A, B \subseteq \omega_1$ are uncountable, there are uncountable $A' \subseteq A$ and $B' \subseteq B$ such that $\{\alpha, \beta\} \in K$ for every $\alpha \in A'$ and $\beta \in B'$. The poset Q of finite $q \subseteq \omega_1$ such that $[q]^2 \subseteq K$ ordered by reverse containment is c.c.c..*

Proof. Let $\langle q_\xi \mid \xi \in \omega_1\rangle$ be a sequence of elements of Q. By refining the sequence of necessary, we may assume that each q_ξ has cardinality n for some $n \in \omega$. Let $q_\xi = \{q_\xi(i) \mid i \in n\}$. Construct uncountable $A_k, B_k \subseteq \omega_1$ for $k < n^2$ such that:

- $A_{k+1} \subseteq A_k$ and $B_{k+1} \subseteq B_k$ if $k < n^2 - 1$ and
- if $k = i + jn$, then for all $\xi \in A_k$ and $\eta \in B_k$, $\{q_\xi(i), q_\eta(j)\} \in K$.

If $\xi \in A_{n^2-1}$ and $\eta \in B_{n^2-1}$, then $q_\xi \cup q_\eta$ is in K. □

Theorem 22.4. *Assume* MA_θ*. If T is a tree with no uncountable branches such that $|T| \leqslant \theta$, then T is a countable union of antichains.*

Proof. Let Q be the poset of all finite antichains in T. By Lemmas 22.2 and 22.3, Q is c.c.c.. By Corollary 22.1, there a countable collection of centered sets $\{G_n \mid n \in \omega\}$ whose union is Q. Set $A_n := \bigcup G_n$, noting that A_n is an antichain. Since $\{t\} \in Q$ for each $t \in T$, $\bigcup\{A_n \mid n \in \omega\} = T$. □

23. Pressing down and verification of the c.c.c.

The pressing down lemma is often useful in verifying the c.c.c. in situations where posets are not σ-linked. Suppose that X is a set and $\mathscr{F}$ is a set consisting of finite subsets of $\mathscr{F}$. We say that $\mathscr{F}$ is a Δ-*system with root* R if $R \subseteq F$ for every $F \in \mathscr{F}$ and $\{F \backslash R \mid F \in \mathscr{F}\}$ is pairwise disjoint and consists of sets of the same cardinality. The next lemma is known as the Δ-*System Lemma*.

Lemma 23.1. *Suppose that $\mathscr{F}$ is a collection of finite sets and $|\mathscr{F}|$ is a regular uncountable cardinal. Then $\mathscr{F}$ contains a Δ-system $\mathscr{F}_0$ such that $|\mathscr{F}_0| = |\mathscr{F}|$.*

Proof. Set $X := \bigcup \mathscr{F}$ and observe that $|X| = |\mathscr{F}|$. Define $\kappa := |\mathscr{F}|$ and observe that, by replacing $\mathscr{F}$ by its image under a bijection, we may assume that $X = \kappa$. Let $\langle F_\xi \mid \xi < \kappa \rangle$ enumerate $\mathscr{F}$ without repetition. Define $f : \kappa \to \kappa$ by $f(\alpha) := \sup\{\max(F_\xi) + 1 \mid \xi \in \alpha\}$ and let $E \subseteq \kappa$ be the set of all limit ordinals α which are f-closed. If $\alpha < \kappa$ is a limit ordinal, let $r(\alpha) = \max((F_\alpha \cap \alpha) \cup \{0\}) + 1$. By the pressing down lemma, there is a stationary $S \subseteq E$ consisting of limit ordinals and an $\gamma < \kappa$ such that $r(\alpha) = \gamma$ if $\alpha \in S$. By shrinking S if necessary, we may assume that for some $n \in \omega$ and $R \subseteq \gamma$:

- $|F_\alpha| = n$ for all $\alpha \in S$;
- $F_\alpha \cap \alpha = R$ for all $\alpha \in S$

To see that $\mathscr{F}_0 := \{F_\alpha \mid \alpha \in S\}$ is a Δ-system, it suffices to check that if $\alpha < \beta$ are in S, then $F_\alpha \cap F_\beta = R$. This follows from the fact that $F_\alpha \cap \alpha = F_\beta \cap \beta = R$ and

$$\max(F_\alpha) < f(\alpha) < \beta \leqslant \min(F_\beta \backslash \beta).$$

□

Corollary 23.2. *Suppose that θ is any cardinal and Q is the poset consisting of all functions $q : D_q \to 2$ such that $D_q \subseteq \theta$ is finite, ordered by $q \leqslant p$ if q extends p. If $X \subseteq Q$ has regular uncountable cardinality κ, then X contains a centered family of cardinality κ.*

Remark 23.3. A subset of this poset is centered if and only if it is linked. If θ is larger than $2^{\aleph_0}$, this poset is not σ-linked.

Proof. By the Δ-System Lemma, there is an $X' \subseteq X$ of cardinality κ such that $\{\mathrm{dom}(q) \mid q \in X\}$ forms a Δ-system with root D. Let $X'' \subseteq X'$ have cardinality κ be such that if $p, q \in X''$, $p \restriction D = q \restriction D$. It follows that X'' is centered. □

Sometimes the Pressing Down Lemma needs to be used more directly. In order to illustrate this, we will prove another consequence of $\mathrm{MA}_{\aleph_1}$.

A *ladder system* on ω_1 is a sequence $\langle C_\alpha \mid \alpha \in \lim(\omega_1)\rangle$ such that if $\alpha \in \lim(\omega_1)$, $C_\alpha \subseteq \alpha$ is cofinal and has ordertype ω. Recall that if f and g are functions with a common domain D, then $f =^* g$ denotes the assertion that $\{x \in D \mid f(x) \neq g(x)\}$ is finite.

Theorem 23.4. *Assume* $\mathrm{MA}_{\aleph_1}$. *If* $\langle C_\alpha \mid \alpha \in \lim(\omega_1)\rangle$ *is a ladder system and* $\langle f_\alpha \mid \alpha \in \lim(\omega_1)\rangle$ *is a sequence of functions such that* $f_\alpha : C_\alpha \to \omega$, *then there is a function* $g : \omega_1 \to \omega$ *such that for all limit ordinals* α, $g \restriction C_\alpha =^* f_\alpha$.

Proof. Let Q be all functions $q : D_q \to \omega$ such that:

- $D_q = \bigcup\{C_\alpha \mid \alpha \in F_q\}$ for some finite $F_q \subseteq \lim(\omega_1)$ and
- if $\alpha \in D_q$, then $q \restriction C_\alpha =^* f_\alpha$.

Order $q \leqslant p$ if q extends p. Observe that for each $\gamma \in \omega_1$, $\{q \restriction \gamma \mid q \in Q\}$ is countable.

We'll first show that Q is property K. Suppose that $\langle p_\xi \mid \xi \in \omega_1\rangle$ is a sequence of conditions from Q and let F_ξ denote F_{p_ξ}. If $\{F_\xi \mid \xi \in \omega_1\}$ is countable, then we can an uncountable $X \subseteq \omega_1$ such that if $\xi, \eta \in X$ such that $p_\xi = p_\eta$. Otherwise, there is an uncountable $X \subseteq \omega_1$ such that $\{F_\xi \mid \xi \in X\}$ forms a Δ-system with root R. For each $\alpha \in \omega_1$, let $\xi_\alpha \in X$ be such that α is not in F_{ξ_α}. Let $E \subseteq \omega_1$ be the closed and unbounded set consisting of all limit ordinals δ such that if $\alpha < \delta$, then $\max(F_{\xi_\alpha}) < \delta$. If $\alpha \in E$, define $r(\alpha)$ to be the least upper bound for $\bigcup\{C_\eta \cap \alpha \mid \eta \in F_{\xi_\alpha}\}$. By the Pressing Down Lemma, there is a stationary $S \subseteq E$ such that r is constantly γ on S. By refining S if necessary, we may assume that if α, β are in S, then $q_\alpha \restriction \gamma = q_\beta \restriction \gamma$. Now observe that if $\alpha < \beta$ are in S, then $\mathrm{dom}(q_{\xi_\alpha}) \subseteq \beta$ and hence $\mathrm{dom}(q_{\xi_\alpha}) \cap \mathrm{dom}(q_{\xi_\beta}) \subseteq \gamma$. It follows that $\{q_{\xi_\alpha} \mid \alpha \in S\}$ is linked.

Claim 23.5. *If* α *is a limit ordinal,* $\{q \in Q \mid \alpha \in F_q\}$ *is dense.*

Proof. Let $p \in Q$ be arbitrary. If $\alpha \in F_q$, there is nothing to show. If not, then $C_\alpha \cap \mathrm{dom}(p)$ is finite. Define

$$q := p \cup (f_\alpha \restriction (C_\alpha \setminus \mathrm{dom}(p)))$$

and observe that $q \leqslant p$ and $q \in D_\alpha$. $\square$

By $\mathrm{MA}_{\aleph_1}$, there is a filter $G \subseteq Q$ which meets D_α for every limit ordinal α. Define

$$g(\xi) := \begin{cases} q(\xi) & \text{if } q \in G \text{ and } \xi \in \mathrm{dom}(q) \\ 0 & \text{otherwise} \end{cases}$$

This function satisfies the conclusion of the theorem. $\square$

24. Forcing and its syntax

Paul Cohen developed the method of forcing to establish that the Continuum Hypothesis is not provable from the axioms of ZFC. This method was refined considerably by Solovay and has become the primary tool for proving that set theoretic hypotheses are consistent with ZFC. In order to get an intuitive understand of what we will formalize, let us begin with two thought experiments. First consider the following "paradox" in probability: if Z is a continuous random variable, then for any possible outcome z in $\mathbb{R}$, the event $Z \neq z$ occurs *almost surely* (i.e. with probability 1). How does one reconcile this with the fact that, in a truly random outcome, every event having probability 1 should occur? Recasting this in more formal language we have that, "for all $z \in \mathbb{R}$, *almost surely* $Z \neq z$", while "*almost surely* there exists a $z \in \mathbb{R}$, $Z = z$."

Next suppose that, for some index set I, $\langle Z_i \mid i \in I \rangle$ is a family of independent continuous random variables. It is a trivial matter that for each pair $i \neq j$, the inequality $Z_i \neq Z_j$ holds with probability 1. For large index sets I, however,

$$|\{Z_i \mid i \in I\}| = |I|$$

holds with probability 0; in fact this event contains no outcomes if I is larger in cardinality than $\mathbb{R}$. In terms of the formal logic, we have that, "for all $i \neq j$ in I, *almost surely* the event $Z_i \neq Z_j$ occurs", while "*almost surely it is false that* for all $i \neq j \in I$, the event $Z_i \neq Z_j$ occurs".

It is natural to ask whether it is possible to revise the notion of *almost surely* so that its meaning remains unchanged for simple logical assertions such as $Z_i \neq Z_j$ but such that it commutes with quantification. For instance one might reasonably ask that, in the second example, $|\{Z_i \mid i \in I\}| = |I|$ should occur *almost surely* regardless of the cardinality of the index set. Such a formalism would describe truth in a necessarily larger model of mathematics, one in which there are new outcomes to the random experiment which did not exist before the experiment was performed.

We will now be more formal. A *forcing* is simply a set Q equipped with a transitive, reflective relation $\leqslant$ which has a greatest element $\mathbf{1}$. This will serve as an abstraction of the set of events of positive probability in a probability space. We will be interested in studying the logical properties of a "generic" filter in Q — something that corresponds to the analysis of a random outcome in probability. The central objects of study in forcing are the *forcing relation* $\Vdash$ and the Q-*names.* These are the abstractions of "almost surely" mentioned above and of random variables, respectively. It will turn out that the formal definitions of $\Vdash$ and of Q-names are not as informative as their properties. We will therefore introduce their properties axiomatically first and then later return to given them a formal definition.

Unless we explicitly state otherwise, we will assume the forcing Q is *separative*: whenever $p, q \in Q$, if $p \not\leqslant q$, then there is a $p' \leqslant p$ such that $p' \perp q$. That is, for all $p, q \in Q$, $p = q$ if and only if the set of conditions compatible with p is the same as the set of conditions compatible with q. This assumption is a minor one: Q is any forcing, we can define $p \equiv q$ if

$$\{r \in Q \mid p \parallel r\} = \{r \in Q \mid q \parallel r\}.$$

The relation $\leqslant$ on Q naturally induces an order on the quotient $Q/\equiv$. Thus we lose little generality in assuming all forcings are separative. If Q is the collection of all Borel subsets of $[0, 1]$ of positive measure, then $(Q, \subseteq)$ is not separative: in this case $p \equiv q$ if p and q differ by a measure 0 set.

Fix for the moment a forcing Q. There are two examples of Q-names which deserve special mention. The first is the "check name": for each set x, there is a Q-name $\check{x}$. This corresponds to a random variable which is constant — it does not depend on the outcome. The other is the Q-name $\dot{G}$ for the generic filter; this corresponds to the random variable representing the outcome of the random experiment.

The *forcing language* associated to Q is the class of all first order formulas in the language of set theory augmented by adding a constant symbol for each Q-name. If q is in Q and ϕ is a sentence in the forcing language, then informally the forcing relation $q \Vdash \phi$ asserts that if the event corresponding to q occurs, then *almost surely* ϕ will be true. In the absence of the definitions of "Q-name" and "$\Vdash$," the following properties can be regarded as axioms which govern the behavior of these primitive concepts. They can be proved from the definitions of Q-names and the forcing relation once they are in hand.

Property 1. *For any x and y and any $p \in Q$, $p \Vdash \check{x} \in \check{y}$ if and only if $x \in y$.*

Property 2. $\mathbf{1} \Vdash \dot{G} \subseteq \check{Q}$ *and for every $p, q \in Q$, $p \Vdash \check{q} \in \dot{G}$ if and only if $p \leqslant q$.*

It is useful to define the following terminology: if there is a z such that $q \Vdash \dot{y} = \check{z}$, then we say that q *decides* $\dot{y}$ (to be z). Similarly, if $p \Vdash \phi$ or $p \Vdash \neg\phi$, then we say that p *decides* ϕ.

Property 3. *For any x, any Q-name $\dot{y}$, and $p \in Q$, if $p \Vdash \dot{y} \in \check{x}$, then there is a $q \leqslant p$ which decides $\dot{y}$.*

Property 4. *If $\dot{x}$ is a Q-name and $p \in Q$, then the collection of all Q-names $\dot{y}$ such that $\mathbf{1} \Vdash \dot{y} \in \dot{x}$ forms a set and the collection of all Q-names $\dot{y}$ such that $\mathbf{1} \Vdash \dot{y} = \dot{x}$ forms a set.*

Remark 24.1. Unlike the other properties, this one is dependent on the definition of Q-name which we will later give.

Property 5. *If $p \in Q$ and ϕ is a formula in the forcing language, then $p \Vdash \neg\phi$ if and only if there is no $q \leqslant p$ such that $q \Vdash \phi$.*

Observe that this property implies that if $p \Vdash \phi$ and $q \leqslant p$, then $q \Vdash \phi$.

Property 6. *If $p \in Q$, then $p \Vdash \exists v\phi(v)$ if and only if there is a Q-name $\dot{x}$ such that $p \Vdash \phi(\dot{x})$.*

Property 7. *For any $q \in Q$, the collection of sentences in the forcing language which are forced by q contains the ZFC axioms, the axioms of first order logic, and is closed under* modus ponens*. Moreover, if the axioms of ZFC are consistent, then so are the sentences forced by q.*

Observe that since the $\mathbf{1}$ forces the Axiom of Extensionality, if x and y are sets and $p \in Q$, then $p \Vdash \check{x} = \check{y}$ if and only if $x = y$. If $\mathbf{1} \Vdash \phi$, then we will sometimes say that "Q forces ϕ" or, if Q is clear from the context, that "ϕ is forced." Similarly, we will write "$\dot{x}$ is a Q-name for..." to mean "$\dot{x}$ is a Q-name and Q forces that $\dot{x}$ is...".

A key aspect of the forcing construction is that, for a given forcing Q, the collection of sentences forced by Q is often a proper extension of ZFC. For instance, if Q is the partial order of all finite partial functions from ω_2 to 2 ordered by $q \leqslant p$ if q extends p, then we will see that Q forces the negation of CH. By Property 7, this implies ZFC is consistent with $\neg$CH.

25. Some further properties of the forcing relation

In order to demonstrate how the properties of the forcing relation can be used, we will prove the some propositions which will be useful. The first illustrates the central feature of the forcing syntax: that $\dot{G}$ is forced to be $\mathbf{V}$-generic.

Proposition 25.1. $\mathbf{1} \Vdash \ulcorner \dot{G}$ *is a filter*$\urcorner$. *Furthermore, for every dense* $D \subseteq Q$, $\mathbf{1} \Vdash \dot{G} \cap \check{D} \neq \varnothing$.

Proof. The proof that $\mathbf{1} \Vdash \ulcorner \dot{G}$ is a filter$\urcorner$ will be a homework exercise. If the second part of the proposition were false for some dense set $D \subseteq Q$, by Property 5 there would exist a $p \in Q$ such that $p \Vdash \dot{G} \cap \check{D} = \varnothing$. Since D is dense, there is a $q \leqslant p$ in D. By Property 2, $q \Vdash \check{q} \in \dot{G}$ and by Property 1, $q \Vdash \check{q} \in \check{D}$. By Properties 6 and 7, $q \Vdash \exists r (r \in \dot{G} \cap \check{D})$ or equivalently $q \Vdash \dot{G} \cap \check{D} \neq \varnothing$. Since $q \leqslant p$, this contradicts Property 5. □

Proposition 25.2. *Suppose that x is a set and $\phi(v)$ is a formula in the forcing language. If for all $y \in x$, $p \Vdash \phi(\check{y})$, then $p \Vdash \forall y \in \check{x}\ \phi(y)$.*

Proof. We will prove the contrapositive. Toward this end, suppose that p does not force $\forall y \in \check{x}\ \phi(y)$. It follows from Property 5 there is a $q \leqslant p$ such that $q \Vdash \neg \forall y \in \check{x}\ \phi(y)$. By Property 7, this is equivalent to $q \Vdash \exists y \in \check{x}\ \neg\phi(y)$. By Property 6, there is a Q-name $\dot{y}$ such that

$$q \Vdash (\dot{y} \in \check{x}) \wedge (\neg\phi(\dot{y})).$$

By Property 7, $q \Vdash \dot{y} \in \check{x}$ and therefore by Property 3, there is a $r \leqslant q$ and a z in x such that $r \Vdash \dot{y} = \check{z}$. But now, by Property 7, $r \Vdash \neg\phi(\check{z})$ and hence by Property 5, p does not force $\phi(\check{z})$. □

Proposition 25.3. *Suppose that $\phi(\bar{v})$ is a formula in the language of set theory with only bounded quantification. If $\bar{x}$ is a tuple of sets and $\phi(\bar{x})$ is true, then $\mathbf{1} \Vdash \phi(\check{\bar{x}})$.*

Proof. The proof is by induction on the length of ϕ. If ϕ is atomic, then this follows from Property 1. If ϕ is a conjunct, disjunct, or a negation, then the proposition follows from Property 7 and the induction hypothesis. Finally, suppose $\phi(\bar{v})$ is of the form $\forall w \in v_i\ \psi(\bar{v}, w)$. If $\forall w \in x_i \psi(\bar{x}, w)$ is true, then for each $w \in x_i$, $\psi(\bar{x}, w)$ is true. By our induction hypothesis, $\mathbf{1} \Vdash \phi(\check{\bar{x}}, \check{w})$ for each $w \in x_i$. By Proposition 25.2, it follows that $\mathbf{1} \Vdash \forall w \in \check{x}_i\ \psi(\check{\bar{x}}, w)$. □

Proposition 25.4. *If $\dot{\alpha}$ is a Q-name, $p \in Q$, and $p \Vdash \ulcorner \dot{\alpha}$ is an ordinal$\urcorner$, then there is an ordinal β such that $p \Vdash \dot{\alpha} \in \check{\beta}$.*

Proof. Let p and $\dot\alpha$ be given. Suppose for contradiction that for every $q \leqslant p$ and every $\beta \in \mathbf{ON}$, $q \not\Vdash \dot\alpha \in \check\beta$. Observe that if β is an ordinal, then by Proposition 25.3 and Property 7, $p \Vdash \check\beta \in \mathbf{ON}$. By Property 7 applied to Theorem 3.3,

$$p \Vdash (\check\beta \in \dot\alpha) \vee (\dot\alpha \in \check\beta) \vee (\dot\alpha = \check\beta).$$

By Properties 5 and 7, it must be that $p \Vdash \check\beta \in \dot\alpha$. But this means that

$$\{\check\beta \mid (\beta \in \mathbf{ON}) \wedge (p \Vdash \check\beta \in \dot\alpha)\}$$

is a proper class, which contradicts Property 4. □

Proposition 25.5. *Suppose that T is a set consisting of finite length sequences, closed under taking initial segments. If there is a forcing Q and some $q \in Q$ forces "there is an infinite sequence σ, all of whose finite initial parts are in T," then such a sequence σ exists.*

Proof. If no such sequence σ exists, then there is a function $\rho : T \to \mathbf{ON}$ such that if s is a proper initial segment of t, then $\rho(t) \in \rho(s)$. Such a ρ certifies the nonexistence of such a σ since such a σ would define a strictly decreasing infinite sequence of ordinals. Observe that the assertion that ρ is a strictly decreasing map from T into the ordinals is a statement about ρ and T involving only bounded quantification. By Proposition 25.3, this statement is forced by every forcing Q. □

There is a special class of forcings for which there is a more conceptual picture of the forcing relation. We begin by stating a general fact about forcings. Recall that a Boolean algebra is *complete* if every subset has a least upper bound.

Theorem 25.6. *For every forcing Q, Q is isomorphic to a dense suborder of the positive elements of a complete Boolean algebra.*

A typical example of a complete Boolean algebra is the algebra of measurable subsets of $[0, 1]$ modulo the ideal of measure zero sets. The algebra of Borel subsets of $[0, 1]$ modulo the ideal of first category sets is similarly a complete Boolean algebra.

Suppose now that Q is the positive elements of some complete Boolean algebra $\mathcal{B}$. If ϕ is a formula in the forcing language, then define the *truth value* $[\![\phi]\!]$ of ϕ to be the least upper bound of all $b \in \mathcal{B}$ such that $b \Vdash \phi$. Observe that if $a \leqslant [\![\phi]\!]$, then a cannot force $\neg\phi$. Hence $[\![\phi]\!]$ forces ϕ. The rules which govern the logical connectives now take a particularly nice form:

$$[\![\neg\phi]\!] = [\![\phi]\!]^{\mathsf{c}} \qquad [\![\phi \wedge \psi]\!] = [\![\phi]\!] \wedge [\![\psi]\!] \qquad [\![\phi \vee \psi]\!] = [\![\phi]\!] \vee [\![\psi]\!]$$

$$[\![\forall v \phi(v)]\!] = \bigwedge_{\dot{x}} [\![\phi(\dot{x})]\!] \qquad [\![\exists v \phi(v)]\!] = \bigvee_{\dot{x}} [\![\phi(\dot{x})]\!]$$

Notice that while $\dot{x}$ ranges over all Q-names in the last equations — a proper class — the collection of all possible values of $[\![\phi(\dot{x})]\!]$ is a set and therefore the last items are meaningful.

In spite of the usefulness of complete Boolean algebras in understanding forcing and also in some of the development of the abstract theory of forcing, forcings of interest rarely present themselves as complete Boolean algebras. While Theorem 25.6 allows us to represent any forcing inside a complete Boolean algebra, defining forcing strictly in terms of complete Boolean algebras would prove cumbersome in practice. We will return to complete Boolean algebras to prove a key result about iterated forcing.

26. Forcing semantics: names and interpretation

We will now turn to the task of giving a formal definition of what is meant by a Q-*name* and $q \Vdash \phi$. This will in turn be used to give a semantic perspective of forcing. Fix a forcing Q.

If x is a set, $\check{x}$ is defined recursively by

$$\{(\check{y}, \mathbf{1}) \mid y \in x\}.$$

Notice that this implicitly depends on Q. Also, define $\dot{G} := \{(\check{q}, q) \mid q \in Q\}$.

A set $\dot{x}$ is a Q-*name* if the following conditions are satisfied:

- every element of $\dot{x}$ is of the form $(\dot{y}, p)$ where $\dot{y}$ is a Q-name and p is in Q and
- for all $(\dot{y}, p) \in \dot{x}$ and all $(\dot{z}, q) \in \dot{y}$, either $q \leqslant p$ or $\dot{z} = \check{u}$ for some u.

Notice that this apparently implicit definition is actually a definition by recursion on rank. It should be clear that for any set x, $\check{x}$ is a Q-name and $\dot{G}$ is a Q-name.

As mentioned in the previous section, the notion of a Q-name is intended to describe a procedure for building a new set from a given filter $G \subseteq Q$. This procedure is formally described as follows. If G is any filter and $\dot{x}$ is any set, define $\dot{x}(G)$ recursively by

$$\dot{x}(G) := \{\dot{y}(G) \mid \exists p \in G\ ((\dot{y}, p) \in \dot{x})\}$$

Again, this is a definition by recursion on rank. In the analogy with randomness, $\dot{x}(G)$ corresponds to evaluating a random variable at a given outcome.

The following gives the motivation for the definitions of $\check{x}$ and $\dot{G}$.

Proposition 26.1. *If H is any filter and x is any set, then $\check{x}(H) = x$.*

Proposition 26.2. *If H is any filter, then $\dot{G}(H) = H$.*

We now turn to the formal definition of the forcing relation. The main complexity of the definition of the forcing relation is tied up in the formal definition of $p \Vdash \dot{x} \in \dot{y}$.

If Q is a forcing and $\dot{x}$ and $\dot{y}$ are Q-names, then we define the meaning of $p \Vdash \dot{x} = \dot{y}$ and $p \Vdash \dot{x} \in \dot{y}$ as follows (the definition is by simultaneous recursion on rank):

(1) $p \Vdash \dot{x} = \dot{y}$ if and only if for all $\dot{z}$ and $p' \leqslant p$,
$$(p' \Vdash \dot{z} \in \dot{x}) \leftrightarrow (p' \Vdash \dot{z} \in \dot{y}).$$

(2) $p \Vdash \dot{x} \in \dot{y}$ if and only if for every $p' \leqslant p$ there is a $p'' \leqslant p'$ and a $(\dot{z}, q)$ in $\dot{y}$ such that $p'' \leqslant q$ and $p'' \Vdash \dot{x} = \dot{z}$.

Notice that the definition of $p \Vdash \dot{x} = \dot{y}$ is precisely to ensure that the Axiom of Extensionality is forced by any condition. The definition of the forcing relation for nonatomic formulas is straightforward and is essentially determined by Properties 1–7:

(1) $p \Vdash \neg\phi$ if there does not exist a $q \leqslant p$ such that $q \Vdash \phi$.
(2) $p \Vdash \phi \wedge \psi$ if and only if $p \Vdash \phi$ and $p \Vdash \psi$.
(3) $p \Vdash \phi \vee \psi$ if there does not exist a $q \leqslant p$ such that $q \Vdash \neg\phi \wedge \neg\psi$.
(4) $p \Vdash \forall v \phi(v)$ if and only if for all $\dot{x}$, $p \Vdash \phi(\dot{x})$.
(5) $p \Vdash \exists v \phi(v)$ if and only if there is an $\dot{x}$ such that $p \Vdash \phi(\dot{x})$.

The following theorem is one of the fundamental results about forcing. It connects the syntactic properties of the forcing relation with truth in generic extensions of models of set theory. If M is a countable transitive model of ZFC, Q is a forcing in M, and $G \subseteq Q$ is an M-generic filter, define

$$M[G] := \{\dot{x}(G) \mid \dot{x} \in M \text{ and } \dot{x} \text{ is a } Q\text{-name}\}.$$

In this context, $M[G]$ is the *generic extension* of M by G and M is referred to as the *ground model*. Notice that

$$M = \{\check{x}(G) \mid x \in M\} \subseteq M[G] \qquad \text{and} \qquad G = \dot{G}(G) \in M[G].$$

The following theorem relates the semantics of forcing (i.e. truth in the generic extension) with the syntax (i.e. the forcing relation).

Theorem 26.3. *Suppose that M is a countable transitive model of ZFC and that Q is a forcing which is in M. If q is in Q, $\phi(\bar{v})$ is a formula in the language of set theory, and $\dot{x}_0, \ldots, \dot{x}_{n-1}$ are in M, then the following are equivalent:*

(a) $q \Vdash \phi(\dot{x}_0, \ldots, \dot{x}_{n-1})$.
(b) $M[G] \models \phi(\dot{x}_0(G), \ldots, \dot{x}_{n-1}(G))$ *whenever $G \subseteq Q$ is an M-generic filter and q is in G.*

Remark 26.4. This theorem can be modified to cover countable transitive models of sufficiently large finite fragments of ZFC. In fact this is crucial if one wishes to give a rigorous treatment of the semantics. By Gödel's second incompleteness theorem, ZFC alone does not prove that there are any set models of ZFC (countable or otherwise).

While we will generally not work with the semantics of forcing, let us note here that it is conventional to use $\dot{x}$ to denote a Q-name for an element x of a generic extension $M[G]$. While such names are not unique, the choice generally does not matter and this informal convention affords a great deal of notational economy.

We will now discuss some notational conventions concerning names. It is frequently the case in a forcing construction that one encounters a Q-name for a function $\dot{f}$ whose domain is forced by some condition to be a ground model set; that is, for some set D, $p \Vdash \mathrm{dom}(\dot{f}) = \check{D}$. A particularly common occurrence is when $D = \omega$ or, more generally, some ordinal. Under these circumstances, it is common to abuse notation and regard $\dot{f}$ as a function defined on D, whose values are themselves names: $\dot{f}(x)$ is a Q-name $\dot{y}$ such that it is forced that $\dot{f}(\check{x}) = \dot{y}$. Notice that if, for some sets A and B, $p \Vdash \dot{f} : \check{A} \to \check{B}$, it need not be the case that $\dot{f}(a)$ is of the form $\check{b}$ for some b in B — i.e. p need not *decide* the value of $\dot{f}(a)$ for a given $a \in A$.

In most cases, names are not constructed explicitly. Rather a procedure is described for how to build the object to which the name is referring. Properties 6 and 7 are then implicitly invoked. For example, if $\dot{x}$ is a Q-name, $\bigcup \dot{x}$ is the Q-name for the unique set which is forced to be equal to the union of $\dot{x}$. Notice that there is an abuse of notation at work here: formally, $\dot{x}$ is a set which has a union y. It need not be the case that y is even a Q-name and certainly one should not expect $\mathbf{1} \Vdash \bigcup \dot{x} = \check{y}$. This is one of the reasons for using "dot notation": it emphasizes the role of the object as a name.

A more typical example of is ω_1, the least uncountable ordinal. Since ZFC proves "there is a unique set ω_1 such that ω_1 is an ordinal, ω_1 is uncountable, and every element of ω_1 is countable," it follows that if Q is any forcing, $\mathbf{1} \Vdash \exists x \phi(x)$, where $\phi(x)$ asserts x is the least uncountable ordinal. In particular there is a Q-name $\dot{x}$ such that $\mathbf{1} \Vdash \phi(\dot{x})$. Unless readability dictates otherwise, such names are denoted by adding a "dot" above the usual notation (e.g. $\dot{\omega}_1$).

Another example is $\mathbb{R}$. Recall that $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to its metric — formally the collection of all equivalence classes of Cauchy sequences of rationals. We use this same formal definition of $\mathbb{R}$ to define $\dot{\mathbb{R}}$: if Q is a forcing, $\dot{\mathbb{R}}$ is the collection of all Q-names for equivalence classes of Cauchy sequences of rational numbers. Notice that $\dot{\mathbb{R}}$ is not the same as $\check{\mathbb{R}}$ and, more to the point, we need not even have that $\mathbf{1} \Vdash \dot{\mathbb{R}} = \check{\mathbb{R}}$ for a given forcing Q. This construction also readily generalizes to define $\dot{X}$ if X is a complete metric space. The Q-name $\dot{X}$ is then the collection of all Q-names $\dot{x}$ such that $\mathbf{1}$ forces that $\dot{x}$ is an equivalence class of Cauchy sequences of elements of $\check{X}$. That is, $\dot{X}$ is a Q-name for the completion of $\check{X}$.

Finally, there are some definable sets which are always interpreted as ground model sets and do not depend on the generic filter. Two typical examples are finite and countable ordinals such as 0, 1, ω, and

ω^2 as well as sets such as $\mathbb{Q}$. In such cases, checks are suppressed in writing the names for ease of readability — we will write $\mathbb{Q}$ and not $\check{\mathbb{Q}}$ or $\dot{\mathbb{Q}}$ in formulae which occur in the forcing language.

27. Chain conditions and the preservation of cardinals and cofinalities

An important question when studying a given forcing is whether the forcing Q preserves cardinals: if κ is a cardinal, does $\mathbf{1} \Vdash_Q \ulcorner\check{\kappa}$ is a cardinal$\urcorner$? If the answer is "yes," we say that Q preserves "κ is a cardinal" or sometimes just "Q preserves κ." A related question is whether cofinalities are preserved: if $\mathrm{cf}(\lambda) \geqslant \kappa$, does $\mathbf{1} \Vdash_Q \mathrm{cf}(\check{\lambda}) \geqslant \check{\kappa}$? Observe that an ordinal κ is a regular cardinal if and only if $\mathrm{cof}(\kappa) = \kappa$. Thus if a forcing preserves cofinalities, then it preserves that regular cardinals are regular cardinals. Since supremums of sets of cardinals are cardinals, forcings which preserve cofinalities preserve all cardinals.

Forcings which satisfy the countable chain condition always preserve cardinals. In fact it will be useful to state a more general result. If κ is a cardinal forcing Q satisfies the κ-c.c. if every antichain in Q has cardinality less than κ. The next lemma is the key to understanding the influence of the κ-c.c..

Lemma 27.1. *Suppose that Q is κ-c.c., X is a set and*

$$p \Vdash \dot{x} \in \check{X}.$$

There is a $Y \subseteq X$ such that $|Y| < \kappa$ and $p \Vdash \dot{x} \in \check{Y}$.

Proof. Define

$$Y := \{y \in X \mid \exists q \leqslant p \ (q \Vdash \dot{x} = \check{y})\}.$$

To see that $|Y| < \kappa$, choose a q_y for each $y \in Y$ such that $q_y \Vdash \dot{x} = \check{y}$. Notice that if $y \neq y'$ are in Y, then q_y and $q_{y'}$ must be incompatible: if $r \leqslant q_y, q_{y'}$, then $r \Vdash \check{y} = \dot{x} = \check{y}'$, which is impossible. Since Q is κ-c.c., $|\{q_y \mid y \in Y\}| < \kappa$ and hence $|Y| < \kappa$. To see that $p \Vdash \dot{x} \in \check{Y}$, let $q \leqslant p$ be arbitrary and find a $r \leqslant q$ such that r decides $\dot{x}$ to be y. Then $y \in Y$ and $r \Vdash \dot{x} \in \check{Y}$. Since q was arbitrary, $p \Vdash \dot{x} \in \check{Y}$. □

Theorem 27.2. *Suppose κ is a regular cardinal and Q is a forcing which satisfies the κ-c.c.. If $\mathrm{cf}(\lambda) \geqslant \kappa$, then $\mathbf{1} \Vdash \mathrm{cf}(\check{\lambda}) \geqslant \check{\kappa}$. In particular Q preserves cardinals which are greater than or equal to κ.*

Proof. Suppose that $\dot{f}$ and $\dot{\delta}$ are Q-names such that

$$\mathbf{1} \Vdash (\dot{\delta} \in \dot{\kappa}) \wedge (\dot{f} : \check{\delta} \to \check{\lambda}).$$

It suffices to find a $\gamma \in \lambda$ such that $\mathbf{1} \Vdash \mathrm{range}(\dot{f}) \subseteq \check{\gamma}$. By Lemma 27.1 and the regularity of κ, there is a $\delta_0 \in \kappa$ such that $\mathbf{1} \Vdash \dot{\delta} \in \delta_0$. For each $\alpha \in \delta_0$, let $A_\alpha \subseteq \lambda$ be such that $|A_\alpha| < \kappa$ and

$$\mathbf{1} \Vdash (\check{\alpha} \in \mathrm{dom}(\dot{f})) \to (\dot{f}(\check{\alpha}) \in \check{A}_\alpha)$$

Set $A := \bigcup\{A_\alpha \mid \alpha \in \delta_0\}$ and observe that $|A| < \kappa$ and hence A is bounded in λ by some γ. By Proposition 25.2,

$$\mathbf{1} \Vdash \forall\alpha \in \operatorname{dom}(\dot{f})\ (\dot{f}(\alpha) \in \check{\gamma}).$$

□

Theorem 27.3 (Cohen)**.** *For any α, ZFC is consistent with $|2^\omega| \geqslant \aleph_\alpha$.*

Proof. Set $\theta = \aleph_\alpha$ and let Q consist of all finite partial functions from $\theta \times \omega$ to 2. As we have seen already, Q is c.c.c. and hence preserves cardinals. In particular, for each ordinal ξ, $\mathbf{1} \Vdash \dot{\aleph}_{\check{\xi}} = \check{\aleph}_\xi$. Define $\dot{g}$ to be the name for the union of $\dot{G}$ and define $\dot{r}_\xi$ so that $\mathbf{1} \Vdash \dot{r}_\xi(n) = \dot{g}(\xi, n)$. Since $\{q \in Q \mid (\xi, n) \in \operatorname{dom}(q)\}$ is dense for each $(\xi, n) \in \theta \times \omega$, $\dot{g}$ is forced to be a total function from $\theta \times \omega$ to 2. Since

$$\{q \in Q \mid \exists n(q(\xi, n) = 0 \neq 1 = q(\eta, n))\}$$

is dense for each $\alpha \neq \beta \in \theta$, $\mathbf{1} \Vdash |\{\dot{r}_\xi \mid \xi \in \check{\theta}\}| \geqslant |\check{\theta}| = \dot{\aleph}_{\check{\alpha}}$. Thus the theory of what is forced by Q contains ZFC and $|2^\omega| \geqslant \aleph_\alpha$. □

It should be noted that while the property of being c.c.c. is far from characterizing the preservation of cardinals, there are forcings which collapse cardinals. For instance, if X is any set and Q is the poset of all finite partial functions from ω to X, then Q forces that X is countable — the union of the generic filter is forced to be a surjection from ω onto X.

28. Closure properties of forcings

Another fundamental question concerning a forcing Q is whether forcing with Q adds new subsets to a ground model set or, more generally, new functions between two ground model sets. Of particular interest is whether forcing with Q adds new subsets of ω or — equivalently — new real numbers.

A poset Q is *κ-closed* if whenever $\langle q_\xi \mid \xi \in \alpha\rangle$ is a decreasing sequence in Q of length less than κ, then there is a $\bar{q} \in Q$ such that $\bar{q} \leqslant q_\xi$ for all $\xi \in \alpha$. An ω_1-closed poset is often said to be *σ-closed* or *countably closed*.

Theorem 28.1. *Suppose that κ is a regular cardinal and Q is a κ-closed forcing. If X is a set and $\dot{s}$ is a Q-name for a sequence of elements of X of length less than κ, then the set of conditions which decide $\dot{s}$ is dense. In particular, Q preserves cardinals and when cofinalities are at least κ.*

Proof. Let $p \in Q$ be arbitrary. By extending p if necessary, we may assume that p decides the length of $\dot{s}$ to be α for some $\alpha \in \kappa$. Construct a decreasing sequence $\langle q_\xi \mid \xi \in \alpha\rangle$ by recursion so that $q_0 \leqslant p$ and for all $\xi \in \alpha$ there is a (unique) x_ξ such that $q_\xi \Vdash \dot{s}(\check{\xi}) = \check{x}_\xi$. Since Q is κ-closed, there is a $\bar{q}$ such that $\bar{q} \leqslant q_\xi$ for all $\xi \in \alpha$. Now $\bar{q} \Vdash \forall \xi \in \alpha\ (\dot{s}(\xi) = \check{x}_\xi)$ and therefore if $t := \langle x_\xi \mid \xi \in \alpha\rangle$, $\bar{q} \Vdash \dot{s} = \check{t}$. □

Corollary 28.2. *If Q is σ-closed, Q forces $\dot{\mathbb{R}} = \check{\mathbb{R}}$ and $\dot{\omega}_1 = \check{\omega}_1$.*

Consider the poset Q of all countable partial functions from ω_1 to 2, ordered by extension. Let $\dot{g}$ be the Q-name for the union of the generic filter. By a standard density argument, $\mathbf{1} \Vdash \dot{g} : \check{\omega}_1 \to 2$. For each $\xi \in \omega_1$, fix a Q-name $\dot{r}_\xi$ for the element of 2^ω such that $\dot{r}_\xi(n) = \dot{g}(\omega \cdot \xi + n)$. For any $s \in 2^\omega$, define D_s to be the set of all q in Q such that for some ξ, $[\omega \cdot \xi, \omega \cdot \xi + \omega)$ is contained in the domain of q and $s(n) = q(\omega \cdot \xi + n)$ for all $n \in \omega$. Clearly D_s is dense and if $q \in D_s$,

$$q \Vdash \exists \xi \in \check{\omega}_1\ (\check{s} = \dot{r}_\xi)$$

It follows that $\mathbf{1} \Vdash |\check{\mathbb{R}}| \leqslant \check{\omega}_1$. Since Q is clearly σ-closed, Q forces $\dot{\mathbb{R}} = \check{R}$ and $\dot{\omega}_1 = \check{\omega}_1$ and hence that CH holds. In fact it is possible to prove more.

Theorem 28.3. *Let Q be the poset of all countable partial functions from ω_1 to 2. Q forces $\diamondsuit$.*

Proof. First notice that since $|\omega_1 \times \omega_1| = |\omega_1|$, the poset of all countable partial functions from $\omega_1 \times \omega_1$ to 2 is isomorphic to Q. We will use this poset instead for convenience. Let $\dot{g}$ be the name for the union of the generic filter. By standard density arguments, $\dot{g}$ is forced to be a total function from $\omega_1 \times \omega_1$ to 2. Define a sequence of Q-names $\langle \dot{A}_\alpha \mid \alpha \in \omega_1 \rangle$ by $\beta \in \dot{A}_\alpha$ if and only if $\beta \in \alpha$ and $\dot{g}(\alpha, \beta) = 1$. It suffices to show that for every $p \in Q$, if $p \Vdash \dot{X} \subseteq \dot{\omega}_1$, then there is a $\delta \in \omega_1$ and a $q \leqslant p$ such that $q \Vdash \dot{X} \cap \check{\delta} = \dot{A}_{\check{\delta}}$.

Set $\delta_0 := \omega$, $p_0 = p$ and construct conditions p_n and countable ordinals δ_n such that:

- $p_{n+1} \leqslant p_n$ decides $\dot{X} \cap \check{\delta}_n$ to be Y_n;
- the domain of p_{n+1} is contained in $\delta_{n+1} \times \delta_{n+1}$.

Notice that this is possible since, because Q is σ-closed, $\mathbf{1} \Vdash \dot{\mathscr{P}}(\delta_n) = \check{\mathscr{P}}(\delta_n)$. Set $\delta := \sup\{\delta_n \mid n \in \omega\}$ and $Y := \bigcup\{Y_n \mid n \in \omega\}$ and observe that if $q \leqslant p_n$ for all n, then $q \Vdash \dot{X} \cap \check{\delta} = \check{Y}$. Define

$$q(\alpha, \beta) := \begin{cases} p_n(\alpha, \beta) & \text{if } (\alpha, \beta) \in \operatorname{dom}(p_n) \\ 1 & \text{if } \beta \in \alpha = \delta \text{ and } \beta \in Y \\ 0 & \text{if } \beta \in \alpha = \delta \text{ and } \beta \notin Y \end{cases}$$

Notice that

$$q \Vdash \dot{X} \cap \check{\delta} = \check{Y} = \dot{A}_{\check{\delta}}$$

as desired. □

Remark 28.4. Baumgartner generalized this argument to show that if Q is a σ-closed forcing which added a new subset of ω_1, then Q forces $\diamondsuit$.

Consider the forcing Q which is obtained by taking the separative quotient of $([\omega]^\omega, \subseteq)$. Observe that if $\langle x_n \mid n \in \omega \rangle$ is a sequence of infinite subsets of ω such that $x_{n+1} \subseteq^* x_n$, then there is an infinite $x \in [\omega]^\omega$ such that $x \subseteq^* x_n$ for all n: for instance set

$$x := \{\min(\bigcap_{i<n} x_i \backslash n) \mid n \in \omega\}$$

In particular, even though $([\omega]^\omega, \subseteq)$ is not σ-closed, Q is σ-closed.

An ultrafilter $\mathscr{U}$ on ω is *selective* if it is nonprinciple and for every $f : \omega \to \omega$ there is a $U \in \mathscr{U}$ such that $f \restriction U$ is either constant or one-to-one. It can be shown that $\mathscr{U}$ is a selective ultrafilter if and only if $\mathscr{U}$ is nonprinciple and whenever $f : [\omega]^d \to k$ for $k, d \in \omega$, then there is a U in $\mathscr{U}$ such that $f \restriction [U]^d$ is constant. Thus selective ultrafilters are also known as *Ramsey ultrafilters.*

Theorem 28.5. *The partial order $([\omega]^\omega, \subseteq)$ forces that $\dot{G}$ is a selective ultrafilter on ω.*

Proof. As noted above, the separative quotient Q of this partial order is σ-closed and in particular does not add new functions from ω to ω. Thus it suffices to show that for every $f : \omega \to \omega$ and every $p \in Q$ there is a $q \leqslant p$ such that

$$q \Vdash \exists y \in \dot{G} \ (\check{f} \restriction y \text{ is constant or one-to-one})$$

Let x be in the equivalence class of p. If there is a k such that $x \cap f^{-1}(k)$ is infinite, set $y := x \cap f^{-1}(k)$. Otherwise $f \restriction x$ is finite-to-one and $y := \{\min(x \cap f^{-1}(k)) \mid k \in \text{range}(f \restriction x)\}$ is infinite. Let q be the equivalence class of y and observe that $q \Vdash \check{y} \in \dot{G}$. □

Remark 28.6. Let R_θ be the forcing consisting of all compact subsets of 2^θ of positive measure, ordered by $\subseteq$. Kunen has shown that if θ is a cardinal greater than $2^{\aleph_0}$, then R_θ forces that there are no selective ultrafilters.

29. Product forcing

We will now discuss product forcing and iterated forcing with the ultimate aims of proving the consistency of $\mathrm{MA}_{\aleph_1}$ with ZFC and of constructing Solovay's model in which all sets of reals are Lebesgue measurable and ZF holds. It will sometimes be informative to take a less formal, more semantic approach to forcing going forward. We will frequently talk about starting with a ground model V of ZFC, taking a V-generic filter G for some forcing Q in V, and forming the generic extension $V[G]$. This can always be formalized syntactically (but sometimes with great notational headache) or semantically by applying the reflection theorem and the Löwenheim-Skolem theorem to find countable transitive models of arbitrary finite fragments of ZFC (but introducing a certain amount of irrelevant baggage). Regardless of the type of formalism, the rigor tends to obscure and distract from the underlying set theory.

First we will discuss product forcing. If P and Q are forcings, what is the effect of forcing with $P \times Q$? It turns out that the following three operations are essentially the same: forcing with P and then Q, forcing with Q and then P, and forcing with $P \times Q$. Let us note the following fact.

Proposition 29.1. *Suppose that P and Q are forcings and $K \subseteq P \times Q$ is a filter. Then $K = G \times H$ for some filters $G \subseteq P$ and $H \subseteq Q$.*

Proof. Let $K \subseteq P \times Q$ be given and define

$$G := \{p \in P \mid \exists q\ ((p,q) \in K)\}$$
$$H := \{q \in Q \mid \exists p\ ((p,q) \in K)\}.$$

Observe that trivially $K \subseteq G \times H$. To see the other containment, suppose $p \in G$ and $q \in H$. Let $q' \in Q$ be such that $(p,q') \in K$ and let $p' \in P$ be such that $(p',q) \in K$. Since K is a filter, there is a $(\bar{p},\bar{q}) \in K$ such that $\bar{p} \leqslant p,p'$ and $\bar{q} \leqslant q,q'$. Since $(\bar{p},\bar{q}) \leqslant (p,q)$ and since K is a filter, $(p,q) \in K$. Since $(p,q) \in G \times H$ was arbitrary, $G \times H \subseteq K$. □

Theorem 29.2. *Suppose that V is a transitive model of ZFC and $P,Q \in V$ are forcings. If $G \subseteq P$ and $H \subseteq Q$ are filters, the following are equivalent:*

(1) G is V-generic and H is $V[G]$-generic.
(2) $G \times H \subseteq P \times Q$ is V-generic.

Proof. To see the forward implication, suppose that $D \subseteq P \times Q$ is dense. Define

$$\dot{E} := \{(\check{q}, p) \mid (p,q) \in D\}.$$

Claim 29.3. *P forces that $\dot{E}$ is dense.*

Proof. Let $p_0 \in P$ be arbitrary and $\dot{q}_0$ be such that $p_0 \Vdash_P \dot{q}_0 \in \check{Q}$. Let p_1 be such that $p_1 \leqslant p_0$ decides $\dot{q}_0$ to be q_1. Since D is dense, there is a $(p, q) \in D$ such that $p \leqslant p_1$ and $q \leqslant q_1$. Now $p \leqslant p_0$ and

$$p \Vdash_P (\check{q} \leqslant \dot{q}_0) \wedge (\check{q} \in \dot{E}).$$

Since p_0 and $\dot{q}_0$ were arbitrary, P forces $\dot{E}$ is a dense subset of $\check{Q}$. $\square$

By the claim, since G is V-generic, $V[G]$ satisfies that $E \subseteq Q$ is dense. Since H is $V[G]$-generic, $V[G]$ satisfies that $H \cap E$ contains some q. But this means precisely that there is $p \in G$ such that $(p, q) \in D$. Thus $(p, q) \in G \times H$.

To see the reverse implication, suppose that $D_0 \subseteq P$ is dense and in V and $D_1 \subseteq Q$ is dense and in $V[G]$. Let $\dot{D}_1$ be a P-name in V whose interpretation by G is D_1 and let $p_0 \in G$ be such that p_0 forces $\dot{D}_1$ is dense. Define

$$D := \{(p, q) \in P \times Q \mid ((p \in D_0) \wedge (p \Vdash_P \check{q} \in \dot{D}_1)) \vee (p \perp p_0)\}$$

Claim 29.4. *D is dense in $P \times Q$.*

Proof. Let $(p_1, q_1) \in P \times Q$ be given. If there is a $p \leqslant p_1$ which is incompatible with p_0, then $(p, q_1) \leqslant (p_1, q_1)$ is in D. Otherwise, p_1 forces $\dot{D}_1$ is dense in $\check{Q}$. Thus there is a Q-name $\dot{q}$ such that

$$p_1 \Vdash_P (\dot{q} \leqslant \check{q}_1) \wedge (\dot{q} \in \dot{D}_1)$$

Since $p_1 \Vdash_P \dot{q} \in \check{Q}$, there is a $p \leqslant p_1$ which decides $\dot{q}$ to be q for some $q \in Q$. Now $(p, q) \in D$ and $(p, q) \leqslant (p_1, q_1)$. Hence D is dense. $\square$

Since $G \times H$ is V-generic, there is a $(p, q) \in G \times H \cap D$. But now $p \in D_0$ and $q \in D_1$. Thus G is V-generic and H is $V[G]$-generic. $\square$

30. Iterated forcing

Suppose now that P is a forcing in a transitive model $\mathbf{V}$ of set theory and Q is a forcing in $\mathbf{V}[G]$. We would like to view the process of forcing first with P and then Q as being equivalent to forcing with a single poset $P*\dot{Q}$. This is useful for a number of reasons. First, it allows us to understand two step generic extensions purely from the ground model. Second, it provides the foundation on which transfinite iterations of forcings are built. While achieving two step generic extensions via a single iterated poset is partly a convenience, it becomes an existential issue for transfinite iterations. For instance if G_{n+1} is generic over $M[G_n]$ for the poset Q_n, what is the candidate for the generic extension of M by $\langle G_n \mid n \in \omega\rangle$? Typically $\bigcup_n M[G_n]$ will not be a model of ZFC and it will not contain $\{G_n \mid n \in \omega\}$. In order to address this issue, we need to better understand two step iterations.

If P is a forcing and $\dot{Q}$ is P-name for a forcing, define $P * \dot{Q}$ to be all pairs $(p, \dot{q})$ such that $p \in P$, $\dot{q}$ is a P-name, and $p \Vdash \dot{q} \in \dot{Q}$. By Property 4, $P * \dot{Q}$ is a set. Define $\leqslant$ on $P * \dot{Q}$ by $(p_1, \dot{q}_1) \leqslant (p_0, \dot{q}_0)$ if $p_1 \leqslant p_0$ and $p_1 \Vdash \dot{q}_1 \leqslant \dot{q}_0$. It is easily checked that $\leqslant$ is both reflexive and transitive. Typically $\leqslant$ is not separative or even antisymmetric. We will implicitly work with the separative quotient, which we will also denote $P * \dot{Q}$. The forcing $P * \dot{Q}$ is called the *iteration of P and $\dot{Q}$*.

Notice that if P and Q are forcings, $P * \check{Q}$ is not the same as $P \times Q$. The set $\{(p, \check{q}) \mid (p, q) \in P \times Q\}$ is is dense in $P * \dot{Q}$, however, and $(p, q) \mapsto (p, \check{q})$ is an isomorphism onto its range. Theorem 29.2 has the following analog for iterations, whose proof is left as an exercise.

Theorem 30.1. *Suppose that M is a transitive model of ZFC and $P * \dot{Q}$ is an iteration of forcings in M. If $K \subseteq P * \dot{Q}$ is a filter, then the following are equivalent:*

(1) $G := \{p \in P \mid \exists \dot{q}\ ((p, \dot{q}) \in K)\}$ is a M-generic filter and $H := \{q \in Q \mid \exists p \in G\ ((p, \dot{q}) \in K)\}$ is a $M[G]$-generic filter.
(2) K is M-generic.

The next theorem is what might be referred to as the fundamental theorem of iterated forcing.

Theorem 30.2. *Suppose that M is a transitive model of ZFC, P is a forcing in M and $G \subseteq P$ is a M-generic filter. If N is a transitive model of ZFC such that $M \subseteq N \subseteq M[G]$, then there is an iteration $P_0 * \dot{Q}$ in M and filters $G_0 \subseteq P_0$ and $H \subseteq Q$ in $M[G]$ such that G_0 is M-generic, H is $M[G_0]$-generic, $N = M[G_0]$ and $M[G] = N[H]$.*

Remark 30.3. By $N \subseteq M[G]$ is a transitive model of ZFC we mean that N is a transitive class from the point of view of $M[G]$ and that $(N, \in)$ satisfies ZFC.

Before we prove this theorem, it will be useful to make a few general observations. The first is that every dense subset of a forcing Q contains a maximal antichain. On the other hand, if $A \subseteq Q$ is a maximal antichain, then the set of conditions which extend some element of A is dense. In particular, if M is a transitive model of ZFC and Q is a forcing in M, a filter $G \subseteq Q$ is M-generic if and only if it intersects every maximal antichain in M.

Next suppose M is a transitive model of set theory, Q is a forcing in M and $Q_0 \subseteq Q$ is dense and in M. If $G \subseteq Q$ if a M-generic filter, then $G \cap Q_0$ is a M-generic filter and if $G \subseteq Q_0$ is a M-generic filter, then the upwards closure of G in Q is a M-generic filter. In particular, generic extensions of M by Q coincide with the generic extensions of M by Q_0.

If $Q_0 \subseteq Q$ are forcings and every maximal antichain in Q_0 is a maximal antichain in Q, then we say that Q_0 is a regular suborder of Q. This is equivalent to the assertion that

$$\mathbf{1} \Vdash_Q \ulcorner \dot{G} \cap \check{Q}_0 \text{ is } \check{\mathbf{V}}\text{-generic}\urcorner$$

If B is a complete Boolean algebra and A is a complete subalgebra of B, then A^+ is a regular suborder of B^+. Finally, we recall that if Q is any separative forcing, Q embeds as a dense suborder of the positive elements of a complete Boolean algebra B. Recall that if ϕ is a formula in the forcing language associated to B^+, $[\![\phi]\!]$ is the maximum element of B such that $[\![\phi]\!] \Vdash \phi$. We are now ready to give the proof of Theorem 30.2.

Proof. By the observations made above, we may assume that P is the positive elements of a complete Boolean algebra B. Let $M \subseteq N \subseteq M[G]$ be given as in the statement of the theorem and let $\phi(v)$ be a formula in the forcing language which defines the class N within $M[G]$. First suppose that $X \in N$ is a set of ordinals and let $\dot{X}$ be the P-name whose evaluation is X. Notice that while $[\![\dot{X} \subseteq \mathbf{ON} \wedge \phi(\dot{X})]\!]$ need not be one, there is a P-name $\dot{Y}$ such that $[\![\dot{X} = \dot{Y}]\!] = [\![\dot{X} \subseteq \mathbf{ON} \wedge \phi(\dot{X})]\!]$ and $[\![\dot{Y} \subseteq \mathbf{ON} \wedge \phi(\dot{Y})]\!] = \mathbf{1}$. Thus we may assume without loss of generality that $[\![\dot{X} \subseteq \mathbf{ON} \wedge \phi(\dot{X})]\!] = \mathbf{1}$. Let $A_{\dot{X}}$ be the complete subalgebra of B generated by $\{[\![\check{\alpha} \in \dot{X}]\!] \mid \alpha \in \mathbf{ON}\}$. Observe that

$$\mathscr{A} := \{A_{\dot{X}} \mid [\![\dot{X} \subseteq \mathbf{ON} \wedge \phi(\dot{X})]\!] = \mathbf{1}\}$$

is contained in $\mathscr{P}(B)$ and hence is a set.

Claim 30.4. *There is an $A_{\dot{Y}} \in \mathscr{A}$ such that every other element of $\mathscr{A}$ is a subset of $A_{\dot{Y}}$.*

Proof. Let $\{\dot{X}_\xi \mid \xi \in \kappa\}$ be a list of P-names such that for each $\xi \in \kappa$, $[\![\dot{X} \subseteq \mathbf{ON} \wedge \phi(\dot{X})]\!] = 1$ and

$$\{A_{\dot{X}} \mid [\![\dot{X} \subseteq \mathbf{ON} \wedge \phi(\dot{X})]\!] = \mathbf{1}\} = \{A_{\dot{X}_\xi} \mid \xi \in \kappa\}.$$

By Proposition 25.4, there is a sufficiently large ordinal λ such that for every $\xi \in \kappa$, $[\![\dot{X}_\xi \subseteq \check{\lambda}]\!] = \mathbf{1}$. Define a P-name $\dot{Y}$ for a subset of $\lambda \cdot \kappa$ by

$$[\![\check{\kappa}\dot{\xi} + \check{\alpha} \in \dot{Y}]\!] = [\![\check{\alpha} \in \dot{X}_\xi]\!].$$

In particular $A_{\dot{X}_\xi} \subseteq A_{\dot{Y}}$ for all $\xi \in \kappa$. By definition $A_{\dot{Y}} \in \mathscr{A}$. □

Let $\dot{Y}$ be as in the claim and let Y be the interpretation of $\dot{Y}$ by G. Let κ be such that $[\![\dot{Y} \subseteq \check{\kappa}]\!] = 1$. Since both Y and $\dot{Y}$ are in N,

$$\{[\![\check{\alpha} \in \dot{Y}]\!] \mid \alpha \in Y\} \cup \{[\![\check{\alpha} \notin \dot{Y}]\!] \mid \alpha \in \kappa \backslash Y\}$$

is a set in N. Let G_0 be the filter generated by this set, noting that $G_0 = G \cap A$. Now observe that if a is any set in N, there is a set of ordinals X in N such that a is in any transitive model of ZFC which has X as an element. Thus we have shown that $N = M[G_0]$.

To see that $M[G]$ is a generic extension of $M[G_0]$, define $I \subseteq B$ to be the set of complements of elements of G_0 in B. I is an ideal and we can define B/I to be the quotient Boolean algebra. Let Q be the positive elements of B/I. We leave it as an exercise that G/I is $M[G_0]$-generic for Q. □

While we will see that the analysis of finite products and iterations of forcings can be very fruitful, many applications of forcing require one to work with a transfinite iteration of forcings. We will only work with the simplest type of transfinite iterated forcing which well suited for working within the class of c.c.c. forcings but nothing more.

31. Finite Support Iterated forcing

A *finite support iteration* of forcings is a sequence $\langle P_\alpha \mid \alpha \in \theta\rangle$ of forcings such that:

- each element of P_α is a sequence of length α;
- if $\alpha \in \beta \in \theta$, then $P_\alpha = \{p\restriction\alpha \mid p \in P_\beta\}$;
- if $\alpha + 1 \in \theta$, then there is a P_α-name $\dot{Q}_\alpha$ such that is in
$$P_{\alpha+1} := \{p^\frown\langle\dot{q}\rangle \mid (p \in P_\alpha) \wedge (p \Vdash_{P_\alpha} \dot{q} \in \dot{Q}_\alpha)\}$$
- if $p \in P_\alpha$, then $\{\xi \in \alpha \mid p(\xi) \neq \dot{\mathbf{1}}\}$ is finite.

The set in the last condition is known as the *support* of p. The orderings are required to satisfy the following conditions:

- if β is a limit ordinal and $p, q \in P_\beta$, then $q \leqslant p$ if and only if for all $\alpha \in \beta$, $q\restriction\alpha \leqslant p\restriction\alpha$.
- if $\beta = \alpha + 1$ and $p, q \in P_\beta$, then $q \leqslant p$ if and only if $q\restriction\alpha \leqslant p\restriction\alpha$ and $q\restriction\alpha \Vdash_{P_\alpha} q(\alpha) \leqslant p(\alpha)$.

It follows that if $\alpha \in \beta \in \theta$ and $p, q \in P_\beta$, then $q \leqslant p$ implies $q\restriction\alpha \leqslant p\restriction\alpha$.

Notice that P_0 always consists of a single element — namely the sequence of length 0. Also, there is a canonical isomorphism between $P_{\alpha+1}$ and $P_\alpha * \dot{Q}_\alpha$ — namely $p \mapsto (p\restriction\alpha, p(\alpha))$. Furthermore, if β is a limit ordinal, then $p \in P_\beta$ if and only if $p\restriction\alpha \in P_\alpha$ for all $\alpha \in \beta$ and the support of p is finite. Thus $\langle P_\alpha \mid \alpha \in \theta\rangle$ is uniquely determined by the sequence $\langle\dot{Q}_\alpha \mid \alpha + 1 \in \theta\rangle$ and indeed transfinite iterations of forcings are typically specified by recursively selecting the sequence of Q_α's. Typically θ is a limit ordinal, in which case there is a unique forcing P_θ such that $\langle P_\alpha \mid \alpha \in \theta + 1\rangle$ is a finite support iteration. P_θ is referred to as the finite support iteration of the iterands $\langle\dot{Q}_\alpha \mid \alpha \in \theta\rangle$. It is often the case that θ is called the length of the iteration.

A fundamental problem in set theory is to determine what effect properties of the iterands in an iteration have on the iteration itself. In the case of finite support iterations, this is fairly straightforward. This is largely because finite support iterations are only useful in iterating c.c.c. forcings and this is a fairly restrictive class. *Countable support iterations* allow for the iteration of a much broader class of forcings but their analysis can be very challenging and the preservation of basic properties such as when no new real numbers are added is still not fully understood (and likely intractable). The most fundamental fact about finite support iterations is contained in the following theorem of Solovay and Tennenbaum.

Theorem 31.1. *Suppose that $\langle P_\alpha \mid \alpha \in \theta\rangle$ is a finite support iteration such that for each α, P_α forces that $\dot{Q}_\alpha$ is c.c.c.. Then P_θ is c.c.c..*

Remark 31.2. One of the most important features of the class of c.c.c. forcings is that they preserve cardinals (and $\aleph_1$ in particular) and are preserved by finite support iterations. It is not difficult to show that if $\langle P_n \mid n \in \omega\rangle$ is a finite support iteration and for infinitely many n, P_n forces that $\dot{Q}_n$ is not c.c.c., then P_ω collapses $\aleph_1$. Forcings which preserve stationary subsets of $[X]^\omega$ for every uncountable X are known as *proper forcings*. The class of proper forcings includes both the class of all c.c.c. forcings and all σ-closed forcings. This class of forcings preserves $\aleph_1$ and is closed under taking countable support iterations.

Proof. We will first show that if $P * \dot{Q}$ is an iteration, P is c.c.c. and $\dot{Q}$ is forced to be c.c.c., then $P * \dot{Q}$ is c.c.c.. To see this, suppose that $\langle(p_\xi, \dot{q}_\xi) \mid \xi \in \omega_1\rangle$ is a sequence of conditions in $P * \dot{Q}$. Since P is c.c.c., there is an $r_0 \in P$ such that

$$r_0 \Vdash_P \ulcorner\dot{\Xi} := \{\xi \in \omega_1 \mid \check{p}_\xi \in \dot{G}\} \text{ is uncountable}\urcorner.$$

Since $\dot{Q}$ is forced to be c.c.c., there are P-names $\dot{\xi}$ and $\dot{\eta}$ such that r_0 forces $\dot{\xi} \neq \dot{\eta}$ are in $\dot{\Xi}$ and $\dot{q}_{\dot{\xi}}$ is compatible with $\dot{q}_{\dot{\eta}}$. Let $r \leqslant r_0$ decide $\dot{\xi}$ and $\dot{\eta}$ to be ξ and η. Since r forces $\check{p}_\xi$ and $\check{p}_\eta$ are in $\dot{G}$, it must be that $r \leqslant p_\xi, p_\eta$ and in particular p_ξ and p_η are compatible. Since r forces $\dot{q}_{\check{\xi}}$ and $\dot{q}_{\check{\eta}}$ are compatible, there is a P-name $\dot{s}$ such that r forces $\dot{s} \in \dot{Q}$ is a lower bound for $\dot{q}_{\check{\xi}}$ and $\dot{q}_{\check{\eta}}$. Now $(r, \dot{s}) \in P * \dot{Q}$ is a lower bound for $(p_\xi, \dot{q}_\xi)$ and $(p_\eta, \dot{q}_\eta)$. It follows that $P * \dot{Q}$ is c.c.c..

We will now prove Theorem 31.1 by induction on θ. If $\theta = \alpha + 1$ for some α, then P_θ is isomorphic to $P_\alpha * \dot{Q}_\alpha$. Since P_α is c.c.c. by our induction hypothesis, the conclusion of the theorem follows from the special case of the theorem for length 2 iterations. Now suppose that θ is a limit ordinal and that $\langle p_\xi \mid \xi \in \omega_1\rangle$ is a sequence of elements of P_θ. Define $D_\xi := \{\alpha \in \theta \mid q_\xi(\alpha) \neq \mathbf{1}\}$. By the Δ-System Lemma, there is an uncountable $\Xi \subseteq \omega_1$ such that $\{D_\xi \mid \xi \in \Xi\}$ forms a Δ-system with root R. Since θ is a limit ordinal, there is an $\alpha \in \theta$ be such that $R \subseteq \alpha$. By our inductive assumption, there are $\xi \neq \eta$ such that $p_\xi \restriction \alpha$ and $p_\eta \restriction \alpha$ have a common lower bound $q_0 \in P_\alpha$. Define

$$q(\beta) := \begin{cases} q_0(\beta) & \text{if } \beta < \alpha \\ p_\xi(\beta) & \text{if } \beta \in D_\xi \backslash \alpha \\ p_\eta(\beta) & \text{if } \beta \in D_\eta \backslash \alpha \\ \mathbf{1} & \text{otherwise} \end{cases}$$

Since $D_\xi \cap D_\eta = R \subseteq \alpha$, q is well defined. Clearly $q \in P_\theta$, $q \leqslant p_\xi$, and $q \leqslant p_\eta$. This completes the proof that P_θ is c.c.c.. $\square$

32. Some bookkeeping lemmas

We would now like to show that, for any regular uncountable cardinal θ, there is always a c.c.c. forcing P which forces MA_θ. The forcing P will be constructed as a finite support iteration of c.c.c. forcings of length $\kappa := 2^\theta$ and will force $2^\theta = 2^{\aleph_0}$. Before we do this, we will need to establish a number of lemmas which allow us to keep track of the tasks we need to accomplish.

First observe that if $P \subseteq Q$ are forcings, then any P-name is a Q-name. If P is a regular suborder of Q, and ϕ is any formula in the forcing language associated to P, then $p \Vdash_P \phi$ if and only if $p \Vdash_Q \phi$ for any $p \in P$. If every condition occurring in a Q-name $\dot{x}$ is in P, then $\dot{x}$ is also a P-name. If there is a natural embedding of P into Q, it is often common to treat P as a suborder of Q. In particular, P-names are regarded as Q-names via this embedding. This issue commonly arises in the context of iterations: P is naturally embedded into $P * \dot{Q}$ as a regular suborder via the map $p \mapsto (p, \dot{\mathbf{1}})$. We will abuse notation and treat P-names as $P * \dot{Q}$-names without further mention. This will also be true for finite support iterations.

Lemma 32.1. *Suppose that μ and κ are infinite cardinals such that $\kappa^{<\mu} = \kappa$ and such that μ is regular. If P is a μ-c.c. poset containing a dense set of cardinality at most κ, then P forces $\kappa^{<\mu} = \kappa$.*

Proof. Observe that every element of $\kappa^{<\mu}$ is a subset of $\kappa \times \mu$ of cardinality less than μ. Since $\mu \leqslant \kappa$ (otherwise $\kappa < \kappa^\kappa \leqslant \kappa^{<\mu}$), it suffices to show that P forces there are κ many subsets of κ of cardinality less than μ. Let $D \subseteq P$ have cardinality at most κ.

Suppose now that $\dot{X}$ is a P-name such that

$$p \Vdash (\dot{X} \subseteq \kappa) \wedge (|\dot{X}| < \mu)$$

for some $p \in P$. For each $\alpha \in \kappa$, set

$$D_\alpha := \{q \in D \mid (q \leqslant p) \wedge (q \Vdash \check{\alpha} \in \dot{X})\}$$

and let $A_\alpha \subseteq D_\alpha$ be an antichain which is maximal with respect to being contained in D_α. Define

$$\dot{Y} := \{(\check{\alpha}, q) \mid (\alpha \in \kappa) \wedge (q \in A_\alpha)\}.$$

Claim 32.2. $p \Vdash \dot{Y} = \dot{X}$.

Proof. This is left as an exercise. $\square$

Observe that, since P is μ-c.c., each A_α has cardinality less than μ. Thus $\dot{Y}$ is a union of sets of fewer than μ sets, each of cardinality less than μ. Since μ is regular, $|\dot{Y}| < \mu$. Also, $\dot{Y}$ is a subset of $\kappa \times D$ and $|\kappa \times D| = \kappa$. We've therefore show than every subset of κ of size less than μ in the generic extension has a P-name which has cardinality less than μ and is a subset of a set of size κ. This completes the proof. $\square$

A P-name $\dot{X}$ such that elements of $\dot{X}$ have the form $(\check{x}, p)$ for $p \in P$ and such that for each x, $\{p \in P \mid (\check{x}, p) \in \dot{X}\}$ is an antichain are sometimes called *nice names*. As the previous lemma shows, nice names are useful in estimating cardinalities in generic extensions. We showed that if $\dot{X}$ is a P-name for a subset of a ground model set, then there is a nice P-name $\dot{Y}$ such that $\mathbf{1} \Vdash \dot{Y} = \dot{X}$.

We will also need the following general fact about cardinal arithmetic.

Theorem 32.3 (König). *Suppose that θ is an infinite cardinal. The cofinality of 2^θ is greater than θ.*

Proof. Let $\langle A_\xi \mid \xi \in \theta\rangle$ be a partition of θ into θ sets of cardinality θ. Suppose that $\mathscr{F}_\xi \subseteq 2^\theta$ for each $\xi \in \theta$ and that $|\mathscr{F}_\xi| < 2^\theta$. Since $\mathscr{P}(A_\xi) = 2^\theta$, there is a $g_\xi : A_\xi \to 2$ such that $g_\xi \neq f \restriction A_\xi$ for any $f \in A_\xi$. Define $g : \theta \to 2$ by $g = \bigcup\{g_\xi \mid \xi \in \theta\}$, noting that g is not in $\mathscr{F}_\xi$ for any $\xi \in \theta$. Thus 2^θ is not the union of θ sets, each of cardinality less than 2^θ. $\square$

Lemma 32.4. *Let θ be an infinite cardinal and suppose that $\langle P_\alpha \mid \alpha \in \gamma\rangle$ is an iteration of c.c.c. posets, each of which is forced to have cardinality at most θ. Then P_γ has a dense subset of cardinality at most* $\max(|\gamma|, \theta)$.

Proof. The proof is by induction on γ. If $\gamma = 0$, there is nothing to show since $|P_0| = 1 < \theta$. If $\gamma = \beta + 1$, let $D \subseteq P_\beta$ be a dense set of cardinality at most $\max(|\beta|, \theta)$ and fix a sequence of names $\langle \dot{q}_\xi \mid \xi \in \theta\rangle$ such that P_β forces $\dot{Q}_\beta = \{\dot{q}_\xi \mid \xi \in \check{\theta}\}$. If $p \in P_\beta$ and $\dot{q}$ is a P_β-name for an element of $\dot{Q}_\beta$, then there is a $\xi \in \theta$ and a $p' \leqslant p$ in D such that $p' \Vdash \dot{q}_\xi = \dot{q}$. In particular,

$$\{p \in P_\alpha \mid (p \restriction \beta \in D) \wedge \exists \xi \in \theta\ (p(\beta) = \dot{q}_\xi)\}$$

is dense and of cardinality at most $|D| \cdot \theta \leqslant \max(|\gamma|, \theta)$. If γ is a limit ordinal, let $D_\alpha \subseteq P_\alpha$ be a dense set of cardinality at most $\max(|\alpha|, \theta)$ for each $\alpha \in \gamma$. Define $D \subseteq P_\gamma$ to be all $p \in P_\gamma$ such that for some $\alpha \in \gamma$, $p \restriction \alpha \in D_\alpha$ and if $\alpha < \beta < \gamma$, $p(\beta) = \mathbf{1}$. It follows that $|D| \leqslant |\gamma| \cdot \max(|\gamma|, \theta) = \max(|\gamma|, \theta)$. $\square$

33. How to force MA_θ

We will now begin the recursive construction of the iteration. This is done by recursively constructing a sequence $\langle \dot{Q}_\alpha \mid \alpha \in \kappa\rangle$, defining as we go P_γ to be the finite support iteration of $\langle \dot{Q}_\alpha \mid \alpha \in \gamma\rangle$. This construction will be carried out so as to satisfy that for each $\alpha \in \kappa$, P_α forces $\dot{Q}_\alpha$ is a c.c.c. forcing of cardinality at most $\check{\theta}$. By Lemma 31.1, this will imply that P_α is c.c.c. for all $\alpha \leqslant \kappa$. Moreover, by Lemma 32.4, P_α will have a dense subset of cardinality less than κ. If $\alpha \leqslant \kappa$, let $\dot{G}_\alpha$ denote the P_α-name for the generic filter and $\dot{H}_\alpha$ be the $P_{\alpha+1}$-name for the $V[\dot{G}_\alpha]$-generic filter in $\dot{Q}_\alpha$.

Now suppose that we've constructed $\langle \dot{Q}_\alpha \mid \alpha \in \gamma\rangle$ for some $\gamma \in \kappa$. By Lemma 32.1, there are P_γ-names $\langle \dot{Q}_{\gamma,\xi} \mid \xi \in \kappa\rangle$ such that if $\dot{Q}$ is a P_γ-name for a partial order on a subset of θ,

$$\{\xi \in \kappa \mid \mathbf{1} \Vdash_{P_\gamma} \dot{Q} = \dot{Q}_{\gamma,\xi}\}$$

is cofinal in κ. Let α and ξ be such that $\gamma = 2^\alpha \cdot (2\xi + 1)$. If $\mathbf{1} \Vdash_{P_\gamma}$ $\ulcorner \dot{Q}_{\alpha,\xi}$ is c.c.c.$\urcorner$, define $\dot{Q}_\gamma = \dot{Q}_{\alpha,\xi}$ if $\gamma = 2^\alpha \cdot (2\xi + 1)$. Otherwise define $\dot{Q}_\gamma$ to be the P_γ-name for the trivial partial order. This completes the recursive definition. The next lemma is the final ingredient to completing the argument.

Lemma 33.1. *Suppose that for each $\xi \in \theta$, X_ξ is a set of cardinality at most θ and $\dot{Y}_\xi$ is a nice P_κ-name such that $\mathbf{1} \Vdash \forall \xi \in \theta\ \dot{Y}_\xi \subseteq \dot{X}_\xi$. There is an $\alpha \in \kappa$ such that for all $\xi \in \theta$, $\dot{X}_\xi$ is a P_α-name.*

Proof. First observe that since $2^\theta = \kappa$, Theorem 32.3 implies the cofinality of κ is greater than θ. For each $\xi \in \theta$, set

$$S_\xi := \{p \in P_\kappa \mid \exists x \in X_\xi((\check{x}, p) \in \dot{Y}_\xi)\}.$$

Since $|X_\xi| \leqslant \theta$ and P_θ is c.c.c., $|S_\xi| \leqslant \theta$. Consequently $S := \bigcup\{S_\xi \mid \xi \in \theta\}$ has cardinality at most θ. Since $\theta < \mathrm{cof}(\kappa)$, there is an $\alpha \in \kappa$ such that S is contained in P_α (in the sense indicated above). It follows that $\dot{X}_\xi$ is a P_α-name for all $\xi \in \theta$. $\square$

Suppose now that $(\dot{Q}, \dot{\leqslant})$ and $\langle \dot{D}_\xi \mid \xi \in \theta\rangle$ are P_θ-names such that $p \in P_\theta$ forces $(\dot{Q}, \leqslant)$ is a c.c.c. poset of cardinality at most θ and for all $\xi \in \check{\theta}$, $\dot{D}_\xi \subseteq \dot{Q}$ is dense. Without loss of generality we name assume that $\dot{Q}$, $\dot{\leqslant}$ and each $\dot{D}_\xi$ are nice P_θ-names. Let $\alpha \in \kappa$ be such that $p \in P_\alpha$ and all of these names are P_α-names. Let $\xi \in \kappa$ be such that $p \Vdash_{P_\alpha} \dot{Q}_{\alpha,\xi} \cong \dot{Q}$. Since $p \Vdash_{P_\theta} \dot{Q}$ is c.c.c., $p \Vdash_{P_\alpha} \ulcorner \dot{Q}$ is c.c.c.$\urcorner$. It follows that p forces $\dot{H}_\alpha \subseteq \dot{Q}_\alpha$ is a filter which is $\{\dot{D}_\xi \mid \xi \in \theta\}$-generic.

34. The Levy Collapse

Suppose that X is a set of ordinals and λ is an infinite regular cardinal. Define $\mathrm{Coll}(\lambda, X)$ to be the collection of all partial functions p defined on $X \times \lambda$ such that the domain of p has cardinality less than λ and if $(\alpha, \xi) \in \mathrm{dom}(p)$, $p(\alpha, \xi) \in \max(\alpha, \omega)$. $\mathrm{Coll}(\lambda, X)$ is ordered by extension. For each infinite $\alpha \in X$, define a $\mathrm{Coll}(\lambda, X)$-name $\dot{e}_\alpha$ for a function on λ by $\dot{e}_\beta(\xi) = \alpha$ if and only if there is a $p \in \dot{G}$ such that $p(\beta, \xi) = \alpha$. Observe that it is forced that $\dot{e}_\alpha$ witnesses $|\check{\alpha}| \leqslant |\check{\lambda}|$. Also $\mathrm{Coll}(\lambda, X)$ is λ-closed and therefore does not add new sequenced of length less than λ.

The main cases of interest are when $X = \kappa$ for some strongly inaccessible cardinal and either $\lambda = \omega$ or $\lambda = \omega_1$. In this case $\mathrm{Coll}(\lambda, \kappa)$ is known as the *Levy collapse of κ to λ^+*. This terminology is standard but is perhaps misleading since, under the assumption that κ is inaccessible, κ remains a cardinal in the generic extension.

Lemma 34.1. *If κ is a regular cardinal,* $\mathrm{Coll}(\omega, \kappa)$ *is κ-c.c..*

Remark 34.2. More generally, if κ is regular and $\kappa^{<\lambda} = \kappa$, then $\mathrm{Coll}(\lambda, \kappa)$ is κ-c.c..

Proof. Suppose that $\langle p_\xi \mid \xi \in \kappa \rangle$ is a sequence of conditions in $\mathrm{Coll}(\omega, \kappa)$. By the Δ-system lemma there is a $\Xi \subseteq \kappa$ of cardinality κ such that $\{\mathrm{dom}(p_\xi) \mid \xi \in \Xi\}$ forms a Δ-system with root R. Let $\alpha \in \kappa$ be such that if $(\beta, n) \in \mathrm{dom}(p_\xi)$, then $\beta \in \alpha$. Since there are fewer than κ finite partial functions from $\alpha \times \omega$ to α, there are $\xi \neq \eta$ in Ξ such that $p_\xi \restriction R = p_\eta \restriction R$. But now $p_\xi \cup p_\eta$ is a lower bound for p_ξ and p_η. Thus $\mathrm{Coll}(\omega, \kappa)$ is κ-c.c.. $\square$

Next we will make some simple but useful observations about the Levy collapse.

Lemma 34.3. *If $X = Y \cup Z$ is a partition of a set of ordinals into two disjoint pieces,* $\mathrm{Coll}(\lambda, X) \cong \mathrm{Coll}(\lambda, Y) \times \mathrm{Coll}(\lambda, Z)$.

Proof. The function $p \mapsto (p \restriction \lambda \times Y, p \restriction \lambda \times Z)$ is an isomorphism. $\square$

A forcing Q is *atomless* if every element of Q has two incompatible extensions. Let $\mathcal{C}$ denote all finite sequences from ω of positive length, ordered by extension.

Proposition 34.4. *If Q is a countable atomless partial order, Q contains a dense subset isomorphic to $\mathcal{C}$.*

Proof. We first must show that if Q is atomless, then for every $p \in Q$, the set of extensions of p contains an infinite antichain. To see this, construct p_n and q_n by recursion so that $p_{n+1}, q_n \leqslant p_n$ are incompatible with $p_0 = p$. It follows that $\{q_n \mid n \in \omega\}$ is an infinite antichain of extensions of p. Now let $Q = \{q_n \mid n \in \omega\}$ and construct $\{p_s \mid s \in \mathcal{C}\}$ by recursion on the length of s. Select p_s for $s \in \mathcal{C}$ of length 1 so as to enumerate a maximal antichain in Q. Given p_s of length $n+1$, let $\{p_{s^\frown\langle i\rangle} \mid i \in \omega\}$ be an infinite maximal antichain of extensions of p_s such that for each i, $p_{s^\frown\langle i\rangle}$ either extends q_n or is incompatible with q_n. Notice that if q is compatible with p_s, then q is compatible with $p_{s^\frown\langle i\rangle}$ for some $i \in \omega$. In particular, for each n, $\{p_s \mid s \in \mathcal{C} \wedge |s| = n\}$ is a maximal antichain in Q. We've ensured that $s \mapsto p_s$ is an isomorphism which preserves incompatibility. To see that the range is dense, let $q_n \in Q$ be given. Since $\{p_s \mid s \in \mathcal{C} \wedge |s| = n+1\}$ is a maximal antichain, there is an s such that p_s is compatible with q_n. By construction, it must be that $p_s \leqslant q_n$. Hence $\{p_s \mid s \in \mathcal{C}\}$ is dense. □

Lemma 34.5. *Suppose that P is a forcing and $\dot{Q}_0$ and $\dot{Q}_1$ are P-names for forcings such that it is forced by* $\mathbf{1}$ *that $\dot{Q}_0$ and $\dot{Q}_1$ have isomorphic dense subsets. Then $P * \dot{Q}_0$ and $P * \dot{Q}_1$ have isomorphic dense subsets.*

Proof. Let $\dot{\phi}$ be a P-name for isomorphism between dense subsets of $\dot{Q}_0$ and $\dot{Q}_1$. If p forces $\dot{q} \in \operatorname{dom}(\dot{\phi})$, let $\dot{\phi}(\dot{q})$ denote a P-name for the value of $\dot{q}$ under $\dot{\phi}$. Let $D \subseteq P * \dot{Q}_0$ denote the set of all $(p, \dot{q})$ such that $p \Vdash \dot{q} \in \operatorname{dom}(\dot{\phi})$. Define $\psi : D \to P * \dot{Q}_1$ by $\psi((p, \dot{q})) = (p, \dot{\phi}(\dot{q}))$. It can be checked that D is dense, the range of D under ψ is dense, and ψ is an isomorphism. □

The next proposition gives a key property of the Levy collapse, known as the absorption property.

Proposition 34.6. *Suppose that κ is a cardinal and Q is a forcing with $|Q| < \kappa$.* $\operatorname{Coll}(\omega, \kappa)$ *and* $Q \times \operatorname{Coll}(\omega, \kappa)$ *contain isomorphic dense subsets.*

Proof. By Lemma 34.3

$$\operatorname{Coll}(\omega, \kappa) \cong \operatorname{Coll}(\omega, \{0\}) \times \operatorname{Coll}(\omega, \kappa \backslash \{0\}) \cong \operatorname{Coll}(\omega, \{0\}) \times \operatorname{Coll}(\omega, \kappa).$$

The partial order $\operatorname{Coll}(\omega, \{0\})$ is just the collection of all finite partial functions from ω to ω. In particular, it is countable and atomless. Observe that

$$Q \times \operatorname{Coll}(\omega, \kappa) \cong (Q \times \operatorname{Coll}(\omega, \{0\}) \times \operatorname{Coll}(\omega, \kappa)$$

is isomorphic to a dense suborder of $\operatorname{Coll}(\omega, \kappa) * (\check{Q} \times \operatorname{Coll}(\omega, \{0\}))$. Furthermore, $\operatorname{Coll}(\omega, \kappa)$ forces $\check{Q} \times \operatorname{Coll}(\omega, \{0\})$ is countable and atomless. The proposition now follows from Lemmas 34.4 and 34.5. □

35. Universally Baire sets

Suppose that B is a Borel subset of $\mathbb{R}$ and Q is a forcing. How to we interpret B in a generic extension by Q? One option is $\check{B}$ but if Q adds new reals, this set will typically not be Borel. In this section, we will give a very general procedure for resolving this issue.

Suppose Q is a forcing, A and B are nonempty sets, and A is countable. Observe that if $\dot{x}$ is a nice Q-name for an element of $\check{B}^{\check{A}}$, then there is a countable collection $\mathscr{D}$ of dense subsets of Q such that if $G \subseteq Q$ is a $\mathscr{D}$-generic filter, then $\dot{x}(G)$ is in B^A.

A subset X of B^A is *Q-universally Baire* if there is a Q-name $\dot{X}$ such that whenever $\dot{y}$ is a nice Q-name for an element of $\check{B}^{\check{A}}$, there is a countable collection $\mathscr{D}$ of dense subsets of Q such that if $G \subseteq Q$ is a $\mathscr{D}$-generic filter then the following are equivalent:

- $\dot{y}(G) \in X$;
- there is a $p \in G$ such that $p \Vdash \dot{y} \in \dot{X}$;
- there is no $p \in G$ such that $p \Vdash \dot{y} \notin \dot{X}$;

We will say that $\dot{X}$ witnesses that X is Q-universally Baire and that *$\mathscr{D}$ certifies $\dot{X}$* for $\dot{y}$.

Proposition 35.1. *If $\dot{X}$ and $\dot{Y}$ are Q-names which witness that $X \subseteq B^A$ is universally Baire, then $\mathbf{1} \Vdash_Q \dot{X} = \dot{Y}$.*

Proof. If the proposition is false, then there is a $p \in Q$ such that $p \Vdash \dot{X} \neq \dot{Y}$ and let $\dot{z}$ be forced by p to be in $(\dot{X} \backslash \dot{Y}) \cup (\dot{Y} \backslash \dot{X})$. By extending p if necessary, we may assume that p decides $\dot{z} \in \dot{X}$ and by exchanging the roles of $\dot{X}$ and $\dot{Y}$ if necessary, we may assume $p \Vdash \dot{z} \in \dot{X} \backslash \dot{Y}$. Let $\mathscr{D}$ be a countable collection of dense subsets of Q which which certifies both $\dot{X}$ and $\dot{Y}$ for $\dot{z}$. Now let $G \subseteq Q$ be a $\mathscr{D}$-generic filter with $p \in G$. Since $p \Vdash \dot{z} \in \dot{X}$, $\dot{z}(G) \in X$ but since $p \Vdash \dot{z} \notin \dot{Y}$, $\dot{z}(G) \notin X$, a contradiction. $\square$

Armed with this proposition, we will fix, for each Q-universally Baire set $X \subseteq B^A$, a Q-name $\dot{X}$ which witnesses that X is universally Baire.

Theorem 35.2. *Suppose that Q is a forcing, A and B are nonempty sets and A is countable. The Q-universally Baire subsets of B^A form a σ-algebra which contains the Borel subsets of B^A, where B is equipped with the discrete topology. Moreover:*

1. *if $U \subseteq B^A$ is open and $\dot{U}$ witnesses U is universally Baire, $\mathbf{1} \Vdash \ulcorner \dot{U}$ is open$\urcorner$.*
2. *if $X \subseteq B^A$ is universally Baire and $Y = B^A \backslash X$, then $\mathbf{1} \Vdash \dot{Y} = \check{B}^{\check{A}} \backslash \dot{X}$.*

(3) *if* $Y = \bigcup\{X_n \mid n \in \omega\}$ *and* $\dot{X}_n$ *witnesses* X_n *is* Q*-universally Baire, then* $\mathbf{1} \Vdash \dot{Y} = \bigcup\{\dot{X}_n \mid n \in \omega\}$.

Proof. If $U \subseteq B^A$ is open, define $\dot{U}$ to be all pairs $(\dot{x}, p)$ such that:

- $\dot{x}$ is a nice Q-name for an element of $\check{B}^{\check{A}} \subseteq \mathscr{P}(\check{A} \times \check{B})$
- for some finite $F \subseteq A$, p decides $\dot{x} \restriction F$ to be some $s : F \to B$ and for every $y \in B^A$ which extends s, $y \in U$.

It should be clear that $\dot{U}$ is forced to be open. Now suppose that $\dot{y}$ is a nice Q-name for an element of $\check{B}^{\check{A}}$. For each finite $F \subseteq A$, let D_F denote the set of all q in Q such that for some $p \in Q$ and $s : F \to B$, we have that $q \leqslant p$ and $((a, s(a)), p) \in \dot{y}$ for every $a \in F$. Observe that if $G \subseteq Q$ is a filter which meets D_F, then $\dot{y}(G)$ is a function whose domain contains F and if $s := \dot{y}(G) \restriction F$, then there is a $q \in G$ such that $q \Vdash \check{s} = \dot{y} \restriction \check{F}$.

If $\dot{X}_n$ witnesses that X_n is universally Baire for all $n \in \omega$ and $\mathscr{D}_n$ certifies $\dot{X}_n$ for some nice Q-name $\dot{y}$, then the Q-name $\dot{Y}$ for $\bigcup\{\dot{X}_n \mid n \in \omega\}$ witness that $Y := \bigcup\{X_n \mid n \in \omega\}$ is universally Baire and $\bigcup\{\mathscr{D}_n \mid n \in \omega\}$ certifies $\dot{Y}$ for $\dot{y}$. Similarly, if $\dot{X}$ witnesses that X is universally Baire and $\mathscr{D}$ certifies $\dot{X}$ for $\dot{y}$, then the name for $\check{B}^{\check{A}} \backslash \dot{X}$ witnesses that $B^A \backslash X$ is universally Baire and $\mathscr{D}$ certifies this name for $\dot{y}$. The remainder of the theorem follows from Proposition 35.1. □

We are now in a position to make the following definitions. Suppose that $\dot{r}$ is a Q-name for an element of $\mathbb{R}$. We say that p forces $\dot{r}$ is a *Cohen real* if for all meager Borel sets $B \subseteq \mathbb{R}$, $p \Vdash \dot{r} \notin \dot{B}$. Similarly, p forces $\dot{r}$ is a *random real* if whenever B is a measure 0 Borel subset of $\mathbb{R}$, $\mathbf{1} \Vdash \dot{r} \notin \dot{B}$. Let $\mathcal{R}$ denote the collection of all compact set subsets of $\mathbb{R}$ of positive measure. The next proposition is left as an exercise.

Proposition 35.3. *The following are true for a forcing* Q *and a* Q*-name for a real number* $\dot{r}$*:*

- *p forces $\dot{r}$ is a Cohen real if and only if p forces $\{\dot{r} \restriction n \mid n \in \omega\} \subseteq \omega^{<\omega}$ is a $\check{\mathbf{V}}$-generic filter.*
- *p forces $\dot{r}$ is a Random real if and only if p forces $\{B \in \mathcal{R} \mid \dot{r} \in \dot{B}\}$ is a $\check{\mathbf{V}}$-generic filter.*

36. Homogeneity and a proof of Solovay's theorem

Are nearly ready to prove the following theorem of Solovay.

Theorem 36.1 (Solovay)**.** *Suppose that κ is an inaccessible cardinal and $\phi(u,v)$ is a formula in the language of set theory. If $\dot{a}$ is a $\mathrm{Coll}(\omega,\kappa)$-name for an element of $\mathbf{ON}^\omega$ and $\dot{X}$ is the $\mathrm{Coll}(\omega,\kappa)$-name for the set $\{x \in \mathbb{R} \mid \phi(x,\dot{a})\}$, then $\mathbf{1}$ forces that $\dot{X}$ is Lebesgue measurable, has the Baire property, and has the perfect set property.*

Recall that if $X \subseteq \mathbb{R}$:

- X is Lebesgue measurable if there is a Borel set B such that $X \triangle B$ is measure 0.
- X has the Baire property if there is a Borel set B such that $X \triangle B$ is meager.
- X has the perfect set property if either X is countable or else X contains a nonempty closed set with non isolated points.

It is possible to show that if $X \subseteq \mathbb{R}$ is definable from an ω-sequence of ordinals, then X is in $\mathbf{L}(\mathbb{R})$, the minimum model of ZF which contains all real numbers. In particular, Solovay showed that if ZFC is consistent with the existence of an inaccessible cardinal, then so is ZF together with the assertion that all subsets of $\mathbb{R}$ are Lebesgue measurable, have the Baire property, and have the perfect set property. It is known that the inaccessible cardinal is required in Solovay's theorem, at least for two of the conclusions.

Theorem 36.2 (Miller)**.** *If every $\mathbf{\Pi}^1_1$ set of reals has the perfect set property, then ω_1 is an inaccessible cardinal in $\mathbf{L}$.*

Theorem 36.3 (Shelah)**.** *If every $\mathbf{\Sigma}^1_3$ set is Lebesgue measurable, then ω_1 is an inaccessible cardinal in $\mathbf{L}$.*

Shelah has shown, however, that if ZFC is consistent then so is ZF together with "All subsets of $\mathbb{R}$ have the Baire property."

In order to prove Solovay's result, we need one more forcing technique. Suppose that $\Theta : P \cong Q$ is an isomorphism. This function induces a class function $\hat{\Theta}$ which maps P-names to Q-names:

$$\hat{\Theta}(\dot{x}) := \{(\hat{\Theta}(\dot{y}), \Theta(p)) \mid (\dot{y}, p) \in \dot{x}\}.$$

Observe that $\hat{\Theta}(\check{x}) = \check{x}$ (though technically the former $\check{x}$ is a P-name and the latter is a Q-name). The next proposition has a routine proof which is left as an exercise.

Proposition 36.4. *If $\Theta : P \cong Q$ is an isomorphism of posets and $\hat{\Theta}$ is the induced map on P-names, then $p \Vdash_P \phi(\dot{x}_i \mid i < n)$ if and only if*

$\Theta(p) \Vdash_Q \phi(\hat{\Theta}(\dot{x}_i) \mid i < n)$ whenever $\phi(v_i \mid i < n)$ is a formula in the language of set theory and $\langle \dot{x}_i \mid i < n\rangle$ is a sequence of P-names.

If Q is a poset and $p \in Q$, set $Q_p := \{q \in Q \mid q \leqslant p\}$. A poset Q is *weakly homogeneous* if for every $p, q \in Q$, there are $p' \leqslant p$ and $q' \leqslant q$ such that $Q_{p'} \cong Q_{q'}$. Proposition 36.4 now yields:

Proposition 36.5. *Suppose that Q is a weakly homogeneous poset, $\phi(v_i \mid i < n)$ is a formula in the language of set theory and $\langle x_i \mid i < n\rangle$ is a sequence of sets. Either $\mathbf{1} \Vdash \phi(\check{x}_i \mid i < n)$ or $\mathbf{1} \Vdash \neg\phi(\check{x}_i \mid i < n)$.*

Proof. If the conclusion of the proposition is false, there are $p, q \in Q$ such that $p \Vdash \phi(\check{x}_i \mid i < n)$ and $q \Vdash \neg\phi(\check{x}_i \mid i < n)$. By extending p and q if necessary, we may assume that $Q_p \cong Q_q$ are isomorphic by some Θ. Since $\hat{\Theta}(\check{x}_i) = \check{x}_i$, we have a contradiction to Proposition 36.4. $\square$

Note that $\mathrm{Coll}(\omega, \kappa)$ is weakly homogeneous: if $p, q \in \mathrm{Coll}(\omega, \kappa)$, we can find $p' \leqslant p$ and $q' \leqslant q$ such that $\mathrm{dom}(p') = \mathrm{dom}(q')$. If $r \leqslant p'$, define $\Theta(r) := (r \backslash p') \cup q'$.

We are now ready to give a proof of Solovay's theorem. We will first argue that it suffices to prove Solovay's theorem when $\dot{a} = \check{a}$ for some $a \in \mathbf{ON}^\omega$. To see this, let $\dot{a}$ be a $\mathrm{Coll}(\omega, \kappa)$-name for an element of $\mathbf{ON}^\omega$. Since $\dot{a}$ is forced to be countable and $\mathrm{Coll}(\omega, \kappa)$ is κ-c.c., there is a $\delta \in \kappa$ such that $\dot{a}$ is a $\mathrm{Coll}(\omega, \delta)$-name. But now

$$\mathrm{Coll}(\omega, \kappa) \cong \mathrm{Coll}(\omega, \delta) \times \mathrm{Coll}(\omega, \kappa \backslash \delta) \equiv \mathrm{Coll}(\omega, \delta) * \mathrm{Coll}(\omega, \check{\kappa})$$

and therefore we can derive the conclusion of Solovay's theorem for $\dot{a}$ by applying the ground model version of Solovay's theorem in the generic extension by $\mathrm{Coll}(\omega, \delta)$.

Lemma 36.6. *Suppose κ is inaccessible, $G \subseteq \mathrm{Coll}(\omega, \kappa)$ is V-generic, and $r \in \mathbb{R} \cap V[G]$. There is a filter $H \subseteq \mathrm{Coll}(\omega, \kappa)$ such that H is $V[r]$-generic and $V[G] = V[r][H]$.*

Proof. Let $\delta \in \kappa$ be such that r is in $V[G \cap \mathrm{Coll}(\omega, \delta)]$ and set $G' := G \cap \mathrm{Coll}(\omega, \delta)$ and $G'' := G \cap \mathrm{Coll}(\omega, \kappa \backslash \delta)$. By Theorem 30.2, there are posets $P \in V$, $Q \in V[r]$ and filters $G_0 \subseteq P$ and $H_0 \subseteq Q$ such that G_0 is V-generic, $V[G_0] = V[r]$, H_0 is $V[r]$-generic, and $V[G'] = V[G_0][H_0]$. Observe that by applying Lemma 34.5 in $V[r]$,

$$Q * \mathrm{Coll}(\omega, \kappa \backslash \delta) \equiv \mathrm{Coll}(\omega, \kappa \backslash \delta) * \check{Q} \equiv \mathrm{Coll}(\omega, \kappa \backslash \delta) * \mathrm{Coll}(\omega, \delta) \equiv \mathrm{Coll}(\omega, \kappa).$$

Therefore there is an $H \subseteq \mathrm{Coll}(\omega, \kappa)$ such that $V[r][G] = V[r][H_0][G'']$. $\square$

Suppose now that we wish to show that $\{x \in \dot{\mathbb{R}} \mid \phi(x, \check{a})\}$ is forced by $\mathbf{1} \in \mathrm{Coll}(\omega, \kappa)$ to be Lebesgue measurable. Let $\dot{E}$ denote the $\mathrm{Coll}(\omega, \kappa)$-name for the union of all Borel measure 0 sets in V. Since κ is inaccessible, the set of ground model measure 0 sets is countable and hence it is forced by $\mathbf{1}$ that $\dot{E}$ is measure 0. Notice that if $\dot{r}$ is any $\mathrm{Coll}(\omega, \kappa)$-name for an element of $\dot{R} \backslash \dot{E}$, then $\mathbf{1} \Vdash \ulcorner \dot{r}$ is a random real over $\check{\mathbf{V}}\urcorner$.

Now consider the truth value $B := [\![\mathbf{1} \Vdash_{\mathrm{Coll}(\omega,\kappa)} \phi(\check{\dot{r}}, \check{a})]\!]$ with respect to forcing with the complete Boolean algebra of the Borel subsets of $\mathbb{R}$ modulo the measure 0 sets. It suffices to show that $\mathbf{1}$ forces that for all $r \in \dot{\mathbb{R}} \backslash \dot{E}$, $r \in \dot{B}$ if and only if $\phi(r, \check{a})$.

Let $G \subseteq \mathrm{Coll}(\omega, \kappa)$ be a V-generic filter and let $r \in \mathbb{R} \cap V[G]$ with $r \notin E$. By Lemma 36.6, there is a $V[r]$-generic filter $H \subseteq \mathrm{Coll}(\omega, \kappa)$ such that $V[r][H] = V[G]$. Since r is a random real over V, $r \in B^{V[G]}$ if and only if $\mathbf{1} \Vdash_{\mathrm{Coll}(\omega,\kappa)} \phi(\check{r}, \check{a})$. By homogeneity of $\mathrm{Coll}(\omega, \kappa)$, this is equivalent to $\mathbf{1}$ does not force $\neg\phi(\check{r}, \check{a})$. Since H is a $V[r]$-generic filter and $V[G] = V[r][H]$, $r \in B^{V[G]}$ if and only if $V[r][H] \models \phi(r, a)$.

This argument adapts *mutatis mutandis* to show that

$$V[G] \models \ulcorner\{r \in \mathbb{R} \mid \phi(r, a)\} \text{ has the Baire property}\urcorner$$

One simply replaces the complete Boolean algebra of Borel sets modulo measure 0 sets with Borel sets modulo meager sets and replaces the notion of a random real with that of a Cohen real.

The proof of the perfect set property is somewhat different. For notational simplicity, we'll prove the perfect set property for subsets of 2^ω. As before, we may assume that $a \in \mathbf{ON}^\omega$ is in V. If

$$\mathbf{1} \Vdash_{\mathrm{Coll}(\omega,\kappa)} \{r \in 2^\omega \mid \phi(r, \check{a})\} \subseteq \check{2^\omega}$$

then we are done since $\mathbf{1} \Vdash_{\mathrm{Coll}(\omega,\kappa)} |\check{2^\omega}| = \aleph_0$. If not, let $\dot{r}$ be a $\mathrm{Coll}(\omega, \kappa)$-name and $p \in \mathrm{Coll}(\omega, \kappa)$ be such that $p \Vdash \dot{r} \notin \check{2^\omega} \wedge \phi(r, \check{a})$. Let $\delta \in \kappa$ be such that $\dot{r}$ is a $\mathrm{Coll}(\omega, \delta)$-name and $p \in r$. In $V[G]$, let $\langle D_n \mid n \in \omega\rangle$ be an enumeration of the dense subsets of $\mathrm{Coll}(\omega, \delta)$ in V. Construct $\langle p_t \mid t \in 2^{<\omega}\rangle$ and $\langle s_t \mid t \in 2^\omega\rangle$ such that:

- if $t \in 2^n$, $p_t \in D_n$ and p_t forces $\check{s}_t \subseteq \dot{r}$,
- if $u \subseteq v$ are in $2^{<\omega}$, then $p_v \leqslant p_u \leqslant p$,
- if $u \perp v$ are in $2^{<\omega}$ then s_u and s_v are incompatible.

Notice that it must be that $|s_t| \geqslant n$ if t is in 2^n. For each $x \in 2^\omega$, let $G_x \subseteq \mathrm{Coll}(\omega, \delta)$ be the filter generated by $\{p_{x \restriction n} \mid n \in \omega\}$. Each G_x is V-generic and hence $\dot{r}(G_x) = \bigcup\{s_{x\restriction n} \mid n \in \omega\}$ satisfies that $\phi(\dot{r}(G_x), a)$ is true. Notice that $x \mapsto \dot{r}(G_x)$ is continuous and injective and its range is contained in $\{r \in 2^\omega \mid \phi(r, a)\}$.

37. Supercompact cardinals

We will now turn to a result of Shelah and Woodin which explains the special foundational role Solovay's model $L(\mathbb{R})$ plays.

Theorem 37.1 (Shelah-Woodin)**.** *Suppose that there is a supercompact cardinal. The theory of $L(\mathbb{R})$ can not be changed by forcing. In particular, every set of reals in $L(\mathbb{R})$ is Lebesgue measurable, has the Baire Property, and has the Perfect Set Property.*

First we will definite the notion of a *supercompact cardinal* and prove some basic facts about them. Suppose that $\mathbf{M}$ and $\mathbf{N}$ are transitive classes. An *elementary embedding* $\mathbf{j} : \mathbf{M} \to \mathbf{N}$ is a class function such that for any formula $\phi(\bar{v})$ in the language of set theory and any tuple of sets $\bar{x}$, $\mathbf{M} \models \phi(\bar{x})$ if and only if $\mathbf{N} \models \phi(\mathbf{j}(\bar{x}))$. If X is any set, then $\mathbf{j}''X := \{\mathbf{j}(x) \mid x \in X\} \subseteq \mathbf{j}(X)$. In particular, $\mathbf{j}(\alpha) \geqslant \alpha$ holds for all ordinals α. Also observe that the restriction of an elementary embedding to the ordinals is an order preserving function. We note the following fact.

Proposition 37.2. *Suppose $\mathbf{M} \subseteq \mathbf{V}$ is a transitive class and $\mathbf{j} : \mathbf{V} \to \mathbf{M}$ is an elementary embedding which is not the identity. If x is a set of minimum rank κ such that $\mathbf{j}(x) \neq x$, then $\mathbf{j}(\kappa) > \kappa$. In particular $\mathbf{j} \restriction \mathbf{ON}$ is not the identity.*

The least ordinal moved by an elementary embedding $\mathbf{j}$ is called the *critical point* of $\mathbf{j}$ and is denoted $\mathrm{crit}(\mathbf{j})$. The next lemma gives a useful characterization of when $\mathbf{j}''X = \mathbf{j}(X)$.

Lemma 37.3. *Suppose that $\mathbf{j} : \mathbf{M} \to \mathbf{N}$ is an elementary embedding with critical point κ. If $X \in \mathbf{M}$, then $\mathbf{M} \models |X| < \kappa$ if and only if $\mathbf{j}''X = \mathbf{j}(X)$.*

Proof. Set $\theta := |X|$ and let $f : \theta \to X$ be a bijection. By elementarity, $\mathbf{j}(f)$ is a bijection from $\mathbf{j}(\theta)$ to $\mathbf{j}(X)$. If $\theta < \kappa$, then $\mathbf{j}(f)$ has domain θ and if $f(\xi) = x$, $\mathbf{j}(f)(\xi) = \mathbf{j}(x)$ and hence the range of $\mathbf{j}(f)$ is $\{\mathbf{j}(x) \mid x \in X\}$. If $\theta \geqslant \kappa$, then κ is not in the range of $\mathbf{j}$. It follows that $\mathbf{j}(f)(\kappa) \in \mathbf{j}(X)$ is not of the form $\mathbf{j}(x)$ for any $x \in X$. To see this, suppose $x \in X$ and let $\xi \in \theta$ be such that $f(\xi) = x$. Then $\mathbf{j}(f)(\mathbf{j}(\xi)) = \mathbf{j}(x)$ and since $\mathbf{j}(f)$ is one-to-one and κ is not in the range of $\mathbf{j}$, $\mathbf{j}(x) = \mathbf{j}(f)(\mathbf{j}(\xi)) \neq \mathbf{j}(f)(\kappa)$. □

If κ is the critical point of an elementary embedding from $\mathbf{V}$ to $\mathbf{M} \subseteq \mathbf{V}$, then we say that κ is *measurable.* If κ is the critical point of an elementary embedding $\mathbf{j} : \mathbf{V} \to \mathbf{M} \subseteq \mathbf{V}$ such that additionally $\mathbf{j}(\kappa) > \lambda$ and $\mathbf{M}^\lambda \subseteq \mathbf{M}$, then we say that κ is *λ-supercompact.* If κ

is *supercompact* if it is λ-supercompact for every ordinal λ. The next proposition gives the first hint at the scale of these cardinals.

Proposition 37.4. *Suppose that κ is the critical point of an elementary embedding $\mathbf{j} : \mathbf{V} \to \mathbf{M} \subseteq \mathbf{V}$. κ is a strongly inaccessible cardinal and moreover $\{\delta \in \kappa \mid \delta$ is strongly inaccessible$\}$ is stationary in κ.*

Proof. Since 0 and ω are the least ordinal and the least limit ordinal in both $\mathbf{V}$ and $\mathbf{M}$, it follows that $0 \neq \kappa$ and $\omega \neq \kappa$. Now suppose that $f : \alpha \to \kappa$ for some $\alpha \in \kappa$. By Lemma 37.3, $\mathbf{j}''f = \mathbf{j}(f)$ and since $f \subseteq \alpha \times \kappa$, $\mathbf{j}''f = f$. In particular, range$(\mathbf{j}(f)) \subseteq \kappa$ and therefore

$$\mathbf{M} \models \exists \beta \in \mathbf{j}(\kappa)(\text{range}(\mathbf{j}(f)) \subseteq \beta).$$

By elementarity of $\mathbf{j}$, there is a $\beta' \in \kappa$ such that range$(f) \subseteq \beta'$. Since f was arbitrary, this implies that κ is a regular cardinal. Next we will show that κ is a strong limit cardinal. Suppose that $\alpha \in \kappa$ and observe that if $A \subseteq \alpha$, Lemma 37.3 implies $\mathbf{j}(A) = A$. By Lemma 37.3, it suffices to show that $\mathbf{j}(\mathscr{P}(\alpha)) = \mathscr{P}(\alpha)$. This follows from $\alpha < \text{crit}(\mathbf{j})$ and the fact that for any set S, $\mathbf{j}(\mathscr{P}(S)) = \mathscr{P}(\mathbf{j}(S))$. ☐

Notice that, *a priori*, the assertion that κ is λ-supercompact is not a formula in the language of set theory. There is, however, an equivalent definition which is purely set theoretic. An ultrafilter $\mathscr{U}$ on $[\lambda]^{<\kappa}$ is *normal* if whenever $U_0 \in \mathscr{U}$ and $r : U_0 \to \lambda$ satisfies $r(M) \in M$ for all $M \in U_0$, there is a set in $\mathscr{U}$ on which r is constant. An ultrafilter $\mathscr{U}$ on $[\lambda]^{<\kappa}$ is *fine* if for every $\xi \in \lambda$, $\{M \in [\lambda]^{<\kappa} \mid \xi \in M\}$ is in $\mathscr{U}$. Note that normal ultrafilters on $[\lambda]^{<\kappa}$ are closed under intersections of cardinality less than κ.

Suppose now that $\mathscr{U}$ is an ultrafilter on a set I. The ultrapower of $\mathbf{V}$ by $\mathscr{U}$ is defined as follows. If $f, g \in \mathbf{V}^I$, define

$$f =_{\mathscr{U}} g \text{ if and only if } \{i \in I \mid f(i) = g(i)\} \in \mathscr{U}$$

$$f \in_{\mathscr{U}} g \text{ if and only if } \{i \in I \mid f(i) \in g(i)\} \in \mathscr{U}.$$

If $f \in \mathbf{V}^I$, define $f/\mathscr{U}$ to be set of all elements of the $=_{\mathscr{U}}$ equivalence class of f which are of minimal rank. Define $\mathbf{V}^I/\mathscr{U}$ to be the class of all $f/\mathscr{U}$ such that $f \in \mathbf{V}^I$. If x is any set, define f_x to be the function with domain I which is constantly x. By Łós's Theorem for ultraproducts, the map $x \mapsto f_x/\mathscr{U}$ defines an elementary embedding of $\mathbf{V}$ into $\mathbf{V}^I/\mathscr{U}$.

If $\mathscr{U}$ is countably complete, then $\in_{\mathscr{U}}$ is well founded. Let $\pi_{\mathscr{U}} : (\mathbf{V}^I, \in_{\mathscr{U}}) \to (\mathbf{M}, \in)$ be the transitive collapse. It follows that $\mathbf{j}_{\mathscr{U}}(x) = \pi_{\mathscr{U}}(f_x)$ defined an elementary embedding $\mathbf{j}_{\mathscr{U}} : \mathbf{V} \to \mathbf{M} \subseteq \mathbf{V}$. This embedding is the identity if and only if $\mathscr{U}$ is a principle ultrafilter.

Proposition 37.5. *If κ and λ are ordinals, then κ is λ-supercompact if and only if there is a fine normal ultrafilter on $[\lambda]^{<\kappa}$.*

Proof. If $\mathbf{j} : \mathbf{V} \to \mathbf{M} \subseteq \mathbf{V}$ is an elementary embedding witnessing that κ is λ-supercompact, then define $\mathscr{U} := \{U \subseteq [\lambda]^{<\kappa} \mid \lambda \in \mathbf{j}(U)\}$. We must show that $\mathscr{U}$ is fine and normal. To see that $\mathscr{U}$ is fine, let $\xi \in \lambda$ be given.

To see that $\mathscr{U}$ is normal, suppose that $U \in \mathscr{U}$ and $r : U \to \lambda$ satisfies $r(a) \in a$ for all $a \in U$.

If $\mathscr{U}$ is a fine normal ultrafilter on $[\lambda]^{<\kappa}$, then it can be checked that the ultrapower embedding $\mathbf{j}_{\mathscr{U}}$ witnesses that κ is λ-supercompact. $\square$

38. Stationary reflection

Suppose that $S \subseteq [\theta]^\omega$. Is there a "small" subset X of θ such that $S \cap [X]^\omega$ is stationary? If $S \cap [X]^\omega$ is stationary, it is common to say that *(the stationarity of) S reflects to X*.

Theorem 38.1. *Suppose that κ is a supercompact cardinal and θ is an arbitrary cardinal. Every stationary subset of $[\theta]^\omega$ reflects to a set X of cardinality less than κ which contains ω_1.*

Proof. Suppose that $S \subseteq [\theta]^\omega$ is stationary and set $\lambda := |[\theta]^\omega|$. Let $\mathbf{j} : \mathbf{V} \to \mathbf{M} \subseteq \mathbf{V}$ witness that κ is λ-supercompact. By Lemma 37.3, if $a \in [\theta]^\omega$, $\mathbf{j}(a) = \mathbf{j}''a$. Define X to be the image of θ under $\mathbf{j}$ and

$$S' := \mathbf{j}''S = \{\mathbf{j}(a) \mid a \in S\} = \{\mathbf{j}''a \mid a \in S\}.$$

Notice that since $\mathbf{j}\restriction\theta$ is a bijection between θ and X, S' is stationary in $[X]^\omega$ and $\omega_1 \subseteq X$. Furthermore, since $|X| = \theta \leqslant \lambda$, $X \in \mathbf{M}$. Consequently $S' \subseteq \mathbf{j}(S) \cap [X]^\omega$ and hence $\mathbf{j}(S) \cap [X]^\omega$ is stationary. Since $\lambda < \mathbf{j}(\kappa)$, by elementarity, there is $Y \subseteq \theta$ such that $[Y]^\omega \cap S$ is stationary, $\omega_1 \subseteq Y$ and $|Y| < \kappa$. $\square$

It turns out that if a supercompact cardinal κ is collapsed to ω_2, this reflection principle persists. For an uncountable cardinal θ, RP_θ is the assertion that stationary subsets of $[\theta]^\omega$ reflect to sets of size $\aleph_1$ which contain ω_1.

Theorem 38.2. *If κ is a supercompact cardinal, then*

$$\mathbf{1} \Vdash_{\mathrm{Coll}(\omega_1,\kappa)} \ulcorner \mathrm{RP}_\theta \textit{ holds for all } \theta > \omega_1 \urcorner.$$

The proof of this theorem is similar to the proof of Theorem 38.1 but it requires some additional lemmas.

Lemma 38.3. *Suppose that $\mathbf{j} : \mathbf{V} \to \mathbf{M} \subseteq \mathbf{V}$ is an elementary embedding with critical point κ and Q is a forcing in $\mathbf{V}$ which is κ-c.c.. If $H \subseteq \mathbf{j}(Q)$ is an $\mathbf{M}$-generic filter and $G := \mathbf{j}^{-1}(H)$, then G is $\mathbf{V}$-generic and $\mathbf{j}$ extends to an elementary embedding from $\mathbf{V}[G]$ to $\mathbf{M}[H]$.*

Proof. First we will show that $G \subseteq Q$ is a $\mathbf{V}$-generic filter. It suffices to show that if $A \subseteq Q$ is a maximal antichain in $\mathbf{V}$, then $G \cap A \neq \emptyset$. Since Q is κ-c.c., $|A| < \kappa$ and hence by Lemma 37.3, $\mathbf{j}(A)$ is the image of A under $\mathbf{j}$. By elementarity, $\mathbf{M} \models \ulcorner\mathbf{j}(A)$ is a maximal antichain$\urcorner$ and hence there is a $q \in H \cap \mathbf{j}(A)$. Let $p \in Q$ be such that $\mathbf{j}(p) = q$, noting that $p \in G \cap A$.

Extend $\mathbf{j}$ so that $\mathbf{j}(\dot{x}(G)) := \mathbf{j}(\dot{x})(H)$. This does not depend on the choice of $\dot{x}$: if $\dot{x}(G) = \dot{y}(G)$, then there is a $p \in G$ such that $p \Vdash_Q \dot{x} = \dot{y}$. By elementarity $\mathbf{j}(p) \Vdash_{\mathbf{j}(Q)} \mathbf{j}(\dot{x}) = \mathbf{j}(\dot{y})$. Since $\mathbf{j}(p) \in$

H, $\mathbf{j}(\dot{x})(H) = \mathbf{j}(\dot{y})(H)$. By the same reasoning, the extension is an elementary embedding. $\square$

Lemma 38.4. *If $S \subseteq [\theta]^\omega$ is stationary and Q is σ-closed, $\mathbf{1} \Vdash_Q \check{S} \subseteq [\check{\theta}]^\omega$ is stationary.*

Proof. Let λ be a sufficiently large regular cardinal that S and Q are in H_λ. Let $p \in Q$ be any condition and $\dot{f}$ be such that $p \Vdash \dot{f} : \check{\theta}^{<\omega} \to \check{\theta}$. We need to find a $q \leqslant p$ and an $a \in S$ such that $q \Vdash_Q \ulcorner \check{a}$ is $\dot{f}$-closed$\urcorner$. Without loss of generality, we may assume that $\dot{f}$ is a nice name and hence in H_λ. By Lemma 10.9, there a countable elementary submodel M of H_λ such that p, $\dot{f}$ and Q are in M and $a := M \cap \theta$ is in S. Let $\langle D_n \mid n \in \omega \rangle$ list the dense subsets of Q which are in M and recursively construct a decreasing sequence $\langle p_n \mid n \in \omega \rangle$ such that $p_0 := p$ and $p_{n+1} \leqslant p_n$ is in $D_n \cap M$. This is possible by elementarity of M. Notice that for any $\bar{\xi} \in \theta^{<\omega} \cap M$, the set of conditions which decide $\dot{f}(\bar{\xi})$ is a dense set. Since it is definable from parameters in M, this dense set is in M. Thus for each $\bar{\xi} \in \theta^{<\omega} \cap M$, there is an n such that p_n decides $\dot{f}(\bar{\xi})$ to be some η. Since η is definable from parameters in M, it is in M as well. Let q be a lower bound for $\{p_n \mid n \in \omega\}$ and observe that q forces $\check{a} \in \check{S}$ is $\dot{f}$ closed. $\square$

Proof of Theorem 38.2. We'll give an informal semantic proof for ease of reading. Let $G \subseteq \mathrm{Coll}(\omega_1, \kappa)$ be $\mathbf{V}$-generic and suppose that $\theta > \omega_1$ is a cardinal and $S \subseteq [\theta]^\omega$ is stationary in $\mathbf{V}[G]$. Set $\lambda := |[\theta]^\omega|$ and fix an elementary embedding $\mathbf{j} : \mathbf{V} \to \mathbf{M} \subseteq \mathbf{V}$ which witnesses that κ is λ-supercompact. Define $X := \{\mathbf{j}(\xi) \mid \xi \in \theta\}$ and observe that since $\mathbf{M}^\lambda \subseteq \mathbf{M}$, X is in M. Since $\mathrm{Coll}(\omega_1, \mathbf{j}(\kappa)) \cong \mathrm{Coll}(\omega_1, \kappa) \times \mathrm{Coll}(\omega_1, \mathbf{j}(\kappa) \backslash \kappa)$, it is possible to find a $\mathbf{V}$-generic $H \subseteq \mathrm{Coll}(\omega_1, \mathbf{j}(\kappa))$ such that $G \subseteq H$. By Lemma 38.3, $\mathbf{j}$ extends to an elementary embedding $\mathbf{j} : \mathbf{V}[G] \to \mathbf{M}[H]$. Since $\mathrm{Coll}(\omega_1, \mathbf{j}(\kappa) \backslash \kappa)$ is σ-closed and $\mathbf{V}[H]$ is a generic extension of $\mathbf{V}[G]$ by $\mathrm{Coll}(\omega_1, \mathbf{j}(\kappa) \backslash \kappa)$, Lemma 38.4 implies S is still stationary in $\mathbf{V}[H]$. Since $\mathbf{M}[H] \subseteq \mathbf{V}[H]$, $\mathbf{M}[H]$ satisfies S is stationary. As in Theorem 38.1, $\mathbf{j}''S \subseteq [X]^\omega$ is stationary and contained in $\mathbf{j}(S) \cap [X]^\omega$. Observe that $|X| \leqslant \lambda < \mathbf{j}(\kappa)$ and hence

$$\mathbf{M}[H] \models |\lambda| = \omega_1 < |\mathbf{j}(\kappa)| = \omega_2.$$

Thus $\mathbf{M}[H]$ satisfies that there is an $X \subseteq \mathbf{j}(\theta)$ of cardinality $\aleph_1$ such that $\omega_1 \subseteq X$ and $\mathbf{j}(S) \cap [X]^\omega$ is stationary. By elementarity of the extended embedding, there is a $Y \subseteq \theta$ of cardinality $\aleph_1$ such that $\omega_1 \subseteq Y$ and $S \cap [Y]^\omega$ is stationary. $\square$

39. Analysis of $\mathrm{NS}^+_{\omega_1}$ using the Reflection Principle

The hypotheses RP_θ have many uses in set theory and it applications. We will focus on its effect on the forcing $\mathrm{NS}^+_{\omega_1}$ — the collection of all stationary subsets of ω_1 ordered by containment. This forcing will play an integral role in our proof of the $\mathbf{L}(\mathbb{R})$ Absoluteness Theorem.

It will be helpful to develop some terminology. For the time being, set $\theta := 2^{\aleph_1}$. If M and N are countable sets, we will say that N is an *ω_1-end extension* of M if $M \subseteq N$ and $M \cap \omega_1 = N \cap \omega_1$. If $\mathscr{A} \subseteq \mathrm{NS}^+_{\omega_1}$ is a maximal antichain, we say M *captures* $\mathscr{A}$ if:

- $M \cap H_{\theta^+} \prec H_{\theta^+}$ is countable;
- $\mathscr{A} \in M$ and there is an $A \in M \cap \mathscr{A}$ such that $M \cap \omega_1 \in A$.

M is *good* if M captures all maximal antichains in $\mathrm{NS}^+_{\omega_1}$ which are an element of M.

Lemma 39.1. *Assume* RP_{2^θ}*. There is a club of countable $M \prec H_{\theta^+}$ such that for every maximal antichain $\mathscr{A} \subseteq \mathrm{NS}^+_{\omega_1}$ in M there is an ω_1-end extension N of M capturing $\mathscr{A}$.*

Proof. Suppose not and let S consist of all countable $M \prec H_{\theta^+}$ such that for some $\mathscr{A}_M \in M$, there is no ω_1-end extension N of M capturing $\mathscr{A}_M$. By assumption S is stationary and by the Pressing Down Lemma, there is a stationary $S_0 \subseteq S$ such that $M \mapsto \mathscr{A}_M$ is constantly $\mathscr{A}$ on S_0 for some $\mathscr{A}$. By RP_{2^θ} there is an $X \subseteq H_{\theta^+}$ such that $\omega_1 \subseteq X$, $|X| = \aleph_1$, and $S_0 \cap [X]^\omega$ is stationary. Let $\langle M_\xi \mid \xi \in \omega_1\rangle$ be a continuous $\subseteq$-chain in $[X]^\omega$ such that $M_\xi \cap \omega_1 = \xi$ for all $\xi \in \omega_1$. It follows that $\Xi := \{\xi \in \omega_1 \mid M_\xi \in S_0\}$ is stationary. Let $A \in \mathscr{A}$ be such that $\Xi \cap A$ is stationary. Let $N \prec H_{\theta^+}$ be such that $\langle M_\xi \mid \xi \in \omega_1\rangle \in N$, $A \in N$ and $\delta := N \cap \omega_1 \in A \cap \Xi$. By elementarity of N, $M_\xi \subseteq N$ whenever $\xi \in \delta$ and by continuity, $M_\delta \subseteq N$. Thus N is an ω_1-end extension of M and N captures $\mathscr{A}$ via A, contradicting that $M \in S_0$. $\square$

Lemma 39.2. *Assume* RP_{2^θ}*. If M is a countable elementary submodel of $H_{(2^\theta)^+}$ and $\mathscr{A} \in M$ is a maximal antichain in $\mathrm{NS}^+_{\omega_1}$, then M has an ω_1-end extension $N \prec H_{(2^\theta)^+}$ which captures $\mathscr{A}$.*

Proof. Let $M \prec H_{(2^\theta)^+}$ and $\mathscr{A} \in M$ be a maximal antichain in $\mathrm{NS}^+_{\omega_1}$. By elementarity, there is a club $E \subseteq [H_{\theta^+}]^\omega$ in M as stipulated in Lemma 39.1. Observe that $M_0 := M \cap H_{\theta^+}$ is in E and therefore there is an ω_1-end extension $N_0 \prec H_{\theta^+}$ of M_0 which captures $\mathscr{A}$. Let $A \in \mathscr{A} \cap N_0$ be such that $N_0 \cap \omega_1 \in A$. Define N to be the set of all $f(A)$ such that $f \in M$ is a function defined on $\mathscr{A}$. We will show that $N \prec H_{(2^\theta)^+}$ is an ω_1-end extension of M.

First notice that $x \in M$ then the function which takes the constant value x on $\mathscr{A}$ is in M and hence $M \subseteq N$. Also, $A \in N$ since the identity function on $\mathscr{A}$ is in M. To see that $N \prec H_{(2^\theta)^+}$, suppose that $\langle f_i \mid i < n\rangle$ are functions defined on $\mathscr{A}$ which are in M and for some ϕ

$$H_{(2^\theta)^+} \models \exists y \ \phi(f_0(A), \dots, f_{n-1}(A), y).$$

Define $\mathscr{B} := \{B \in \mathscr{A} \mid H_{(2^\theta)^+} \models \exists y \ \phi(f_0(B), \dots, f_{n-1}(B), y)\}$ noting that $\mathscr{B}$ is in M by elementarity and $A \in \mathscr{B}$. Let $g \in M$ be a function defined on $\mathscr{A}$ such that $H_{(2^\theta)^+} \models \phi(f_0(B), \dots, f_{n-1}(B), g(B))$ whenever $B \in \mathscr{B}$. We have $g(B) \in N$ and $H_{(2^\theta)^+} \models \exists y \ \phi(f_0(A), \dots, f_{n-1}(A), y)$, verifying the Tarski-Vaught criterion for elementarity of N.

To see $N \cap \omega_1 = M \cap \omega_1$, let $\alpha \in N \cap \omega_1$ and fix a function $f \in M$ with $f(A) = \alpha$. Define $g : \mathscr{A} \to \omega_1$ by $g(B) = f(B)$ if $f(B) \in \omega_1$ and $g(B) = 0$ otherwise. Then $g(A) = f(A)$ and $g \in M \cap H_{\theta^+} = M_0$. Since g and A are in N_0, $\alpha = g(A) \in N_0 \cap \omega_1 = M \cap \omega_1$.

$\square$

Lemma 39.3. *Assume* RP_{2^θ}*. For every stationary set S, the set of good $M \prec H_{\theta^+}$ with $M \cap \omega_1 \in S$ is stationary.*

Proof. Let $S \subseteq \omega_1$ be stationary and define $\Gamma \subseteq [H_{\theta^+}]^\omega$ to consist of all good M with $M \cap \omega_1 \in S$. Let $M \prec H_{(2^\theta)^+}$ be countable with $M \cap \omega_1 \in S$ and S in M. Since Γ is definable from parameters in M, $\Gamma \in M$. By iterating Lemma 39.2, we can find an ω_1-end extension N of M which is good. Since $N \cap H_{\theta^+} \in \Gamma$ and $\Gamma \in N$, Γ is stationary. $\square$

Lemma 39.4. *Assume* RP_{2^θ}*. If $\langle \mathscr{A}_n \mid n \in \omega\rangle$ is a sequence of antichains in $\mathrm{NS}^+_{\omega_1}$, then for every $p \in \mathrm{NS}^+_{\omega_1}$, there is a $q \leqslant p$ such that for every n, q is compatible with at most $\aleph_1$ elements of $\mathscr{A}_n$.*

Proof. Let p be given and define Γ to be the set of all good $M \prec H_{\theta^+}$ such that $\{\mathscr{A}_n \mid n \in \omega\} \in M$ and $M \cap \omega_1 \in p$. Γ is stationary and hence RP_{2^θ} implies that there is a continuous $\subseteq$-increasing sequence $\langle N_\xi \mid \xi \in \omega_1\rangle$ such that $N_\xi \cap \omega_1 = \xi$ and $q := \{\xi \in \omega_1 \mid N_\xi \in \Gamma\}$ is stationary. Notice that $q \subseteq p$. It suffices to show that if $A \in \mathscr{A}_n$ is compatible with q, then $A \in N_\xi$ for some $\xi \in \omega_1$. For each $\xi \in q \cap A$, let $B_\xi \in N_\xi \cap \mathscr{A}_n$ be such that $\xi \in B_\xi$. By the Pressing Down Lemma, there is B an a stationary $C \subseteq q \cap A$ such that if $\xi \in C$, $B_\xi = B$. It follows that $C \subseteq B \cap A$ and since $\mathscr{A}_n$ was assumed to be an antichain, it must be that $A = B$. $\square$

Lemma 39.5. *Suppose that $p \Vdash_{\mathrm{NS}^+_{\omega_1}} \dot{f} \in \widecheck{\mathbf{V}^{\omega_1}}$ and that $\mathscr{A} \subseteq \mathrm{NS}^+_{\omega_1}$ is a maximal antichain of conditions which decide $\dot{f}$. If at most $\aleph_1$ elements of $\mathscr{A}$ are compatible with p, then there is an $h \in \mathbf{V}^{\omega_1}$ such that $p \Vdash_{\mathrm{NS}^+_{\omega_1}} \check{h} =_{\dot{G}} \dot{f}$.*

Proof. Let $\{A_\xi \mid \xi \in \omega_1\}$ list the elements of $\mathscr{A}$ which are compatible with p and let f_ξ be such that $A_\xi \Vdash_{\mathrm{NS}^+_{\omega_1}} \dot{f} = \check{f}_\xi$. Define $h(\xi) = f_\eta(\xi)$ where $\eta < \xi$ is minimal such that $\xi \in A_\eta$; if no such η exists, define $h(\xi) = 0$. Suppose that $q \leqslant p$ decides $\dot{f}$, noting that $q \Vdash_{\mathrm{NS}^+_{\omega_1}} \dot{f} = \check{f}_\eta$ for some $\eta < \omega_1$. Since $\mathscr{A}$ is an antichain, q is contained in A_η modulo a stationary set. Since $q \Vdash \check{A}_\eta \in \dot{G}$, it suffices to show that

$$B := \{\xi \in A_\eta \mid h(\xi) \neq f_\eta\}$$

is nonstationary. If $\xi \in B$, then there is an $\eta' < \eta$ such that $\xi \in A_{\eta'}$ and $h(\xi) = f_{\eta'}(\xi) \neq f_\eta(\xi)$. If B were stationary, there would be a single η' and a stationary $B' \subseteq B$ such that for all $\xi \in B'$, $\xi \in A_{\eta'}$ and $h(\xi) = f_{\eta'}(\xi)$. But this means $B' \subseteq A_\eta \cap A_{\eta'}$ while $f_\eta \neq f_{\eta'}$, contradicting that $\{A_\xi \mid \xi \in \omega_1\} \subseteq \mathscr{A}$ is an antichain. $\square$

Proposition 39.6. *Assume* RP_{2^θ}. *The following are forced by* $\mathrm{NS}^+_{\omega_1}$*:*

- $\widetilde{\mathbf{V}^{\omega_1}}/\dot{G}$ *is well founded;*
- *the transitive collapse of* $\widetilde{\mathbf{V}^{\omega_1}}/\dot{G}$ *is closed under* ω*-sequences and in particular contains* $\mathbb{R}$*.*

Proof. Suppose that $\langle \dot{f}_n \mid n \in \omega\rangle$ is a sequence of $\mathrm{NS}^+_{\omega_1}$-names for elements of $\widetilde{\mathbf{V}^{\omega_1}}$ and let $p \in \mathrm{NS}^+_{\omega_1}$ be arbitrary. Let $\mathscr{A}_n \subseteq \mathrm{NS}^+_{\omega_1}$ be a maximal antichain of conditions which decide $\dot{f}_n$. By Lemma 39.4, there is a $q \leqslant p$ such that q is compatible with at most $\aleph_1$ elements of $\mathscr{A}_n$ for all n. By Lemma 39.5, there are functions $h_n \in \mathbf{V}^{\omega_1}$ such that for all n, $q \Vdash_{\mathrm{NS}^+_{\omega_1}} \check{h}_n =_{\dot{G}} \dot{f}_n$. Define $g \in \mathbf{V}^{\omega_1}$ by $g(\xi) = \{h_n(\xi) \mid n \in \omega\}$. It follows that q forces that for all $x \in \widetilde{\mathbf{V}^{\omega_1}}$, $x \in_{\dot{G}} \check{g}$ if and only if $x =_{\dot{G}} \check{h}_n$ for some n if and only if $x =_{\dot{G}} \dot{f}_n$ for some n.

Since $\widetilde{\mathbf{V}^{\omega_1}}/\dot{G}$ is forced to satisfies the same theory as $\check{\mathbf{V}}$, $\widetilde{\mathbf{V}^{\omega_1}}/\dot{G}$ models the Axiom of Foundation. Therefore q forces $\widetilde{\mathbf{V}^{\omega_1}}/\dot{G} \models \ulcorner \check{g}$ has an $\in_{\dot{G}}$ minimal element$\urcorner$. Thus q forces $\langle \dot{f}_n \mid n \in \omega\rangle$ has an $\in_{\dot{G}}$-minimal element. Since p was arbitrary, $\mathbf{1} \Vdash_{\mathrm{NS}^+_{\omega_1}} \ulcorner\widetilde{\mathbf{V}^{\omega_1}}/\dot{G}$ is well founded$\urcorner$. Notice that this also establishes the second conclusion: if $\pi_{\dot{G}}$ is the collapsing isomorphism, $\mathbf{1} \Vdash_{\mathrm{NS}^+_{\omega_1}} \pi_{\dot{G}}(\check{g}) = \langle \pi_{\dot{G}}(\dot{f}_n) \mid n \in \omega\rangle$. $\square$

40. The $\mathbf{L}(\mathbb{R})$ Absoluteness Theorem

If Q is a forcing, we will say that $\mathrm{Th}(\mathbf{L}(\mathbb{R}))$ *is absolute* for Q if whenever ϕ is a sentence in the language of set theory, $\mathbf{L}(\mathbb{R}) \models \phi$ if and only if $\mathbf{1} \Vdash_Q \ulcorner \mathbf{L}(\dot{\mathbb{R}}) \models \phi \urcorner$. The $\mathbf{L}(\mathbb{R})$ Absoluteness Theorem asserts that if there is a supercompact cardinal, then $\mathrm{Th}(\mathbf{L}(\mathbb{R}))$ is absolute for every forcing. We'll begin with some observations.

We will eventually show that if κ is a supercompact cardinal, then $\mathrm{Th}(\mathbf{L}(\mathbb{R}))$ is absolute for $\mathrm{Coll}(\omega, \kappa)$. Suppose for a moment that we've established this. If Q is in V_κ, then by Proposition 34.6, $Q * \mathrm{Coll}(\omega, \kappa)$ is forcing equivalent to $\mathrm{Coll}(\omega, \kappa)$. Also Proposition 38.3 implies that $\mathbf{1} \Vdash_Q \ulcorner \kappa \text{ is supercompact} \urcorner$. Thus if ϕ is any sentence, then $\mathbf{L}(\mathbb{R}) \models \phi$ if and only if $\mathbf{1} \Vdash_{\mathrm{Coll}(\omega,\kappa)} \ulcorner \mathbf{L}(\mathbb{R}) \models \phi \urcorner$ if and only if $\mathbf{1} \Vdash_Q \ulcorner \mathbf{1} \Vdash_{\mathrm{Coll}(\omega,\kappa)} \ulcorner \mathbf{L}(\mathbb{R}) \models \phi \urcorner \urcorner$. Next suppose that Q is any forcing and let $\mathbf{j} : \mathbf{V} \to \mathbf{M} \subseteq \mathbf{V}$ be an elementary embedding witnessing that κ is $|\mathscr{P}(Q)|$-supercompact. Let $Q_0 = \mathbf{j}''Q$ and observe that $Q_0 \in \mathbf{M}$ and if $\dot{r}$ is a nice Q_0-name for a real, then $\dot{r}$ is in $\mathbf{M}$. It follows that $\mathbf{V} \models \ulcorner p \Vdash_{Q_0} \ulcorner \mathbf{L}(\mathbb{R}) \models \phi \urcorner \urcorner$ if and only if $\mathbf{M} \models \ulcorner p \Vdash_{Q_0} \ulcorner \mathbf{L}(\mathbb{R}) \models \phi \urcorner \urcorner$. By elementarity, $\mathbf{M} \models \ulcorner \mathbf{j}(\kappa) \text{ is supercompact} \urcorner$ and therefore that $\mathbf{M} \models \ulcorner \mathrm{Th}(\mathbf{L}(\mathbb{R})) \text{ is absolute for } Q_0 \urcorner$. It follows that $\mathrm{Th}(\mathbf{L}(\mathbb{R}))$ is absolute for $Q_0 \cong Q$.

We will now turn to proving that $\mathrm{Th}(\mathbf{L}(\mathbb{R}))$ is absolute for $\mathrm{Coll}(\omega, \kappa)$ if κ is supercompact. Since $\mathrm{Coll}(\omega, \kappa)$ is weakly homogeneous, it is sufficient to show that for some forcing P:

- $\mathrm{Th}(\mathbf{L}(\mathbb{R}))$ is absolute for P and
- P forces there is an $H \subseteq \mathrm{Coll}(\omega, \kappa)$ which is a $\check{\mathbf{V}}$-generic filter such that $\dot{\mathbb{R}} \subseteq \mathbf{V}[H]$.

The forcing P will be of the form $\mathrm{Coll}(\omega_1, \kappa) * \mathrm{NS}^+_{\omega_1} * \dot{Q}$ for some $\dot{Q}$ such that $\mathrm{Coll}(\omega_1, \kappa) * \mathrm{NS}^+_{\omega_1}$ forces $\dot{Q}$ does not add reals.

Observe that $\mathrm{Th}(\mathbf{L}(\mathbb{R}))$ is absolute for forcings which do not add new reals and so the absoluteness of $\mathrm{Th}(\mathbf{L}(\mathbb{R}))$ for the iteration reduces to showing that

$$\mathbf{1} \Vdash_{\mathrm{Coll}(\omega_1,\kappa)} \ulcorner \mathrm{Th}(\mathbf{L}(\mathbb{R})) \text{ is absolute for } \dot{\mathrm{NS}}_{\omega_1} \urcorner.$$

Since $\mathrm{Coll}(\omega_1, \kappa)$ forces RP_{2^θ}, this is a consequence of Proposition 39.6.

Lemma 40.1. *Suppose that* $\mathbf{j} : \mathbf{V} \to \mathbf{M}$ *witnesses that* κ *is* 2^κ*-supercompact. If* $H \subseteq \mathrm{Coll}(\omega_1, \mathbf{j}(\kappa)) * \dot{\mathrm{NS}}^+_{\omega_1}$ *is* $\mathbf{V}$*-generic, then* $G := \mathbf{j}^{-1}(H) \subseteq \mathrm{Coll}(\omega_1, \kappa) * \dot{\mathrm{NS}}^+_{\omega_1}$ *is* $\mathbf{V}$*-generic and* $\mathbf{j}$ *extends to an elementary embedding of* $\mathbf{V}[G]$ *into* $\mathbf{M}[H]$*.*

Proof. Let G_0 and H_0 be the projections of G and H to $\mathrm{Coll}(\omega_1,\kappa)$ and $\mathrm{Coll}(\omega_1,\mathbf{j}(\kappa))$, respectively. By Lemma 38.3, G_0 is $\mathbf{V}$-generic and $\mathbf{j}$ extends to an embedding of $\mathbf{V}[G_0]$ into $\mathbf{M}[H_0]$. Working in $\mathbf{V}[G_0]$, let Γ be the set of all good $M \prec H_{\theta^+}$ such that $M \cap \omega_1 \in B$.

By RP_{2^θ} and arguing as in the proof of Lemma 39.1, there is a continuous $\subseteq$-chain $\langle M_\nu \mid \nu \in \omega_1\rangle$ such that $M_\nu \cap \omega_1 = \nu$ and $C := \{\nu \in \omega_1 \mid M_\nu \in \Gamma\}$ is stationary. Set $M_{\omega_1} := \bigcup\{M_\nu \mid \nu \in \omega_1\}$. If suffices to show that if $\xi \in \omega_1$ and $A \in \mathscr{A}_\xi \backslash M_{\omega_1}$, then $A \cap C$ is nonstationary. Define a function r on $\{M_\nu \mid \nu \in A \cap C\}$ by letting $r(M_\nu)$ be the element of $\mathscr{A}_\xi \cap M_\nu$ which has ν as an element. If $A \cap C$ were stationary, there would be an $A' \in \mathscr{A}_\xi \cap M_{\omega_1}$ such that $r^{-1}(A')$ is stationary. But then $A' \cap A$ is stationary, a contradiction. $\square$

Lemma 40.2. *If $G \subseteq \mathrm{Coll}(\omega_1,\kappa) * \mathrm{NS}^+_{\omega_1}$ is a $\mathbf{V}$-generic filter and $r \in \mathbf{V}[G]$, then there is a $Q_0 \subseteq \mathrm{Coll}(\omega_1,\kappa) * \mathrm{NS}^+_{\omega_1}$ in $\mathbf{V}$ such that:*

- $\mathbf{V} \models |Q_0| < \kappa$;
- $\mathbf{V} \models \ulcorner Q_0$ *is a regular suborder of* $\mathrm{Coll}(\omega_1,\kappa) * \mathrm{NS}^+_{\omega_1}\urcorner$;
- $r \in \mathbf{V}[G \cap Q_0]$.

Proof. It suffices to show that for every $p \in \mathrm{Coll}(\omega_1,\kappa) * \mathrm{NS}^+_{\omega_1}$, there is a $\mathbf{V}$-generic filter G satisfying the conclusion of the lemma such that $p \in G$. Let $\dot r$ be a $\mathrm{Coll}(\omega_1,\kappa) * \mathrm{NS}^+_{\omega_1}$-name such that $p \Vdash \dot r \in \dot{\mathbb{R}}$. Let $\mathbf{j} : \mathbf{V} \to \mathbf{M} \subseteq \mathbf{V}$ witness that κ is 2^κ-supercompact and set $Q_0 := \mathbf{j}'' \mathrm{Coll}(\omega_1,\kappa) * \mathrm{NS}^+_{\omega_1}$. Since $|Q_0| \leqslant 2^\kappa$, $\mathscr{P}(Q_0) \subseteq \mathbf{M}$. Let $H \subseteq \mathrm{Coll}(\omega_1,\mathbf{j}(\kappa)) * \mathrm{NS}^+_{\omega_1}$ be $\mathbf{V}$-generic with $\mathbf{j}(p) \in H$ and set

$$G := \{p \in \mathrm{Coll}(\omega_1,\kappa) * \mathrm{NS}^+_{\omega_1} \mid \mathbf{j}(p) \in H\}$$

noting that $p \in G$. By Lemma 40.1, G is $\mathbf{V}$-generic and hence $H \cap Q_0$ is $\mathbf{M}$-generic. Since H was arbitrary, it follows that $\mathbf{M}$ satisfies Q_0 is a regular suborder of $\mathrm{Coll}(\omega_1,\mathbf{j}(\kappa)) * \mathrm{NS}^+_{\omega_1}$. By elementarity, $\mathbf{j}(\dot r)(H) = \mathbf{j}(\dot r(G))$. Since $\mathbf{j}$ is the identity on hereditarily countable sets, $\mathbf{j}(\dot r(G)) = \dot r(G)$. Thus $\mathbf{j}(\dot r)(H) = \mathbf{j}(\dot r)(H \cap Q_0)$. We've established that $\mathbf{M}[H]$ satisfies Q_0 is a regular suborder of $\mathrm{Coll}(\omega_1,\mathbf{j}(\kappa)) * \mathrm{NS}^+_{\omega_1}$ of cardinality at most $2^\kappa < \mathbf{j}(\kappa)$ and that $\dot r(H) = \dot r(H \cap Q_0)$ is in $\mathbf{M}[H \cap Q_0]$. The conclusion of the lemma now follows by elementarity of $\mathbf{j}$. $\square$

The next proposition finishes the proof.

Proposition 40.3. *Suppose that κ is a supercompact cardinal. There is a $\mathrm{Coll}(\omega_1,\kappa) * \mathrm{NS}_{\omega_1}$-name $\dot Q$ for a forcing with the properties that:*

- *forcing with $\dot Q$ does not add new reals and*
- *forcing with $\dot Q$ adds a $\mathbf{V}$-generic filter $\dot H \subseteq \mathrm{Coll}(\omega,\kappa)$ such that $\dot{\mathbb{R}} \subseteq \mathbf{V}[\dot H]$.*

Proof. Let $G \subseteq \mathrm{Coll}(\omega_1, \kappa) * \mathrm{NS}_{\omega_1}$ be a $\mathbf{V}$-generic filter and work in $\mathbf{V}[G]$. Define $Q \in \mathbf{V}[G]$ to be the set of all $q = (H_q, \delta_q)$ such that $\delta_q \in \kappa$ and $H_q \subseteq \mathrm{Coll}(\omega, \delta)$ is a $\mathbf{V}$-generic filter. Define $q \leqslant p$ if $\delta_p \leqslant \delta_q$ and $H_p = H_q \cap \mathrm{Coll}(\omega, \delta_p)$.

Claim 40.4. *For all $\nu \in \kappa$, $\{q \in Q \mid \nu \leqslant \delta_q\}$ is dense in Q.*

Proof. Let $p \in Q$ be arbitrary. If $\nu \leqslant \delta_p$, there is nothing to show. Otherwise $\mathrm{Coll}(\omega, \nu) \cong \mathrm{Coll}(\omega, \delta_p) \times \mathrm{Coll}(\omega, \nu \backslash \delta_p)$, which is countable. Since κ is inaccessible in $\mathbf{V}[H_p]$ and $\mathscr{P}(\mathrm{Coll}(\omega, \nu\backslash\delta_p)) \cap \mathbf{V}[H_p]$ is countable, there is a filter $K \subseteq \mathrm{Coll}(\omega, \nu\backslash\delta_p)$ which is $\mathbf{V}[H_p]$-generic. If we define

$$H_q := \{s \in \mathrm{Coll}(\omega, \nu) \mid (s \restriction \delta_p \times \omega \in H_p) \wedge (s \restriction (\nu\backslash\delta_p) \times \omega \in K)\}$$

and $\delta_q := \nu$, then H_q is $\mathbf{V}$-generic and $q \leqslant p$. $\square$

Claim 40.5. *For all $r \in \mathbb{R} \cap \mathbf{V}[G]$, $\{q \in Q \mid r \in \mathbf{V}[H_q]\}$ is dense in Q.*

Proof. Let $p \in Q$ be arbitrary. Let $s \in \mathbb{R}$ be such that H_p and r are in $\mathbf{V}[s]$. Applying Lemma 40.2, there is a regular suborder $Q_0 \subseteq \mathrm{Coll}(\omega_1, \kappa) * \mathrm{NS}_{\omega_1}$ in $\mathbf{V}$ such that $|Q_0| < \kappa$ and $s \in \mathbf{V}[G \cap Q_0]$. By Proposition 37.4, there is a $\nu \in \kappa$ such that $\mathbf{V}$ satisfies ν is an inaccessible cardinal greater than $|Q_0|$. Since $Q_0 * \mathrm{Coll}(\omega, \nu)$ is forcing equivalent to

$$\mathrm{Coll}(\omega, \nu) \cong \mathrm{Coll}(\omega, \delta_p) \times \mathrm{Coll}(\omega, \nu\backslash\delta_p),$$

there is a $\mathbf{V}$-generic filter $H \subseteq \mathrm{Coll}(\omega, \nu)$ containing H_p that $G \cap Q_0$ is in $\mathbf{V}[H]$. By an application of Lemma 36.6 in $\mathbf{V}[H_p]$ and using that $\mathrm{Coll}(\omega, \nu\backslash\delta_p) \cong \mathrm{Coll}(\omega, \nu)$, there is an $H_q \subseteq \mathrm{Coll}(\omega, \nu)$ such that $H_p \subseteq H_q$ and $G \cap Q_0 \in \mathbf{V}[H_q]$. $\square$

It just remains to show that forcing with Q does not add new reals. Let $p \in Q$ and $\dot{r}$ be a nice Q-name for a real. We will find a $q \leqslant p$ which decides $\dot{r}$. Observe that Q and $\dot{r}$ are both in $H_{\aleph_2}$. By Proposition 37.4,

$$S := \{\nu \in \omega_1 \mid \mathbf{V} \models \ulcorner \nu \text{ is inaccessible} \urcorner\}$$

is stationary. Let M be a countable elementary submodel of $H_{\aleph_2}$ such that Q, p, and $\dot{r}$ are in M and $\nu := M \cap \omega_1$ is in S. Construct a decreasing sequence $\langle p_n \mid n \in \omega\rangle$ in $M \cap Q$ such that $p_0 := p$ and $\{p_n \mid n \in \omega\}$ meets every dense subset of Q which is in M. By Claim 40.4, $\sup\{\delta_{p_n} \mid n \in \omega\} = \nu$. Define $\delta_q := \nu$ and $H_q := \bigcup\{H_{p_n} \mid n \in \omega\}$. To see that H_q is $\mathbf{V}$-generic, suppose that $A \subseteq \mathrm{Coll}(\omega, \nu)$ is a maximal antichain in $\mathbf{V}$. Since ν is inaccessible in $\mathbf{V}$, $A \subseteq \mathrm{Coll}(\omega, \alpha)$ for some $\alpha \in \nu$. If n is such that $\alpha \leqslant \delta_{p_n}$, then $A \cap H_{p_n}$ is nonempty. Thus $q \in Q$ is a lower bound for $\{p_n \mid n \in \omega\}$. Since $\dot{r}$ is in M and q is a lower bound for an M-generic filter, q decides $\dot{r}$. $\square$