Abstract

In this paper, we will describe the space spanned by the angle-sums of polytopes, recorded in the $\alpha$-vector. We will consider the angles sums of simplices and the angles sums and face numbers of simplicial polytopes and general polytopes. We will construct families of polytopes whose angle sums span the spaces of polytopes defined by the Gram and Perles equations, analogues of the Euler and Dehn-Sommerville equations. We show that the dimensions of the affine span of the space of angle sums of simplices is $\left\lfloor \frac{d-1}{2} \right\rfloor + 1$, and that of the angle sums and face numbers of simplicial polytopes and general polytopes are $d - 1$ and $2d - 3$ respectively.

1 Introduction

One of the motivating questions in the combinatorial study of polytopes is whether a given set of combinatorial data arises from a given class of polytopes. This has been studied in depth using the $f$-vector, which counts the number of faces of a polytope of each dimension, and using the flag $f$-vector, which counts the number of flags of faces, that is chains of faces in the face-poset ordered by inclusion. We will study this question by considering the angle sums of polytopes, which quantify a geometric aspect of polytopes. We will also introduce a method to construct polytopes while controlling the angle sums.

Let $P$ be any $d$-polytope. For any face $F$ of $P$, we consider a small $d$-dimensional ball centered at an interior point of $P$. This ball should be small enough that it only intersects faces which contain $F$. The interior angle of $F$ in $P$, denoted by $\beta(F, P)$, is the fraction of this ball which is contained in $P$. The angle sums of $P$ are defined for $0 \leq i \leq d$ as

$$\alpha_i(P) = \sum_{i \text{-faces } F \subseteq P} \beta(F, P).$$

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If we write $f_i(P)$ for the number of $i$-faces of $P$, we then define the following for a $d$-polytope $P$:

- \( \alpha \)-vector: \((\alpha_0(P), \alpha_1(P), \ldots, \alpha_d(P))\).
- \( f \)-vector: \((f_0(P), f_1(P), \ldots, f_d(P))\).
- \( \alpha-f \)-vector: \((\alpha_0(P), \alpha_1(P), \ldots, \alpha_d(P), f_0(P), f_1(P), \ldots, f_d(P))\).

It is well-known that there are equations on the $f$-vector. For general polytopes, it is known that the only linear relation on the $f$-vector is Euler’s formula [2]:

$$
\sum_{i=0}^{d} (-1)^i f_i(P) = 1 \text{ for any } d\text{-polytope } P.
$$

The only relations on the $f$-vectors of simplicial polytopes are the Dehn-Sommerville equations [2]: For any simplicial polytope $P$ and $-1 \leq k \leq d - 2$,

$$
\sum_{j=k}^{d-1} (-1)^j \binom{j+1}{k+1} f_j(P) = (-1)^{d-1} f_k(P).
$$

It is also known that there are relations on the $\alpha$-vector. The Gram equation is an analog of Euler’s formula [2]:

$$
\sum_{i=0}^{d-1} (-1)^i \alpha_i(P) = (-1)^{d+1} \text{ for any } d\text{-polytope } P.
$$

These equations are the only linear equations on the $\alpha$-vector of general polytopes.

Perles proved an analog of the Dehn-Sommerville relations [G,PS]: For any simplicial polytope $P$ and $0 \leq k \leq d - 2$, we have

$$
\sum_{j=k}^{d-1} (-1)^j \binom{j+1}{k+1} \alpha_j(P) = (-1)^d (\alpha_k(P) - f_k(P)).
$$

In this paper we will consider the spaces spanned by the $\alpha$-vectors of simplices and the $\alpha-f$-vectors of simplicial polytopes and general polytopes. We will construct families of polytopes whose $\alpha$-vectors span the spaces of $\alpha$-vectors defined by the Gram and Perles equations. In the second section, we will construct these polytopes and in the third section, we show that they span the spaces defined by the equations on angles sums and face numbers mentioned above. That is, we show that the dimensions of the affine span of the space of $\alpha$-vectors of simplices is \(\left\lfloor \frac{d-1}{2} \right\rfloor + 1\), and that of the $\alpha-f$-vectors of simplicial polytopes and general polytopes are $d - 1$ and $2d - 3$ respectively. This describes the $\alpha$-vectors of simplices and the $\alpha$-vectors of a given combinatorial type.

## 2 Construction of Polytopes

We will define two construction operations, the pyramid and prism, that create polytopes with varying angle sums. Each polytope will be constructed from a
polytope of dimension one lower. This is similar to the construction done by Bayer and Billera [1], although rather than bipyramids, we will build the dual, prisms. For a \((d - 1)\)-polytope \(Q\), we will denote a \(d\)-pyramid based on it as \(PQ\) and the \(d\)-prism over it as \(BP\), following Bayer and Billera. However, since we are interested in geometric aspects of the construction, we will fix the geometry of the polytopes and not just the combinatorics.

The \textit{prism} over the \((d - 1)\)-polytope \(Q\), denoted \(BQ\), is \(Q \times I\), where \(I = [0, k]\) for some \(k\). Then, any \(i\)-face, \(F\), of \(BQ\) is either an \(i\)-face of one of \(Q \times 0\) or \(Q \times k\), or, for some \((i - 1)\)-face \(G \subseteq Q\), \(F = G \times I\), which is perpendicular to both \(Q \times 0\) and \(Q \times k\). If \(F\) is a face of this latter type, then \(\beta(F, BQ) = \beta(G, Q)\). No angles change as the distance between the two copies of \(Q\) varies, so the angle sums do not depend on \(k\), but only on the prism construction. More specifically, we have the following equations on the \(f\)-vector and angle sums:

\[
\begin{align*}
f_0(BQ) &= 2f_0(Q) \quad (1) \\
f_i(BQ) &= 2f_i(Q) + f_{i-1}(Q) \quad \text{for } 1 \leq i \leq d \quad (2) \\
\alpha_0(BP) &= \alpha_0(P) \quad (3) \\
\alpha_i(BP) &= \alpha_i(P) + \alpha_{i-1}(P) \quad \text{for } 1 \leq i \leq d. \quad (4)
\end{align*}
\]

Figure 1: The polytope \(Q\); \(BQ\), the prism over \(Q\); and \(PQ\), the pyramid over \(Q\).

Now we define the \textit{pyramid} over the polytope \(Q\), which we denote \(PQ\). We start by placing a \((d - 1)\)-dimensional polytope \(Q\) in the hyperplane \(x_d = 0\) in \(\mathbb{R}^d\). We then place a vertex \(v\) along the line perpendicular to \(\mathbb{R}^{d-1}\) through the centroid of \(Q\) so that it has \(d\)th coordinate \(k > 0\) and take the convex hull of \(v\) and \(Q\). An \(i\)-face of \(PQ\) is either an \(i\)-face of \(Q\) and part of the base, or the convex hull of \(v\) and an \((i - 1)\)-face of \(Q\), faces which we will refer to as sides. The angles formed between the sides and faces in the base increase as \(k\) does. For this reason, we may denote the pyramid by \(P_kQ\) to specify the height of \(v\). All \(d\)-simplices can be formed by taking a pyramid over a line segment \(d - 1\) times. For any \(k\), the pyramid operation
has the following effect on the $f$-vector:

$$
f_0(PQ) = f_0(Q) + 1 \tag{5}
$$

$$
f_i(PQ) = f_i(Q) + f_{i-1}(Q) \text{ for } 1 \leq i \leq d \tag{6}
$$

We note two limiting cases of the pyramid operation: the case as $k$ tends toward 0 and the case where $k$ tends toward infinity. We will denote these constructions by $P_0Q$ and $P_\infty Q$, respectively. Although neither is actually a $d$-pyramid, one can easily find the limits of the angle sums as $k$ tends to 0 or infinity, and we will define these values as the angle sums for $P_0Q$ and $P_\infty Q$.

For $P_0Q$, all angles made between the base and sides tend to 0, so any interior angles at proper faces of the base are 0, and the interior angle at the base and at faces including the apex, $v$, are all $\frac{1}{2}$. Therefore, all the angles sums are dependent on the $f$-vector of the base $Q$. Then we have:

$$
\alpha_0(P_0(Q)) = \frac{1}{2} \tag{7}
$$

$$
\alpha_i(P_0(Q)) = \frac{1}{2} f_{i-1}(Q) \text{ for } 1 \leq i \leq d - 2 \tag{8}
$$

$$
\alpha_{d-1}(P_0(Q)) = \frac{1}{2} f_{d-2}(Q) + \frac{1}{2} \tag{9}
$$

For $P_\infty Q$, angles between the sides and base tend to right angles, so for any face $G \subseteq Q$, $\beta(G, P_\infty Q) = \frac{1}{2} \beta(G, Q)$. For faces, $F \subseteq P_\infty Q$ which are the convex hull of a face $G \subseteq Q$ and $v$, the interior angle at $G$ is the same as it was at $F$, i.e., $\beta(F, P_\infty Q) = \beta(G, Q)$. Therefore:

$$
\alpha_0(P_\infty Q) = \frac{1}{2} \alpha_0(Q) \tag{10}
$$

$$
\alpha_i(P_\infty Q) = \frac{1}{2} \alpha_i(Q) + \sum_{i-1 \text{ faces } G \subseteq Q} \beta(G, Q) \tag{11}
$$

$$
= \frac{1}{2} \alpha_i + \alpha_{i-1}(Q) \text{ for } 1 \leq i \leq d - 1. \tag{12}
$$

Since the values of the angle sums vary continuously as $k$ does, we can find pyramids with angle sums which are arbitrarily close to those of $P_0Q$ and $P_\infty Q$.

## 3 Spans of $\alpha$ and $\alpha$-$f$-vectors

Using the prism and pyramid constructions, we can now build families of polytopes with affinely independent $\alpha$-vectors. We will use these families to span the spaces of $\alpha$-vectors and $\alpha$-$f$-vectors defined by the Gram and Perles equations.

First we prove a lemma to show that we can inductively increase the dimension of our families of polytopes.

**Lemma:** If $\epsilon > 0$ is given and $Q_1, Q_2, \ldots, Q_k$ are polytopes with affinely independent $\alpha$-vectors, then you can choose an $M_\epsilon$ such that for any $N > M_\epsilon$, ...
(i) \[ |\alpha_j(P_NQ_i) - \alpha_j(P_\infty Q_i)| < \epsilon \forall i \forall j. \]

(ii) the \(\alpha\)-vectors of \(P_NQ_1, P_NQ_2, \ldots, P_NQ_k\) are affinely independent

Similarly, an affinely independent set of \(\alpha\)-vectors based on these polytopes exists in any higher dimension.

**Proof:** From the preceding formulas, we know that:

\[ \alpha(P_\infty Q_i) = A\alpha(Q_i)^T \]

where

\[
A = \begin{bmatrix}
\frac{1}{2} & 0 & 0 & \ldots & 0 & 0 \\
1 & \frac{1}{2} & 0 & \ldots & 0 & 0 \\
0 & 1 & \frac{1}{2} & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 1 & \frac{1}{2} \\
0 & 0 & 0 & \ldots & 0 & 1
\end{bmatrix}.
\]

The invertibility of the matrix \(A\) guarantees that the affine span of the \(\alpha(P_\infty Q_i), i = 1..k\), has dimension \(k - 1\) and are therefore affinely independent. Since the angles sums are continuous, we can choose \(M_i\) so that \(\alpha_j(P_NQ_i)\) is within the given \(\epsilon\) of \(\alpha_j(P_\infty Q_i)\). Then we can choose \(M = \max_{i=1..k} M_i\). Then for any \(N > M\), \(|\alpha_j(P_NQ_i) - \alpha_j(P_\infty Q_i)| < \epsilon \forall i \forall j\).

Since affine independence is an open condition, we can choose \(\epsilon\) small enough that, for \(N > M\), the \(\alpha\)-vectors of \(P_NQ_1, P_NQ_2, \ldots, P_NQ_k\) are affinely independent. To get a similar family of polytopes with affinely independent \(\alpha\)-vectors in a higher dimension, we can iterate this process any finite number of times. If \(M_1, M_2, \ldots, M_m\) are the necessary values of \(N\) at each stage, we can choose \(M = \max_{i=1..m} M_i\) and then for \(N > M\), the \(\alpha\)-vectors of \((P_N)^mQ_1, (P_N)^mQ_2, \ldots, (P_N)^mQ_k\) will be affinely independent, where \((P_N)^m\) has the obvious meaning of applying \(P_N\) \(m\) times in succession.

**Theorem 1:** The affine span of the \(\alpha\)-vectors of simplices is \(\lfloor \frac{d-1}{2} \rfloor\)-dimensional.

**Proof:** Let \(A\) be the affine space spanned by the \(\alpha\)-vectors of simplices. We first prove that \(\lfloor \frac{d-1}{2} \rfloor\) is an upper bound on the dimension of \(A\) and then use the family of polytopes constructed in the lemma to show this bound is achieved.

If \(P\) is a simplex, we know that \(f_i = \binom{d+1}{i+1}\). So the Perles equations become:

\[
\sum_{j=k}^{d-1} (-1)^j \binom{j+1}{k+1} \alpha_j(P) = (-1)^d \binom{d+1}{k+1} \alpha_k(P) = (-1)^d \binom{d+1}{k+1}
\]

or

\[
\sum_{j=k}^{d} (-1)^j \binom{j+1}{k+1} \alpha_j(P) = (-1)^d \alpha_k(P) \text{ for } -1 \leq k \leq d-1 \text{ since } \alpha_d(P) = 1.
\]
We will define the latter equation as $S_k$.

For $S_{d-2n}$, the $\alpha_{d-2n}(P)$ terms on either side cancel, giving only a relation on higher-dimensional angle sums. However, the equations $S_{d-1}, S_{d-3}, \ldots, S_{d-2\left\lceil \frac{d}{2} \right\rceil -1}$ are independent since for $m$ odd, and $\alpha_{d-m}$ only appears in $S_r$ for $r \geq d-m$. Since the $\alpha$-vector is $d+1$-dimensional and all $\alpha$-vectors lie in the plane of $\alpha_d = 1$, we get:

$$\dim(A) \leq d + 1 - \left( \left\lfloor \frac{d}{2} \right\rfloor + 1 \right) - 1 = \left\lfloor \frac{d-1}{2} \right\rfloor.$$  

We will construct a set of simplices whose $\alpha$-vectors are affinely independent by induction on $d$. The case is very simple for $d = 1$ and $d = 2$ since the simplices, a line segment and a triangle (denoted $P$ and $P^2$, respectively) provide the one element needed for the basis. By the lemma, if we have a set of simplices with affinely independent $\alpha$-vectors in dimension $d-2$, say $Q_1, Q_2, \ldots, Q_{\left\lceil \frac{d-1}{2} \right\rceil}$, then there is an $N$ such that we have an affinely independent set of the same size in dimension $d$ given by $(P_N)^2 Q_1, (P_N)^2 Q_2, \ldots, (P_N)^2 Q_{\left\lceil \frac{d-1}{2} \right\rceil}$ and $|\alpha_j(P_N Q_i) - \alpha_j(P_N Q_i)| < \epsilon \forall i \forall j$ for a given $\epsilon$. To get the $\left\lfloor \frac{d-1}{2} \right\rfloor + 1$ required simplices we need only find one more simplex at each induction step that is affinely independent of the polytopes produced in this manner.

We claim that, for small enough $\delta$, $(P_3)^d$ is such a simplex. We can choose $\delta$ so that $\alpha_0((P_3)^d) \geq \frac{1}{2} - \epsilon$. By the choice of $N$ at each step, $\alpha_0(P_N Q) \leq \frac{1}{2} \alpha_0(Q) + \epsilon$. Then since $\alpha_0(P) = 1$ and $\alpha_0(P^2) = \frac{1}{2}$ we can show by induction that $\alpha_0((P_N)^2 Q) \leq \frac{1}{2} + \frac{3\epsilon}{2}$ as long as $Q$ is constructed by pyramid and prism operations. Then we can choose $\epsilon$ small enough so that all the values of $\alpha_0$ on the polytopes constructed by the lemma are strictly less than that of $(P_3)^d$, and therefore $\alpha((P_3)^d)$ is not an affine combination of the $\alpha$-vectors of the polytopes constructed by the lemma. If it were, we could write the $\alpha_0((P_3)^d)$ as an affine combination of the other $\alpha_0$ values, giving:

$$\frac{1}{2} - \epsilon \leq \alpha_0((P_3)^d) = \sum_{i=1}^{\left\lfloor \frac{d-1}{2} \right\rfloor} \lambda_i \alpha_0((P_N)^2 Q_i)$$

where

$$\sum_{i=1}^{\left\lfloor \frac{d-1}{2} \right\rfloor} \lambda_i = 1.$$  

But if $\epsilon < \frac{1}{10}$, $\alpha_0((P_N)^2 Q_i) < \frac{1}{2} - \epsilon \leq \alpha_0((P_3)^d)$ for all $i$, this is impossible. \hfill \square

Similarly, using the Dehn-Sommerville equations as well as the Perles equations, we can show that the affine span of the $\alpha$-$f$-vectors of simplicial polytopes has at most dimension $d-1$. Since it is known that the affine span of the $f$-vectors of simplicial polytopes is of dimension $\left\lfloor \frac{d}{2} \right\rfloor$, and a spanning basis can be built including a simplex (see [1]), the following theorem holds.
Theorem 2: The affine span of the $\alpha$-$f$-vectors of simplicial polytopes has dimension $d - 1$. The space is spanned by $\lceil \frac{d+1}{2} \rceil$ simplices, as in Theorem 1, and $\left\lfloor \frac{d}{2} \right\rfloor$ non-simplex combinatorially independent simplicial polytopes.

It is worth noting that in spanning the space defined by the combinatorics and angle sums of simplicial polytopes, the dimensions beyond the combinatorics can all be found as angle sums of simplices. This means that once we have the variance in geometry of the simplex, all other variance in the class of simplicial polytopes can be described by variance in combinatorial dimensions.

We can similarly build a set of polytopes whose $\alpha$-$f$-vectors span the whole space to which the Euler and Gram equations limit these vectors.

Theorem 3: The affine span of the $\alpha$-$f$-vectors of general polytopes has dimension $2d - 3$ for $d \geq 2$.

Proof: The Euler and Gram equations provide two independent equations on the $\alpha$-$f$-vectors. We also know that $\alpha_0(P) = 1$, $f_d(P) = 1$, and $\alpha_{d-1}(P) = \frac{1}{d} f_{d-1}(P)$ for all polytopes $P$. As long as $d > 1$, these equations are independent. Therefore, the span of the $\alpha$-$f$-vectors is at most $2d + 2 - 5 = 2d - 3$ if $d \geq 2$.

To show this whole space is spanned, we will again proceed inductively on $d$.

The statement is true in two dimensions, since the $\alpha$-$f$-vectors of the triangle and the square (denoted $P^2$ and $BP$, respectively) are $(\frac{1}{2}, \frac{3}{2}, 1, 3, 3, 1)$ and $(1, 2, 1, 4, 4, 1)$. This gives an affine span of dimension 1.

Suppose the statement is true for dimension $d$. That is, there are $2d - 2$ $d$-polytopes with affinely independent $\alpha$-$f$-vectors: $Q_1, Q_2, \ldots, Q_{2d-2}$. Then by the lemma, we know that there is an $N$ such that the $\alpha$-$f$-vectors of $P_NQ_1, P_NQ_2, \ldots, P_NQ_{2d-2}$ are affinely independent. We claim that these polytopes, together with $B^{d-1}P^2$ and $B^dP$ give $2(d + 1) - 2$ polytopes with affinely independent $\alpha$-$f$-vectors.

We first show that the $\alpha$-$f$-vector of $B^{d-1}P^2$ is affinely independent of the $P_NQ_i$ for $i = 1 \ldots 2d - 2$. This can be seen by considering the $f_0$ entry of the $\alpha$-$f$-vectors of these polytopes. We know that the prism operation doubles the number of vertices in the polytope while the pyramid operation only adds one vertex. Therefore, $f_0(B^kQ) = 2^k f_0(Q)$. This means that when building polytopes with the pyramid and prism operations, the $d$-polytope that maximizes the number of vertices is $B^{d-1}P$, with $f_0(B^{d-1}P) = 2^d$. But then

$$f_0(B^{d-1}P^2) = 2^{d-1}(3) > 1 + 2^d = f_0(P_NB^{d-1}P),$$

where $P_NB^{d-1}P$ has the largest number of vertices of any $(d + 1)$-pyramid built by the prism and pyramid constructions. Since $f_0(B^{d-1}P^2)$ is larger than any of the $f_0$ entries of the affinely independent $\alpha$-$f$-vectors given by the lemma, it cannot be an affine combination of these polytopes.

To show that the $\alpha$-$f$-vector of $B^dP$ is affinely independent of the others, we use the fact that the prism operation preserves $\alpha_0$; that is, $\alpha_0(B^dP) = \alpha_0(P) = 1$. Since $\alpha_0(P_NQ) \leq \frac{1}{2} + \epsilon$ and $\alpha_0(B^{d-1}P^2) = \frac{1}{2}$, we cannot write the $\alpha$-$f$-vector of $B^dP$ as an affine combination of the vectors for $B^{d-1}P^2$ and the $P_NQ_k$ for $k = 1 \ldots 2d - 2$. 


Therefore, the set of \(d-1\)-polytopes \(B^d P, B^{d-1} P^2, P_N Q_1 P_N Q_2, \ldots, P_N Q_{2d-2}\) has affinely independent \(\alpha\)-\(f\)-vectors and the dimension of the affine space is \(2(d+1)-3\) in dimension \(d+1\).

It should be noted that the set of polytopes which span the space of \(\alpha\)-\(f\)-vectors have significant duplication in the \(\alpha\)-vectors. For instance, the polytopes \(P_\infty B^k P\) and \(B^k P^2\) have the same angle sums for all \(k \geq 1\).

This work strengthens the correspondence between the geometric structure and the combinatorial structure of polytopes. The Gram and Perles equations are close analogues of Euler’s formula and the Dehn-Sommerville equations. Also, in this paper, we have shown the the affine dimensions closely correspond. The affine span of the \(\alpha\)-vectors of \(d\)-simplices has the same dimension as the span of the \(f\)-vectors of simplicial \(d-1\)-polytopes. Likewise, the affine span of the \(\alpha\)-\(f\)-vectors of simplicial \(d\)-polytopes has the same dimension as the span of the \(f\)-vectors of \(d\)-polytopes.

It would be interesting to speculate whether there is a deeper significance to this relationship.

References


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