INTRODUCTION TO BIJECTIONS

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1. Sets

This document aims to give the reader an introduction to bijections of sets. Bijections allow us to say that there are the same number of two kinds of objects, even if we don't know how many there are. The setting we will work in is called *set theory.* A *set* is just any collection of distinct objects. Sets are typically specified by listing their elements, and usually denoted like

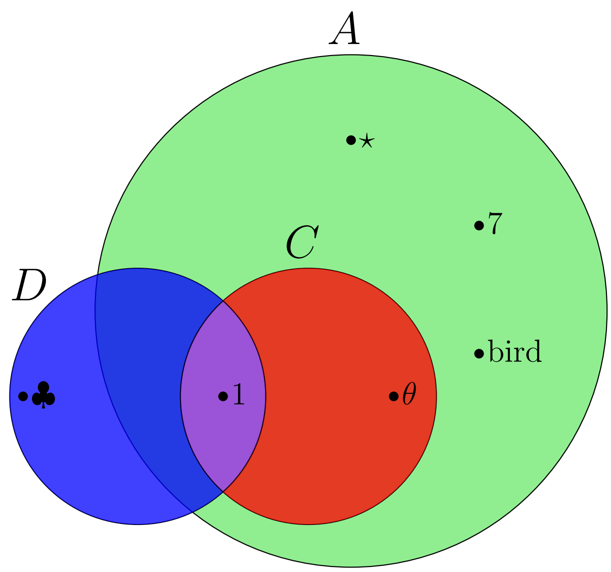
$$A = \{1, 7, \theta, \text{bird}, \star\}.$$

This is read as: "A is the set containing the numbers 1 and 7, the greek letter θ, the word 'bird', and a star". The order of elements in a set does not matter. If

$$B = \{\theta, 7, \star, \text{bird}, 1\},$$

then $A = B$ as sets. Note that since the objects in a set must be distinct, sets cannot have repeats. The set $\{1, 1, 1, 1, 1\}$ is the same as the set $\{1\}$.

Membership of an element in a set is usually written with the 'in' sign $\in$, such as $\theta \in A$. This is read as "The element θ is a member of the set A." A subset of a set T is a smaller collection of elements of T. For instance, if $C = \{1, \star\}$ then C is a subset of A, written $C \subseteq A$. Note that $D = \{1, \clubsuit\}$ is not a subset of A, since it has an element $\clubsuit$ that does not live in A. Graphically this looks like



A set can be either *finite* or *infinite* depending on how many elements it has. If a set A is finite, denote by $|A|$ the number of elements of A.

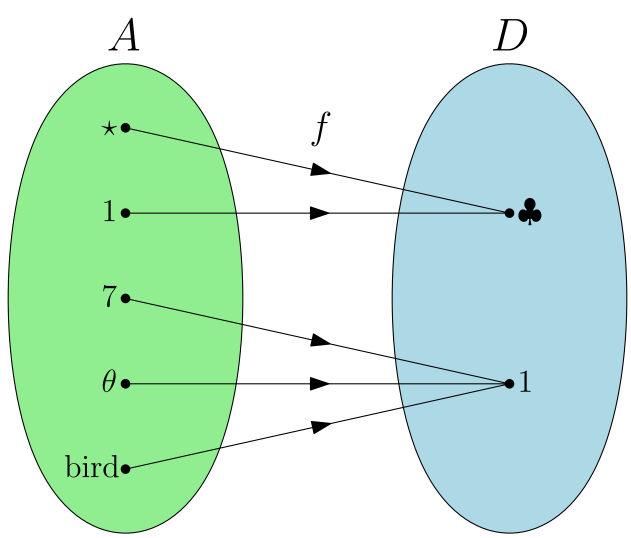
2. Functions

Now that we understand sets, we move on to the main object of this lesson: functions between sets. A *function* f from a set A to a set B is simply a rule that associates to each element of A an element of B. Such a function will be denoted $f : A \to B$. The element $b \in B$ associated to an element $a \in A$ is called the image of a under f, and we usually write $b = f(a)$ as you are probably used to seeing when dealing with functions.

Example 2.1. Let $A = \{\theta, 7, \star, \text{bird}, 1\}$ and $D = \{1, \clubsuit\}$ as before. Define $f : A \to D$ by the rules:

$$\begin{aligned} f(\theta) &= 1 & \qquad f(\star) &= \clubsuit \\ f(7) &= 1 & \qquad f(1) &= \clubsuit \\ f(\text{bird}) &= 1 \end{aligned}$$

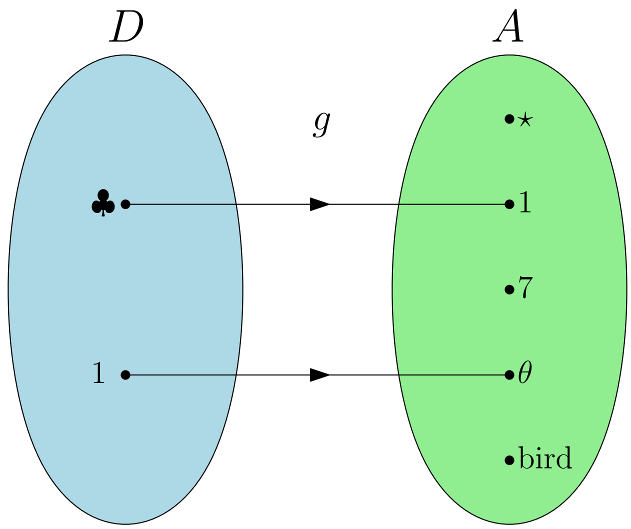
We can illustrate f as



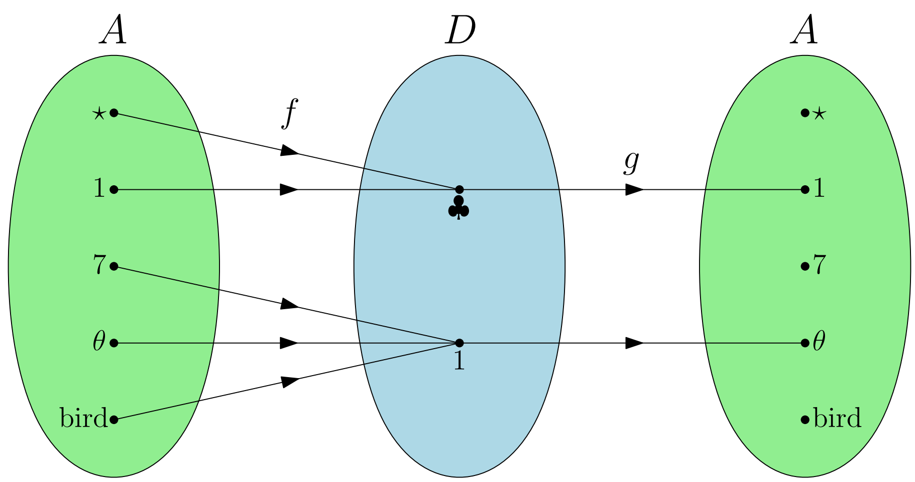
Alternatively, define $g : D \to A$ by

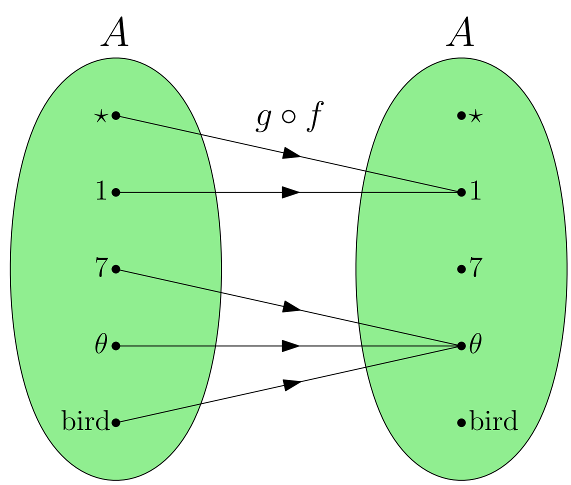
$$g(\clubsuit) = 1$$
$$g(1) = \theta,$$

which is illustrated as

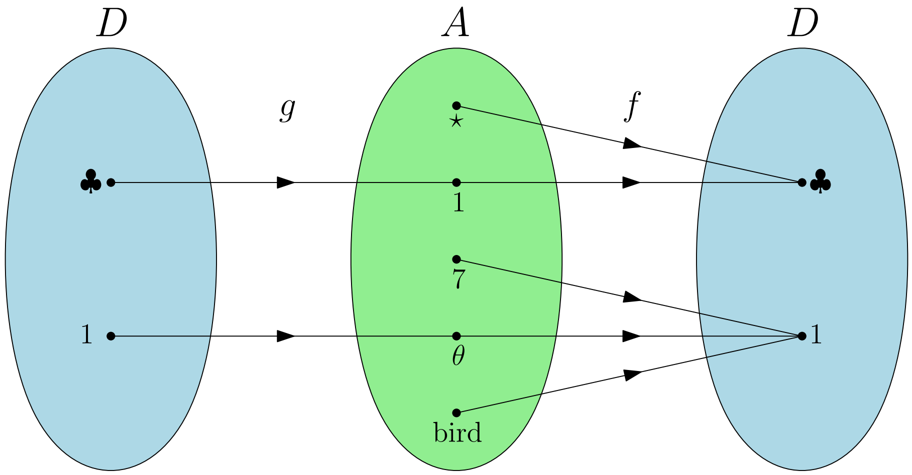


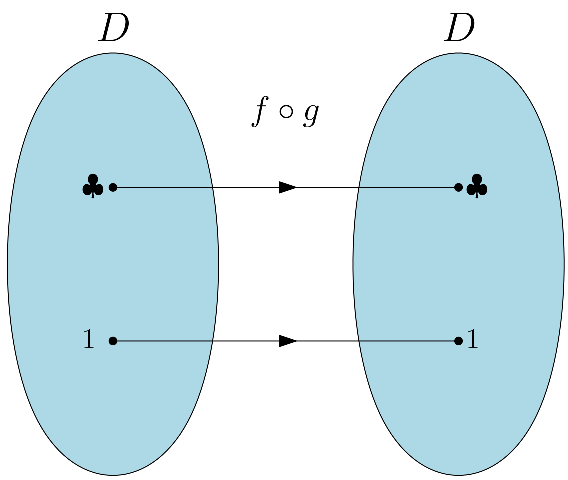
We may compose functions any $f : A \to B$ and $g : B \to C$ to get a function denoted $g \circ f : A \to C$ by simply plugging the outputs of f into g. That is, $(g \circ f)(a) = g(f(a))$. For example, using the specific sets A, D and functions f, g above, we get the following picture for $g \circ f : A \to A$:





Alternatively, we compose in the opposite order to get $f \circ g : D \to D$.





In this lesson, we will only be concerned with pairs of functions between two sets. The following definition describes a special property of certain pairs of functions.

Definition 2.2. Let $f : A \to B$ and $g : B \to A$ be functions. We say f and g are *inverses* if for any $a \in A$ and $b \in B$, we have

$$g(f(a)) = a \quad \text{and} \quad f(g(b)) = b.$$

In this case, we call the functions f and g *bijections* between A and B.

Think of g as a way to perfectly undo f. Note that the specific function f described in Example 2.1 is not a bijection: $f(\star) = f(1) = \clubsuit$, so an inverse would not know where to send $\clubsuit$ in order to undo f. Alternatively, $g \circ f$ does not send $\star$ back to $\star$.

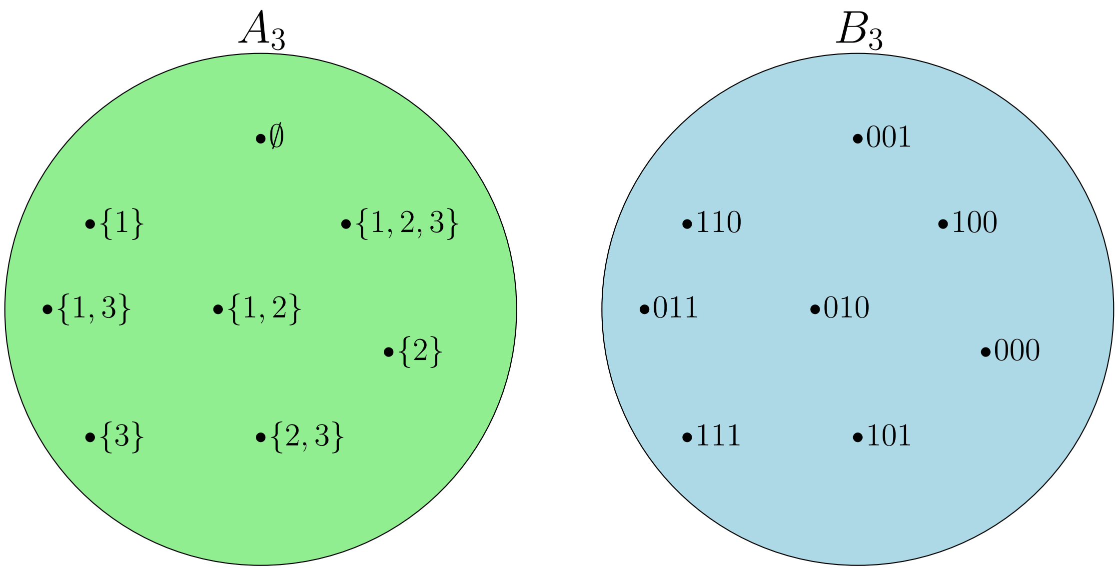
3. Bijections

In this section we give concrete examples of bijections and explain what they are used for. We start with a simple question. Pick any positive integer $[n]$ and let $[n]$ denote the set $\{1, 2, 3, \ldots, n\}$. How many subsets of $[n]$ are there?

Example 3.1. If $n = 2$, then $[2] = \{1, 2\}$. Let's count the subsets! There is always a special subset called the *empty set* which has no elements, usually denoted $\emptyset$. There is also the entire set [2]. What choices lie in between? Well, we can either have 1 in our subset, or we can have 2. Thus the subsets of [2] are $\emptyset$, $\{1\}$, $\{2\}$, and $\{1, 2\}$. There are 4!

Exercise 3.2. Count all the subsets of [3] and [4]. How many do you think there are for any $[n]$? Can you prove your guess is true?

We can solve this problem using a bijection between two sets! On the one hand, we have the set we will call A_n whose elements are all the subsets of $[n]$. On the other hand, we will consider the set B_n whose elements will be all sequences of length n of the numbers 0 and 1. Such sequences are called *binary sequences* (of length n). The case $n = 3$ is illustrated below:



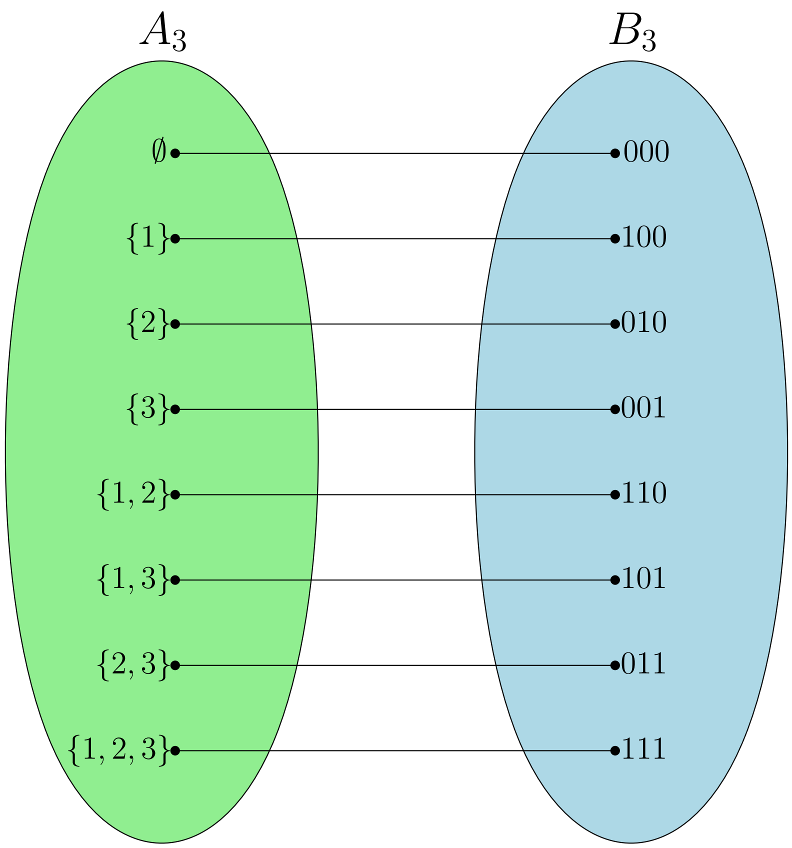
We now describe a bijection $f : A_n \to B_n$. Let $S \in A_n$, so that $S \subseteq [n]$. We must construct a sequence $f(S)$ of length n using zeros and ones. To do so, for each $i \in [n]$, let the ith entry of $f(S)$ be 1 if $i \in S$ and 0 if $i \notin S$. For example if $n = 3$, $f(\{1, 3\}) = 101$.

To prove f is a bijection, we describe the proposed inverse function $g : B_n \to A_n$ of f. Take a binary sequence $b = b_1 b_2 \cdots b_n$, that is where each b_i is zero or one. Define $g(b)$ to be the set containing the indices of the ones ones in b. For example, if $n = 3$, $g(110) = \{1, 2\}$.

To know that f is truly a bijection, we must check that

$$(f \circ g)(b) = b \quad \text{and} \quad (g \circ f)(S)$$

for each $b \in B_n$ and $S \in A_n$. Well, $f(g(b))$ is the binary sequence with ones at positions specified by the members of the set $g(b)$, which itself contains the elements corresponding to the positions of the ones in b. Thus, $f(g(b)) = b$. Conversely, $g(f(S))$ is the set whose elements are the positions of the ones in $f(S)$, which are exactly the elements of S. Thus, $g(f(S)) = S$.



This example shows the use for bijections: bijections tell us two sets are the same size! We formally record this fact below. A bijection between two sets is a perfect correspondence between their elements! To answer the original question of how many subsets of $[n]$ there are, all we have to do is count the length n binary sequences! There are n slots in a sequence, and we have two choices for each: zero or one. Thus, there $2 \times 2 \times 2 \times \cdots \times 2 = 2^n$ binary sequences.

Lemma 3.3. *If $f : A \to B$ is a bijection of sets, then $|A| = |B|$.*

Proof. Let g be the inverse of f. Write $f(A)$ for the subset of B hit by f That is, $f(A)$ is all the elements of B that can be written as $f(a)$ for some $a \in A$. Since f has an inverse, different elements a, a' of A cannot map to the same element $b \in B$ (if they did, how would you know what $g(b)$ should be?). Then $|A| = |f(A)|$. Since $f(A) \subseteq B$, definitely $|f(A)| \leq |B|$. Thus $|A| \leq |B|$. But you can make the exact same argument using g instead of f to conclude that $|B| = |g(B)| \leq |A|$. Hence, $|A| = |B|$. □

We now give a few simpler examples of bijections of finite and infinite sets.

Example 3.4. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of *natural numbers.* Consider

$$\mathbb{E} = \{2, 4, 6, 8, \ldots\}$$

the subset of all even natural numbers, and

$$\mathbb{O} = \{1, 3, 5, 7, \ldots\}$$

the subset of all odd natural numbers. We will show that there are bijections between any two of $\mathbb{N}$, $\mathbb{E}$, and $\mathbb{O}$.

Define $f : \mathbb{N} \to \mathbb{E}$ by $f(n) = 2n$, and define $g : \mathbb{N} \to \mathbb{O}$ by $g(n) = 2n - 1$. Both these maps have inverses, and are bijections. If $m \in \mathbb{E}$, then

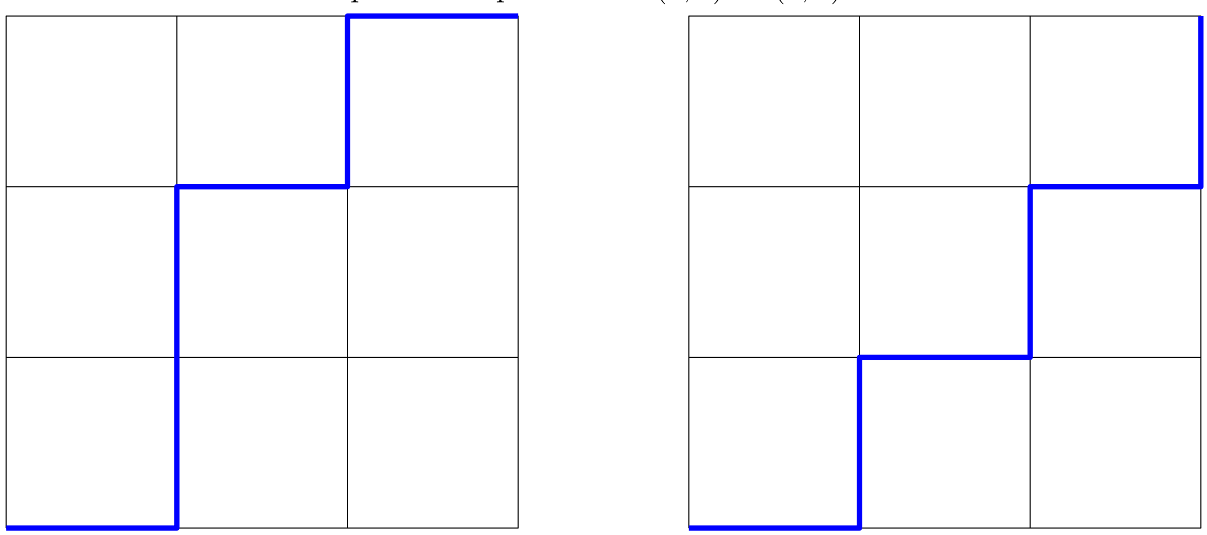
$$f^{-1}(m) = m/2.$$

Note that $m/2$ will always be an actual integer since $m \in \mathbb{E}$. Similarly, if $m \in \mathbb{O}$, then

$$g^{-1}(m) = \frac{m+1}{2}.$$

Can you see why the output of g is always an integer?
Can you find a similar bijection $h : \mathbb{E} \to \mathbb{O}$? What is the inverse of your map?
What does all this say about the relative sizes of $\mathbb{N}$, $\mathbb{E}$, and $\mathbb{O}$? What do you think of this?

Example 3.5. A *lattice path* is a sequence of length 1 east and north steps in the plane. Shown below are two example lattice paths from $(0, 0)$ to $(3, 3)$:



Let's try to biject lattice paths to subsets of some set. First, we can simplify things by ignoring the picture and thinking about the crucial data behind each path: a sequence of two kinds of steps. Let's call the two steps E and N respectively. Then the two lattice paths above can be compactly written as $ENNENE$ and $ENENEN$. If a lattice path goes from $(0, 0)$ to (m, n) what can we say about it? How many N's will it have? How many E's? Try an example if you're not sure!

With this information in hand, a lattice path is basically a binary sequence. Can you now find a set that lattice paths to subsets of?

4. Exercises

Exercise 4.1. Fix any $n > 1$ and any $1 \leq k < n$. Provide a bijection between the size k subsets of $[n]$ and the size $n - k$ subsets of $[n]$.

Exercise 4.2. Find a bijection between the whole numbers

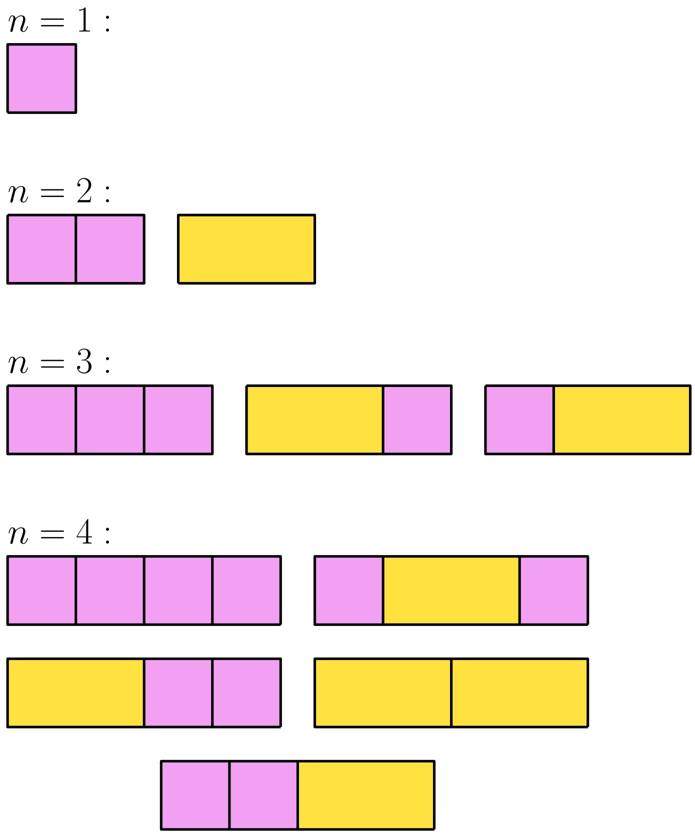
$$\mathbb{W} = \{0, 1, 2, 3, 4, \ldots\}$$

and the integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

You don't need to find a formula for your function, you just need to be able to explain a pattern to how each number maps.

Exercise 4.3. We will define two sets of objects depending on a choice of size n. Let $\mathcal{T}_n$ denote the number of tilings of a row of n boxes using tiles of size 1×1 and 1×2. For $n = 1, 2, 3, 4$, we have the following tilings.



On the other hand, let B_n be the set of length n binary sequences such that the 1's are socially distancing (1's in the sequences cannot be next to each other). For $n = 1, 2, 3$, we have the followings sequences:

$$n = 1 : \quad \{0, 1\}$$

$$n = 2 : \quad \{00, 10, 01\}$$

$$n = 3 : \quad \{000, 001, 010, 100, 101\}$$

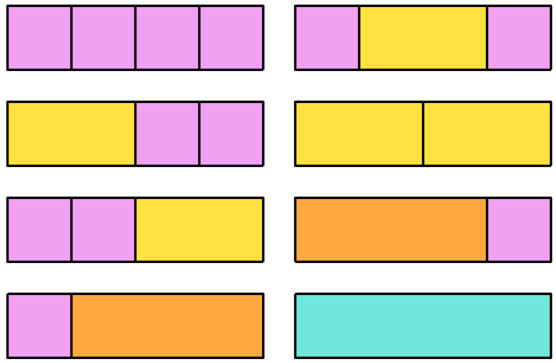
For each $n \geq 1$, can you find a bijection between B_n and $\mathcal{T}_{n+1}$?

Exercise 4.4. A *composition* of n is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive integers such that $\alpha_1 + \cdots + \alpha_k = n$. In this exercise, we will show that there 2^{n-1} compositions of n.

Can you find a bijection between compositions of n and tilings of n boxes using tiles of sizes $1 \times 1, 1 \times 2, \ldots, 1 \times n$? For example, the eight compositions of 4 are

$$(1, 1, 1, 1), (2, 1, 1), (1, 2, 1), (1, 1, 2), (2, 2), (3, 1), (1, 3), (4).$$

The tilings of 4 boxes using tiles of size $1 \times 1, 1 \times 2, 1 \times 3$, and 1×4 are



Now, find a bijection between tilings of n boxes using tiles of size $1 \times 1, 1 \times 2, \ldots, 1 \times n$ and binary sequences of length $n - 1$. Hint: try solving Exercise 4.3 first. Explain why this shows there are 2^{n-1} compositions of n.

If you know about binomial coefficients $\binom{n}{k}$, what can you say about their relationship to subsets of $[n]$ with a given size? What happens to compositions with a particular number of entries under your bijection? How many compositions of n into exactly k parts are there for each k?

Exercise 4.5. Let $\mathbb{R}$ be the set of all *real numbers*, numbers that are representable by decimals. For example, this includes the integers, the rational numbers (fractions), and infinite decimals like π or $\sqrt{2}$. Show that the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is a bijection. Is the map $g(x) = x^2$ a bijection also?

References

[1] R. P. Stanley. Bijective proof problems. `http://www-math.mit.edu/~rstan/bij.pdf`, 2009.