CATALAN NUMBERS

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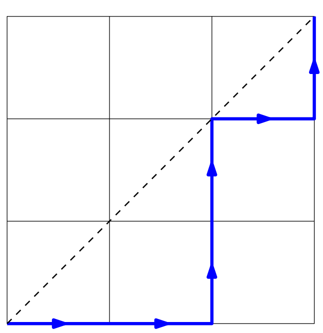
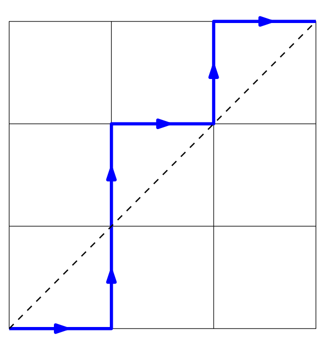
Contents

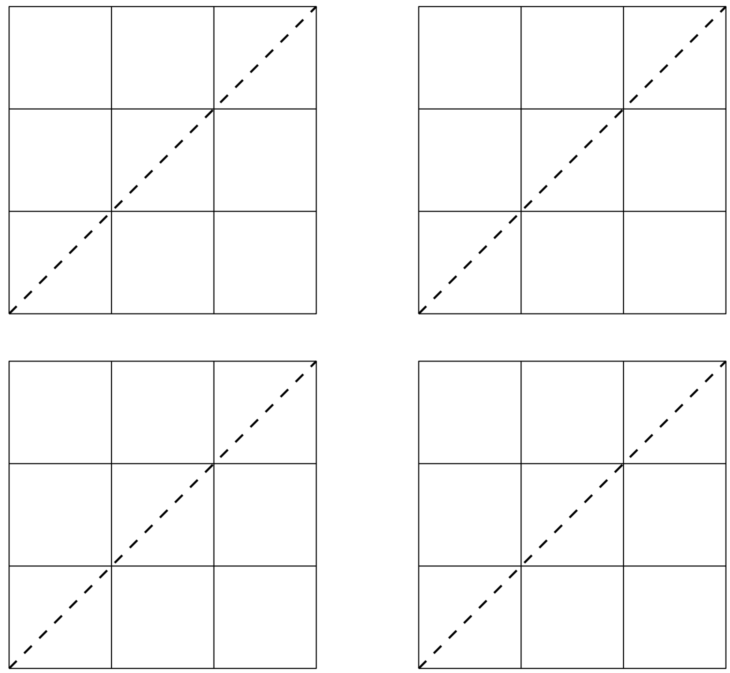
1. Catalan Numbers

This document aims to give the reader an introduction to Catalan numbers, a sequence of numbers arising throughout mathematics. We will explore some of the myriad of objects counted by these numbers using the machinery of bijections. We begin with one such family of objects called Dyck paths, and use them to get a formula for the Catalan numbers.

A *lattice path* is a sequence of length 1 east and north steps in the plane starting at $(0,0)$. A *Dyck path* is a lattice path that does not go above the line $y = x$. Shown below are two example lattice paths in the 3×3 grid. The one on the left is a Dyck path, but the one on the right is not.

Can you find the other four Dyck paths in the 3×3 grid?



Definition 1.1. For each $n \geq 1$, define the *Catalan number* C_n to be the number of Dyck paths in the $n \times n$ grid.

We will work toward giving a formula for C_n. First, recall the binomial coefficients.

Definition 1.2. For nonnegative integers n and k, the *binomial coefficient* $\binom{n}{k}$ is the number

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

which counts the number of ways to choose k objects at once out of a bag of n objects.

In other words, $\binom{n}{k}$ counts the number of size k subsets of $[n] = \{1, 2, \ldots, n\}$.

Example 1.3. The size 3 subsets of $[5] = \{1, 2, 3, 4, 5\}$ are

$$\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}.$$

Correspondingly,

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)(2 \times 1)} = 10$$

To find a formula for the Catalan numbers, we begin with the following fact:

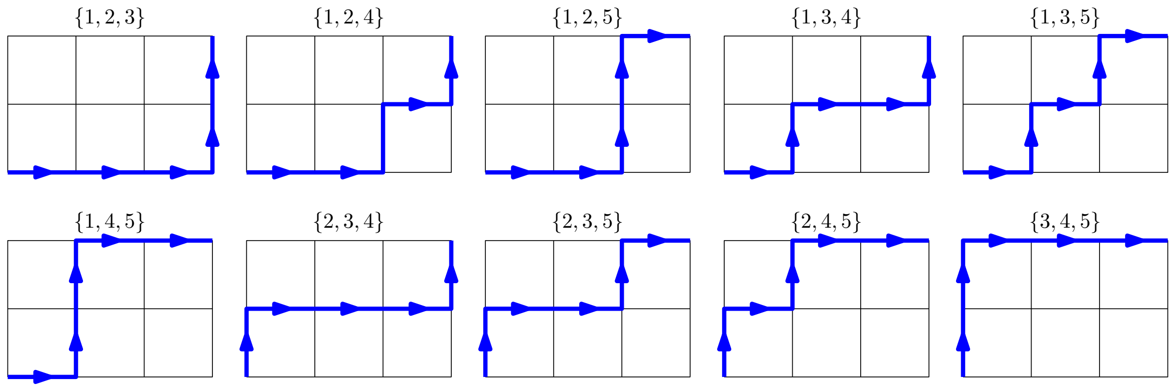
Lemma 1.4. *For any m and n, there are $\binom{m+n}{m}$ lattice paths from $(0,0)$ to (m,n) in the $m \times n$ grid.*

Proof. Think of a lattice path as a sequence of letters indicating the steps, N for north and E for east. To reach (m,n), a lattice path must have exactly m E's and n N's, but can have them in any order. We can biject paths to size m subsets of $[m+n]$. Think of $[m+n]$ as a line of empty boxes. In each box we can place either an E or an N. We think of E in box i as meaning "put i in the subset" and N as meaning "throw away i". For example,

$$NEENNNNEENE \text{ corresponds to the subset} \{2, 3, 8, 9, 11\} \text{ of } [11].$$

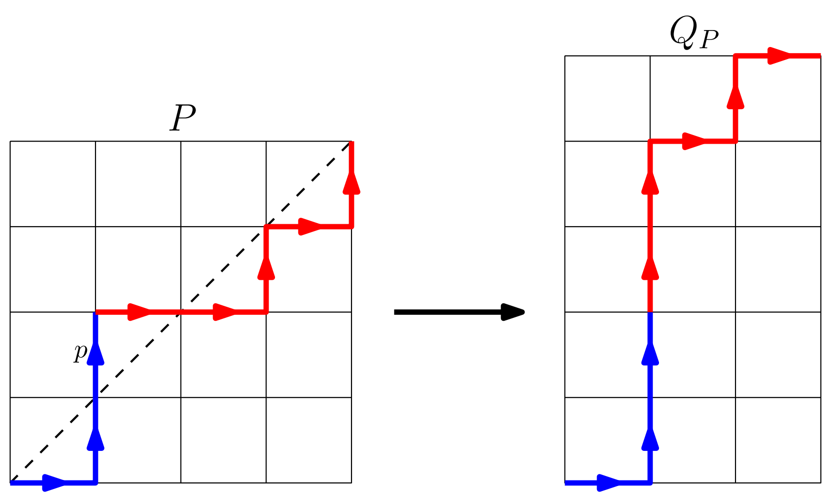
□

For example, for $(m,n) = (3,2)$, the bijection with size 3 subsets of $[5]$ is



Proposition 1.5. *The lattice paths in the $n \times n$ grid that cross the line $y = x$ are in bijection with the lattice paths in the $(n-1) \times (n+1)$ grid.*

Proof. Given a lattice path P in the $n \times n$ grid crossing the line $y = x$, we construct a lattice path Q_P in the $(n-1) \times (n+1)$ grid. Let p be the first step of P that crosses the line $y = x$. Let Q_P agree with P up to p, but after p swap the direction of each edge. For example:

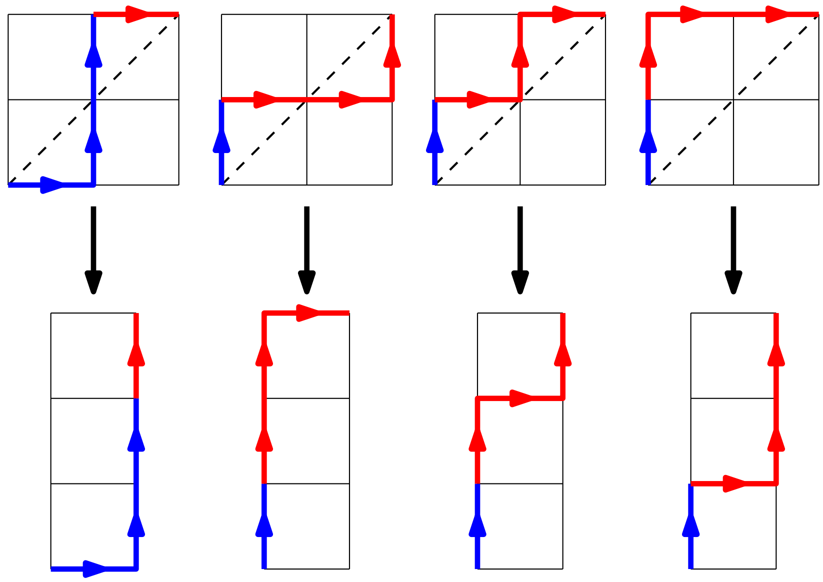


In the blue portion of P, there will always be exactly one more north steps than east steps by the definition of p. Since there are exactly n north steps and n east steps in P, there must be exactly one more east steps than north steps in the red portion of P.

Thus, both the blue and red portions of Q_P will have one more north steps than east steps. Thus, Q_P will have $n-1$ total east steps and $n+1$ total north steps. In other words, Q_P will be a lattice path in the $(n-1) \times (n+1)$ grid.

For the inverse bijection, just draw in the $n \times n$ diagonal in the $(n-1) \times (n+1)$ grid and do the same reflection again! □

For $n = 2$, there are four non-Dyck paths, and they biject to all lattice paths in the 1×3 grid:



We can now describe the Catalan numbers.

Theorem 1.6. *The Catalan numbers are given by the formula*

$$C_n = \binom{2n}{n} - \binom{2n}{n-1}$$

Proof. C_n counts the number of lattice paths in the $n \times n$ grid that do not cross the line $y = x$. There are $\binom{2n}{n}$ total lattice paths. Of these, there are the same number as bad paths as there are lattice paths in the $(n-1) \times (n+1)$ grid. That is, there are $\binom{2n}{n-1}$. Hence,

$$C_n = \#\text{ of total paths} - \#\text{ of bad paths} = \binom{2n}{n} - \binom{2n}{n-1}.$$

□

The first few Catalan numbers are $C_0 = 1$ (basically a definition), $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $C_5 = 42$. What is so special about these numbers, you might ask. It turns out they count many, many completely different objects! We will look at a handful of these in the next section.

2. Objects Counted by Catalan Numbers

In this section, introduce various families of mathematical objects and give bijections between them to show that they are counted by Catalan numbers. We begin with ballot sequences.

2.1. **Ballot Sequences.** A *ballot sequence* of length $2n$ is a sequence with n -1's and n 1's such that every partial sum is nonnegative. Equivalently, as you read the sequence left-to-right, you always encounter at least as many $1's$ as -1's. For example, $(1, -1, 1, 1, -1, -1, -1, 1)$ has partial sums 1, 0, 1, 2, 1, 0, -1, and 0, so it is not a ballot sequence. For $n = 3$, there are five ballot sequences:

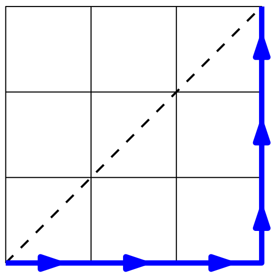
$$(1,1,1,-1,-1,-1) \quad (1,1,-1,1,-1,-1) \quad (1,1,-1,-1,1,-1)$$
$$(1,-1,1,1,-1,-1) \quad (1,-1,1,-1,1,-1)$$

We show that ballot sequences are counted by Catalan numbers through a bijection to Dyck paths!

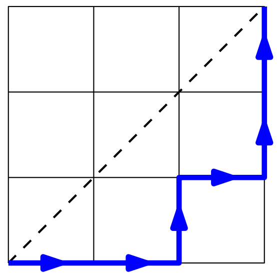
Proposition 2.1. *There are C_n ballot sequences of length $2n$.*

Proof. Given a ballot sequence b, replace each 1 by E and each -1 by N to get a lattice path P. We claim P is actually a Dyck path. Since the partial sums of b are nonnegative, in any initial segment of b there are at least as many 1's as -1's. Then in each initial segment of D, there are at least as many E's as N's. Thus, D never goes above the line $y = x$. Reversing this process gives a map from Dyck paths to ballot sequences. ☐

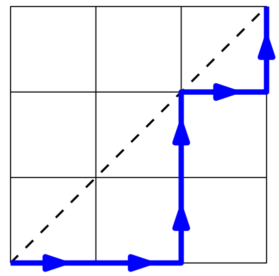
When $n = 3$, we get the bijection:



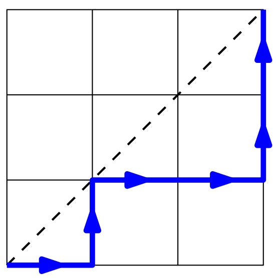
$(1, 1, 1, -1, -1, -1)$



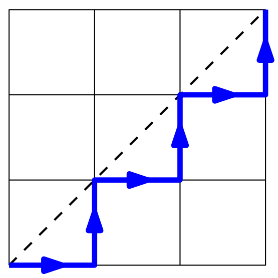
$(1, 1, -1, 1, -1, -1)$



$(1, 1, -1, -1, 1, -1)$



$(1, -1, 1, 1, -1, -1)$



$(1, -1, 1, -1, 1, -1)$

2.2. **Parenthesizations.** Fix a string of $n + 1$ letters. A *parenthesization* is an insertion of $n - 1$ open parentheses and $n - 1$ close parentheses so that the resulting expression only involves products of two adjacent things at a time. For $n = 2$, we are looking at the string abc and inserting one pair of parentheses. This can be done in two ways:

$$a(bc) \quad \text{and} \quad (ab)c.$$

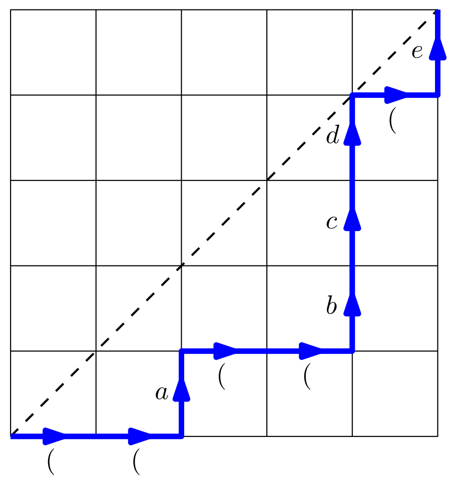
Can you find five parenthesizations of $abcd$?

Proposition 2.2. *There are C_n parenthesizations of $n + 1$ letters.*

Proof. We give a bijection to Dyck paths. Take a parenthesization of $n+1$ letters and remove all the right parentheses and the very last letter to get a string of length $2n-1$ consisting of $n-1$ open parentheses and the first n letters. Note that this is bijective since we can recover the full parenthesization from this information. Try an example to see how you do this:

Add a '(' to the beginning of the string. To get a Dyck path, replace each '(' by E and each letter by N. For example,

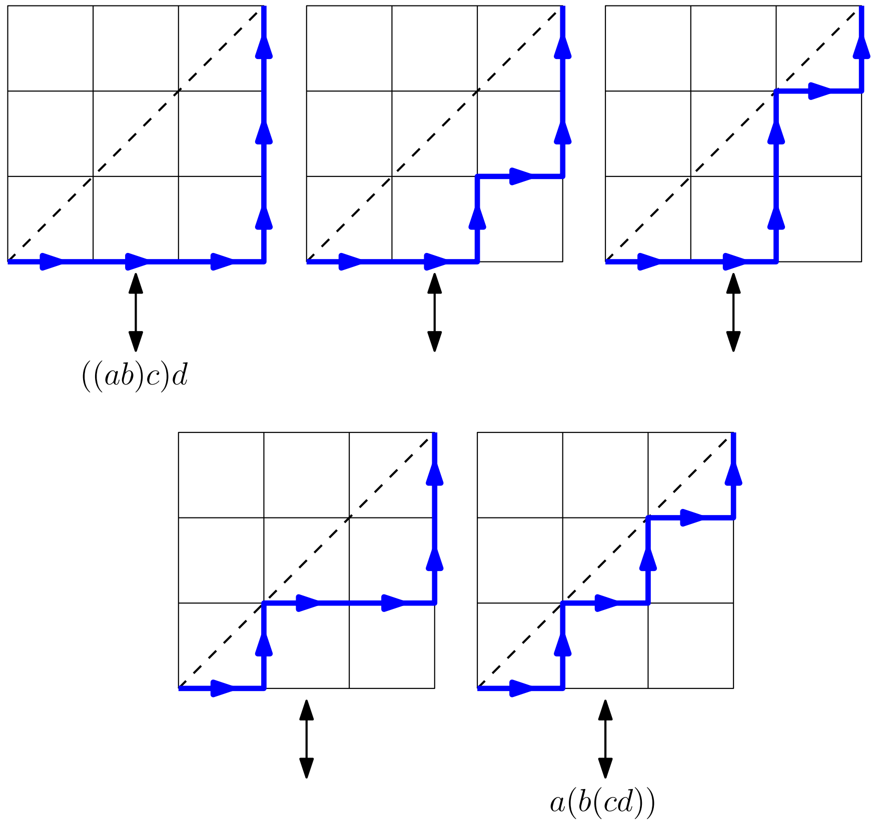
$(a((bc)d))(ef)$ $\longrightarrow$



For this not to be a Dyck path, at some point prior to the end of the path there would have to be more N steps than E steps. Then at the corresponding point in the original parenthesization (which will be before the last letter), there would have been at least two more letters than '('s (two, since we added a '(' while building the path). In any valid parenthesization, there can only be at most one more letter than open parenthesis at any point, otherwise we will end up trying to multiply three or more things at once. To convince yourself of this, try writing down 6 letters and only 4 open (and closed if you want) parentheses in various ways in the space below. Can you make it so that at least two more letters than open parentheses occur prior to the last letter? What goes wrong?

This completes a map from parenthesizations to Dyck paths. Explain how you can reverse this map to go from Dyck paths to parenthesizations. This will complete the bijective proof. ∎

Using the five parenthesizations of $abcd$ you found earlier, complete the $n = 3$ bijection:

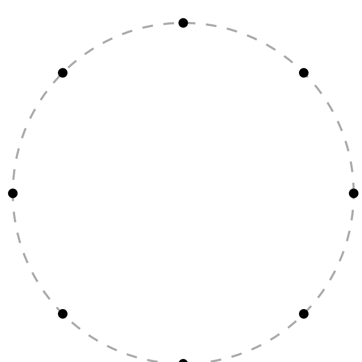


3. Exercises

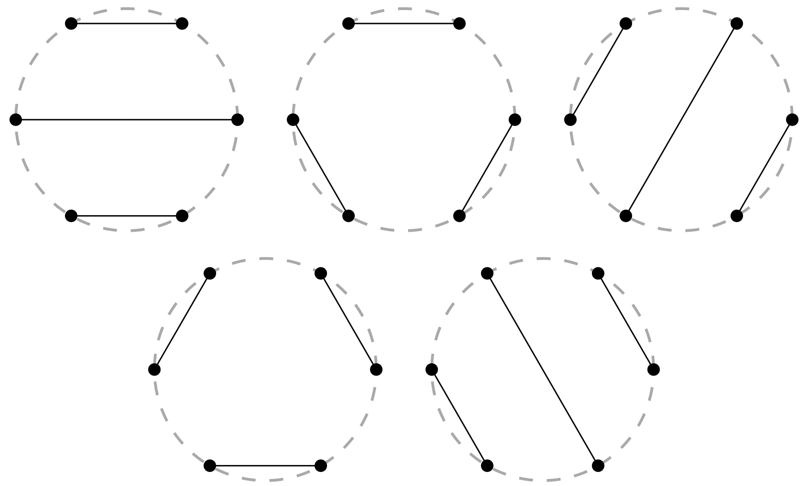
For the exercises, we will define more objects counted by Catalan numbers and ask you to find bijections to prove it. For each type of object, we will show you the case $n = 3$ for you to play around with.

3.1. **Nonintersecting Chords.**

Imagine $2n$ points in the plane lying on a circle.



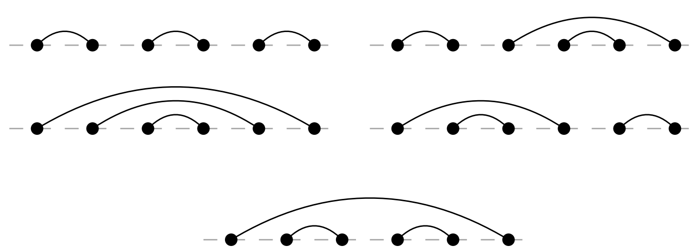
Consider all the ways of connecting the $2n$ points by n chords (line segments lying inside the circle) such that no two chords ever cross. For $n = 3$, there are five ways to do this:



Exercise 3.1. For each n, find a bijection between the arrangements of nonintersecting chords on $2n$ points and the Dyck paths in the $n \times n$ grid. Use your bijection to conclude that there are C_n arrangements of nonintersecting chords on $2n$ points for each n.

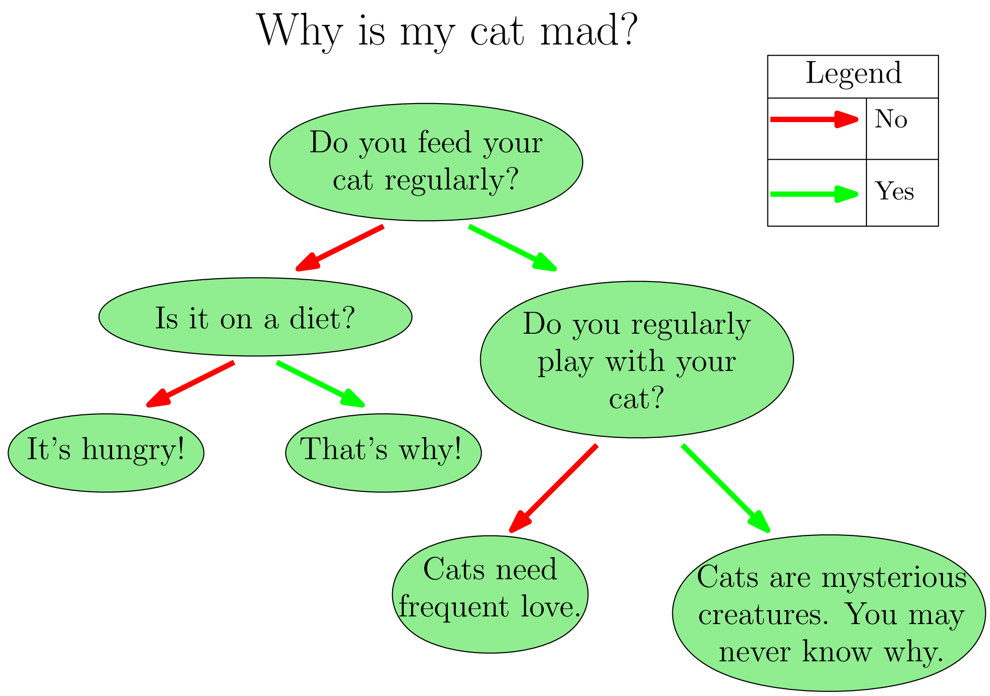
3.2. **Noncrossing Arcs.** Imagine $2n$ points in the plane lying on a horizontal line.

Consider all the ways of connecting the $2n$ points by n arcs lying above the line such that each arc connects exactly two points and no arcs ever cross. For $n = 3$, there are five ways to do this:



Exercise 3.2. For each n, find a bijection between the arrangements of noncrossing arcs on $2n$ points and the arrangements of nonintersecting chords on $2n$ points. Use your bijection to conclude that there are C_n arrangements of noncrossing arcs on $2n$ points for each n.

3.3. **Binary and Full Binary Trees.** Binary trees are a type of graph (think of a diagram consisting of points and lines, not the graph of a function or a data plot) that represent a sequence of yes/no choices. Consider the following feline diagnostic tool as a motivating example.

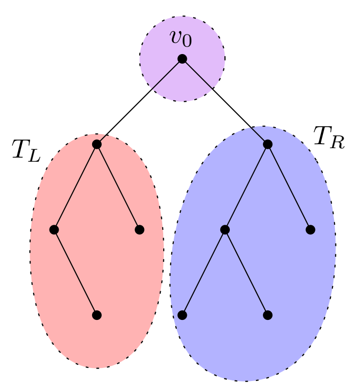


You begin at the top of the graph. At each point, you have at most two choices for how to answer. Each choice takes you further down the graph. When you reach a point with no more choices, you stop.

We now formally describe graphs with this kind of structure, called binary trees. The definition works recursively, defining a binary tree of a given size in terms of smaller binary trees. In general, it is not required that there are exactly two choices at each vertex. Binary trees with this property are called full binary trees.

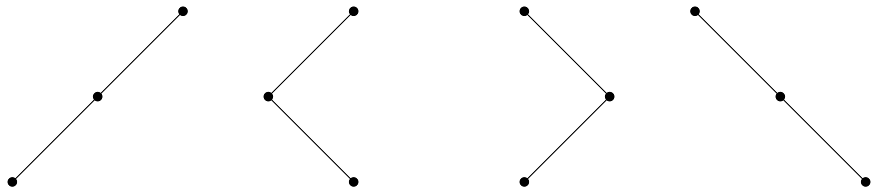
Recall that a graph is a set of points, called *vertices*, and line segments between them, called *edges*. *Binary trees* are the members of a family of graphs defined recursively:

- The empty graph (no vertices or edges) is a binary tree.
- A nonempty binary tree consists of a root vertex v_0, a left-subtree T_L, and a right subtree T_R such that both T_L and T_R are binary trees, and v_0 has an edge to the roots of both.



Exercise 3.3. Explain in the example above why T_L is itself a binary tree. Where is its root? Why are its left and right subtrees binary trees?

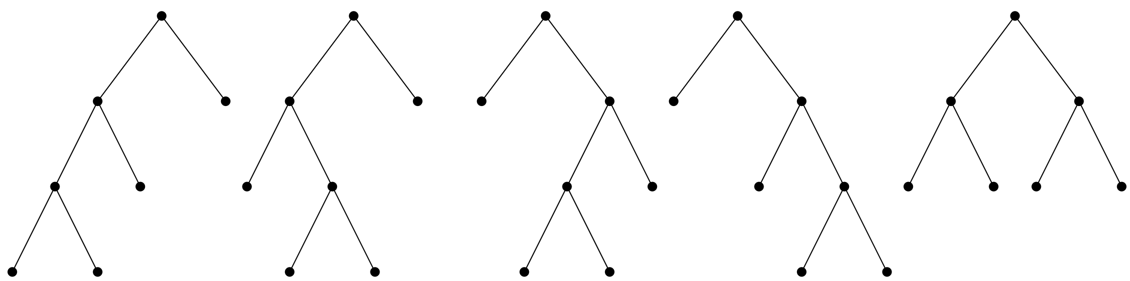
Here are the five binary trees on 3 vertices:



Let T be a binary tree, and fix a vertex v in T. The vertices that have an edge with v and are farther away from the root vertex of T are called the *children* of v.

Exercise 3.4. Prove that in a binary tree, each vertex has 0, 1, or 2 children.

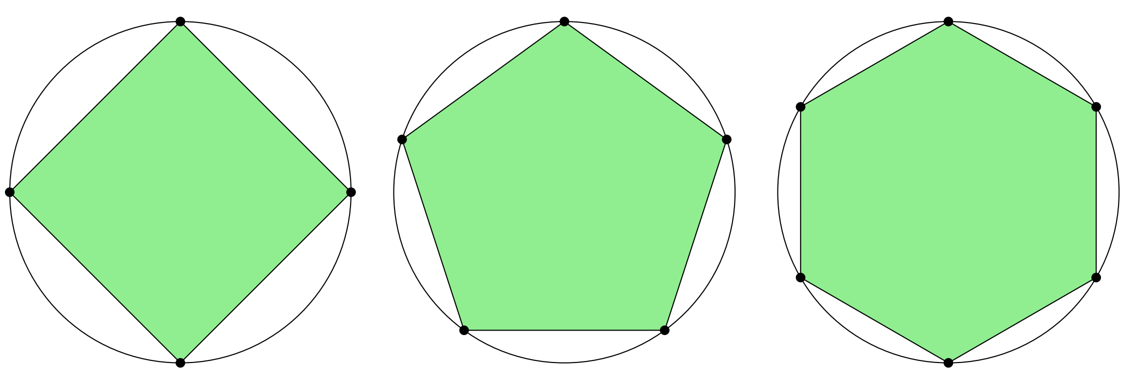
A *full binary tree* is a binary tree where each vertex has either zero or two children. Here are the five full binary trees on 7 vertices:



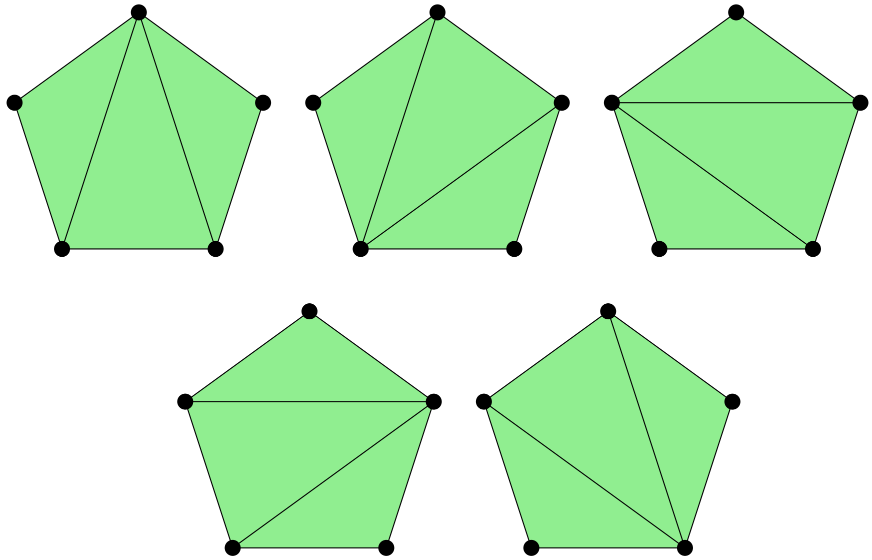
Exercise 3.5. For each n, find a bijection between the full binary trees on $2n+1$ vertices and the parenthesizations of $n+1$ letters. Use your bijection to conclude that there are C_n full binary trees on $2n+1$ vertices for each n.

Exercise 3.6. Find a bijection between the full binary trees on $2n+1$ vertices and the binary trees on n vertices.

3.4. **Triangulations.** An n-sided polygon, or n-*gon* for short, is obtained by connecting n distinct points (again called *vertices*) on a circle with line segments to their closest neighbors, and then filling in the enclosed region. Here are n-gons for $n = 4, 5, 6$:

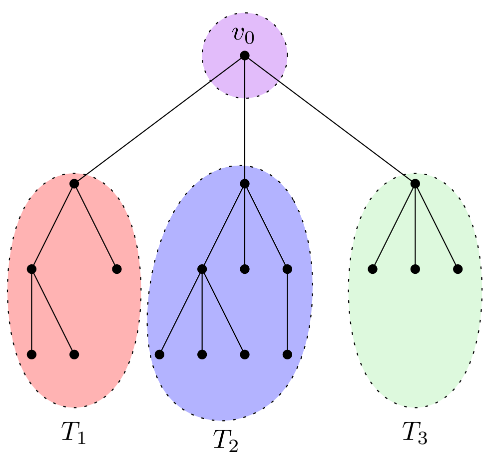


A *diagonal* of an n-gon is a line segment between two vertices that are not neighbors on the circle. A *triangulation* of an n-gon is a set of $n-3$ diagonals that don't intersect inside the polygon and that chop the polygon up into triangles. As you may be expecting by now, there are five triangulations of a 5-gon:



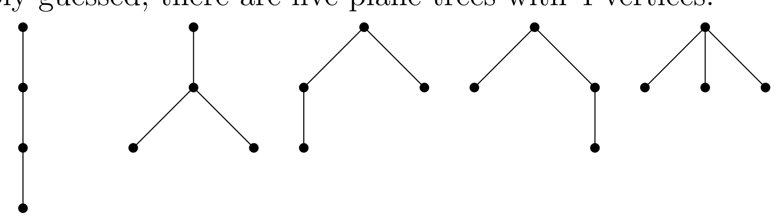
Exercise 3.7. Find a bijection between triangulations of n-gons and either binary trees on n vertices or full binary trees on $2n+1$ vertices. Conclude that there are C_{n-2} triangulations of an n-gon.

3.5. **Plane Trees.** Similar to binary trees, *plane trees* are the members of a family of graphs defined recursively. Every plane tree P has a root vertex v_0 (and so is nonempty). Either this is all of T, or T has subtrees $T_1, \ldots, T_k$ all of which are plane trees whose roots have an edge to v_0.



Exercise 3.8. Explain in this example why T_2 is itself a plane tree. Where is its root? What are its subtrees, and why are they plane trees?

As you probably guessed, there are five plane trees with 4 vertices:

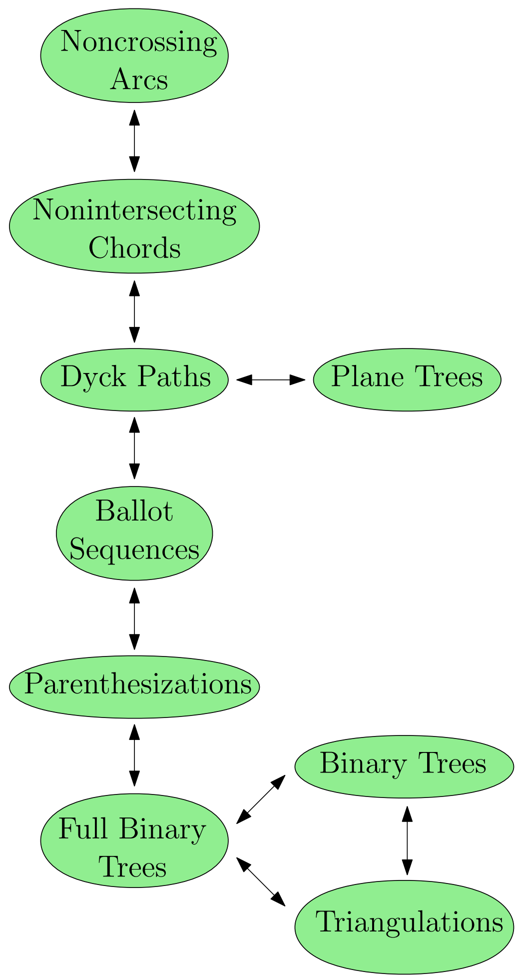


Exercise 3.9. Find a bijection between plane trees with $n+1$ vertices and Dyck paths in the $n \times n$ grid.

Exercise 3.10. Find a bijection between plane trees with $n+1$ vertices and binary trees with n vertices.

4. Even More Catalan!

So far, we have the following chart of objects and bijections:



Can you add any more bijections to this chart?

For more Catalan objects than you can imagine, the book [2] describes over 200 different objects counted by the Catalan numbers!

References

[1] Marc Renault. Four proofs of the ballot theorem. *Mathematics Magazine*, 80, 12 2007.
[2] R. P. Stanley. *Catalan Numbers*. Cambridge University Press, 2015.