RECURRENCES

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This document aims to give the reader an introduction to the use of recurrence relations in combinatorics. Recurrences can be used to count families of objects that depend on an integer parameter. We will define recurrences and use them to explore various objects. We begin by returning to subsets of $[n]$.

1. A Recurrence for Subsets

Recall the set $[n] = \{1, 2, 3, \ldots, n\}$. A *subset* of $[n]$ is a collection of these numbers with no repeats. For instance, if $n = 2$, then we could take $\{1\}$, $\{2\}$, $\{1, 2\}$, or we could take neither 1 nor 2, giving the *empty set*, denoted $\emptyset$. So for $n = 2$, there are 4 subsets.

How many subsets of $[n]$ are there in general? We are going to count them in a different way than you might have seen before. Let a_n denote the number of subsets of $[n]$. Our strategy is to relate a_n to a_{n-1} and work backwards to figure out what a_n is.

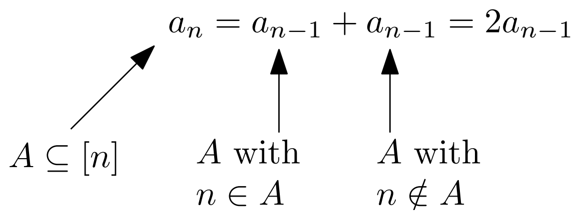
The trick is to make an obvious statement: Given a subset A of $[n]$, either $n \in A$ (n is in A) or $n \notin A$ (n is not in A). Exactly one of these must be true. For example, when $n = 3$ we have

$$\begin{array}{rr} \emptyset, & \{3\}, \\ \{1\}, & \{1,3\}, \\ \{2\}, & \{2,3\}, \\ \{1,2\}, & \{1,2,3\}. \end{array}$$

Let's now look at both cases.

If $n \in A$, then A consists of a subset of $[n-1]$ together with the element n. This shows that the subsets of $[n]$ which contain n are in bijection with the subsets of $[n-1]$ via adding or removing n. Consequently, there are a_{n-1} subsets of $[n]$ containing n.

On the other hand, the subsets of $[n]$ not containing n are exactly the subsets of $[n-1]$, so there are a_{n-1} of these as well. Whatever a_n may be, we have shown that



The nice thing is that we were nonspecific about which n we were using, so what we proved should hold for all appropriate n. Replacing n by $n-1$, we get

$$a_n = 2a_{n-1} = (2 \times 2)a_{n-2}.$$

What if we keep going like this? What are the limits of this line of thinking? Is it true that

$$a_2 = 2a_1 = (2 \times 2)a_0 = (2 \times 2 \times 2)a_{-1} = \cdots = 2^{100}a_{-98}?$$

Our original question "How many subsets of $[n]$ are there?" only makes sense when n is nonnegative, so negative indices do not make sense. So the best we can do is say

$$a_n = 2a_{n-1} = 2^2a_{n-2} = 2^3a_{n-3} = \cdots 2^{n-1}a_1 = 2^na_0.$$

Okay, so if we know a_0, then we know a_n for any $n > 0$. What is a_0?

Well, by definition a_0 is the number of subsets of $[0]$, a set with no elements (so $\emptyset$). Since $[0] = \emptyset$ has no elements, the only subset it can have is itself. Thus $a_0 = 1$, and $a_n = 2^n$ for all $n \geq 0$.

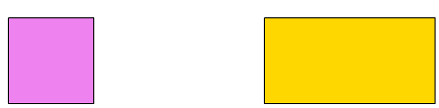
The trick behind this solution method was to relate the subsets of $[n]$ with subsets of a smaller set, in this case $[n-1]$. With this in mind, we make the following definition.

Definition 1.1. Given a sequence of numbers $b_0, b_1, b_2, \ldots$, a *recurrence relation* is an equation relating b_n to any of $b_{n-1}, b_{n-2}, \ldots, b_{n-k}$ for some fixed k and each $n \geq k$.

Recurrence relations can indicate internal structure within an object of size n in some collection of objects. There are many different types of recurrence relations that arise, and many approaches to solving them explicitly. We will focus mostly on finding and explaining recurrences in this document. We first turn our attention to a famous recurrence.

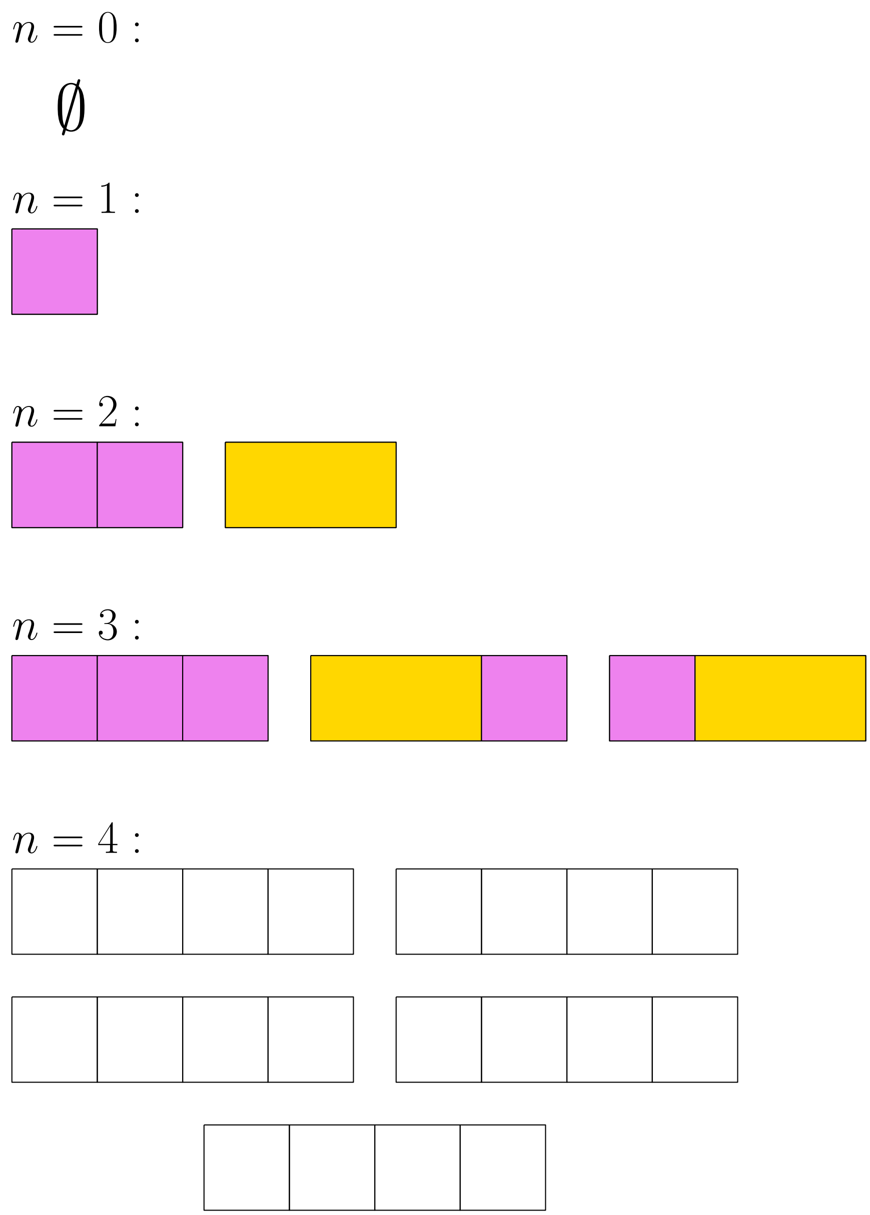
2. Fibonacci Numbers

Imagine a line of n adjacent boxes. Consider two types of domino tiles, a 1×1 square and a 1×2 rectangle.



1×1 tile $\qquad$ 1×2 tile

How many ways are there to exactly cover the n boxes using the two types of tiles? This number is called the nth Fibonacci number, and is denoted F_n. The cases for $n \leq 3$ are shown below. Can you fill in the five tilings of $n = 4$ boxes?



So far we have $F_0 = 1$ (the only way to tile zero boxes is by doing nothing), $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, and $F_4 = 5$. If we were to play around some more, we would find $F_5 = 8$, $F_6 = 13$, and $F_7 = 21$. That's 1, 1, 2, 3, 5, 8, 13, 21. Do you see a pattern in these numbers? Can you write your pattern as a recurrence relation for F_n? When you have your guess in hand, turn over to see the solution.

Theorem 2.1. *The Fibonacci numbers satisfy the recurrence*

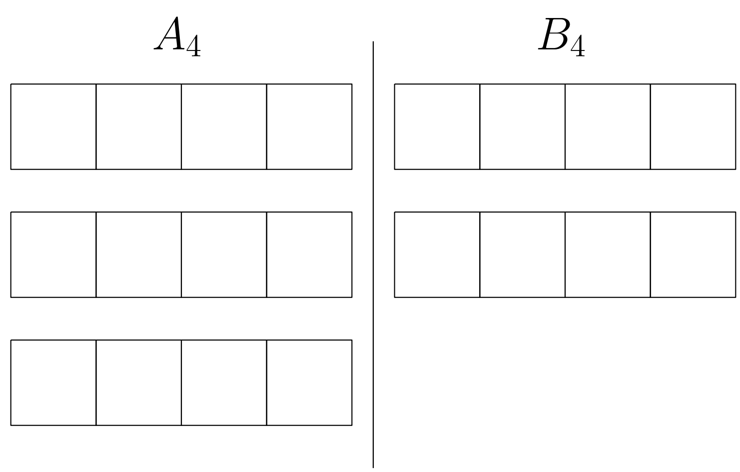
$$F_n = F_{n-1} + F_{n-2} \text{ for } n > 1$$

with $F_0 = 1$ *and* $F_1 = 1$.

How do we go about proving this? The key is to bring it from the level of numbers of tilings to the level of the tilings themselves. The goal is to split the tilings of n boxes into two separate groups: a group that is essentially the same as the tilings of $n-1$ boxes, and a group that is essentially the same as the tilings of $n-2$ boxes. Counting the groups will exactly give the result.

Proof. Let $\mathcal{T}_n$ denote the set of tilings of n boxes by 1×1 and 1×2 tiles, so F_n is the number of elements of $\mathcal{T}_n$. Separate the tilings into sets A_n and B_n as follows: A_n will consist of all tilings whose leftmost tile is 1×1 and B_n will consist of all tilings whose leftmost tile is 1×2.

For $n = 4$, draw in the tilings from before in the appropriate set:



Clearly, A_n and B_n have no common members, so

$$F_n = |\mathcal{T}_n| = |A_n| + |B_n|.$$

To complete the proof, can you find bijections from A_n to $\mathcal{T}_{n-1}$ and from B_n to $\mathcal{T}_{n-2}$? This will show that

$$|A_n| = |\mathcal{T}_{n-1}| = F_{n-1} \quad \text{and} \quad |B_n| = |\mathcal{T}_{n-2}| = F_{n-2}.$$

□

3. Catalan Numbers

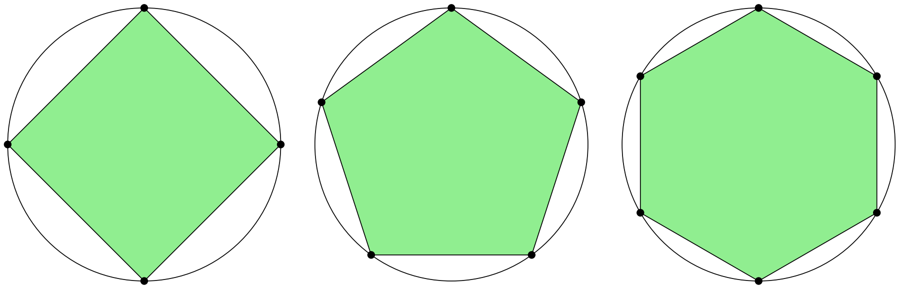
We now revisit the Catalan numbers C_n. Recall that C_n counts many many things, and is given by the formula

$$C_n = \binom{2n}{n} - \binom{2n}{n-1},$$

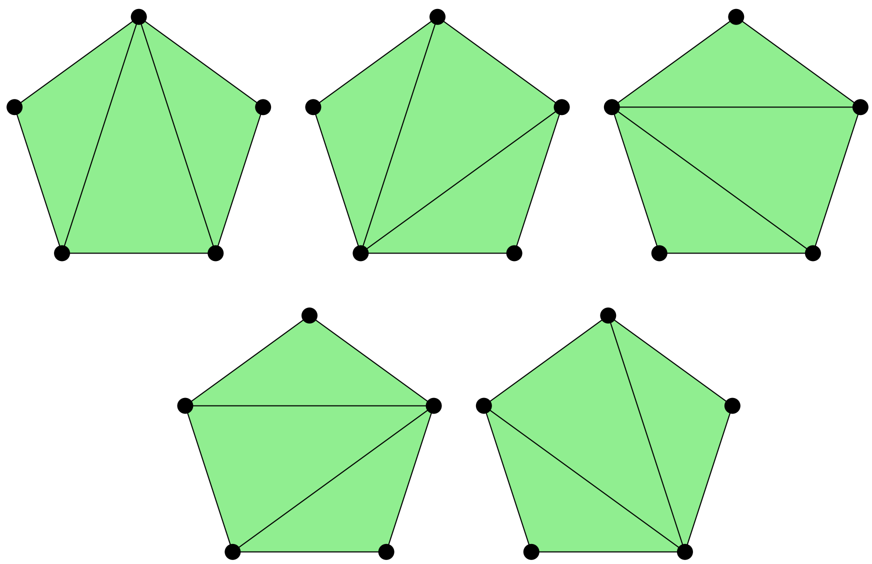
which we discussed by counting Dyck paths. For example, the first few Catalan numbers are $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $C_5 = 42$, $C_6 = 132$, $C_7 = 429$, $C_8 = 1430$, and $C_9 = 4862$. Additionally, even though $n = 0$ does not work with the formula, C_0 is just defined to be 1 (the recurrence later will show why this is a good choice).

Do you see a pattern with these numbers? Their pattern is much more involved. We will find a recursion satisfied by the Catalan numbers. Since we have so many objects to choose from, we will work with triangulations first. We will ask you to reinterpret the recurrence later for different objects in an exercise.

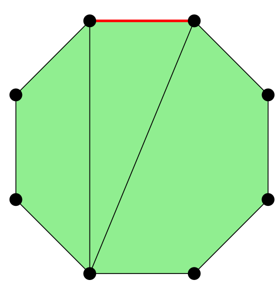
Recall that an n-sided polygon (n-*gon* for short) is obtained by connecting n distinct points (called *vertices*) on a circle with line segments to their closest neighbors, and then filling in the enclosed region. Here are n-gons for $n = 4, 5, 6$:



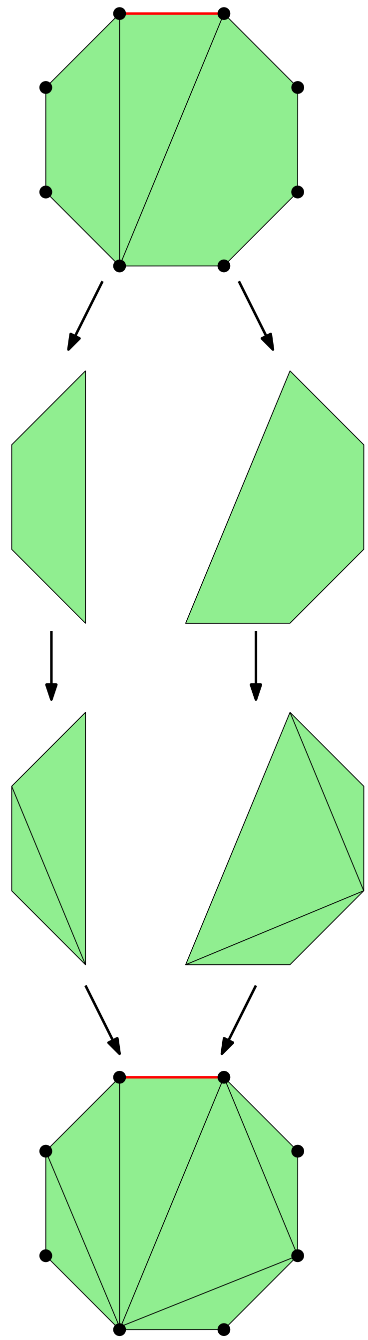
A *diagonal* of an n-gon is a line segment between two vertices that are not neighbors on the circle. A *triangulation* of an n-gon is a set of $n-3$ diagonals that don't intersect inside the polygon and that chop the polygon up into triangles. There are C_{n-2} triangulations of an n-gon. Recall $C_3 = 5$:



So how do we go about trying to find a recurrence? For the Fibonacci numbers, we broke the objects into groups and identified each group with smaller objects. We will do something similar here. Start with an n-gon and imagine placing diagonals so that the top edge is in a triangle:



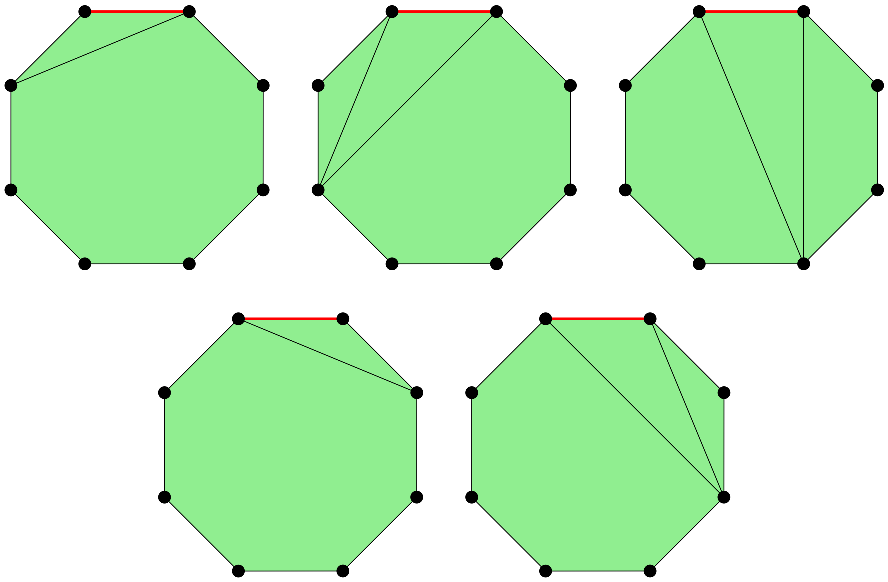
How many ways are there to finish the triangulation of this 8-gon? The key insight is that no diagonal can cross the triangle you made. So if you took some scissors and the cut out your first triangle, there would be no harm in triangulating the smaller polygons that came from the cut however you wanted and then gluing them back together.



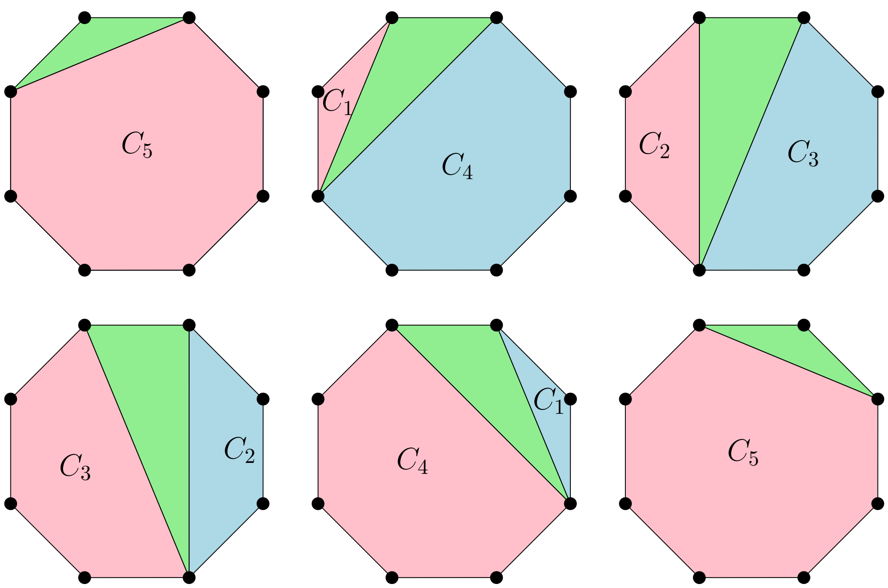
So how many ways were there to finish this triangulation? Well, one for each pair of ways to triangulate a 4-gon and a 5-gon! There are C_2 ways to triangulate a 4-gon and C_3 ways to triangulate a 5-gon, so if we insisted on starting with this particular first triangle, we could obtain

$$C_2 \times C_3 = 2 \times 5 = 10$$

triangulations of our original 8-gon! That's not all the triangulations of the 8-gon though. What if we had picked a different first triangle using that top edge?



None of these options will have any triangulations in common: they all do something different with the top edge! So how do we count up all the possibilities from each of these cases? Well,



Thus

$$\begin{aligned} C_6 &= C_5 + C_1C_4 + C_2C_3 + C_3C_2 + C_4C_1 + C_5 \\ &= 42 + (1 \times 14) + (2 \times 5) + (5 \times 2) + (14 \times 1) + 42 \\ &= 132. \end{aligned}$$

As a reality check, this agrees with our earlier formula:

$$C_6 = \binom{2 \times 6}{6} - \binom{2 \times 6}{5} = 924 - 792 = 132.$$

The ideas we used above to work through an 8-gon can be applied to any n, giving the following recurrence. To make the formula prettier, we use here the definition we made earlier of $C_0 = 1$:

Theorem 3.1. *The Catalan numbers C_n for $n > 0$ satisfy the recurrence*

$$C_n = C_0C_{n-1} + C_1C_{n-2} + C_2C_{n-3} + \cdots + C_{n-2}C_1 + C_0C_{n-1}$$

4. Exercises

Exercise 4.1. A *permutation* of $[n]$ is an arrangement of $1, 2, \ldots, n$ in any order. For instance, there are are 6 permutations of [3], namely 123, 132, 213, 231, 312, and 321. Let a_n denote the number of permutations of $[n]$, with $a_1 = 1$. Prove that $a_n = na_{n-1}$ for all $n \geq 1$ and use this to find a formula for a_n.

Exercise 4.2. Recall the binomial coefficients

$$\binom{n}{k} = \# \text{ subsets of } [n] \text{ with size } k.$$

Without using the formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

which you might have seen previously, prove the recurrence relation

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Exercise 4.3. A *set partition* of $[n]$ is a decomposition of $[n]$ into disjoint nonempty subsets. For instance

$$[8] = \{1,3\} \cup \{2,6,7\} \cup \{4\} \cup \{5,8\}$$

is a set partition of [8]. You can think of the partition as a set of subsets $\{\{1,3\},\{2,6,7\},\{4\},\{5,8\}\}$, since the order of the subsets does not matter. The subsets in the decomposition are called *blocks* of the partition.

Denote by $S(n,k)$ the number of set partitions of $[n]$ into exactly k blocks, with $S(0,0)$ defined to be 1. For example, $S(4,2) = 3$ since the set partitions of [4] with two parts are

$$\{\{1,2\},\{3,4\}\}, \quad \{\{1,3\},\{2,4\}\}, \quad \text{and} \quad \{\{1,4\},\{2,3\}\}.$$

Prove that for all $n, k \geq 1$,

$$S(n,k) = kS(n-1,k) + S(n-1,k-1).$$

Exercise 4.4. Let $B(n)$ denote the total number of set partitions of $[n]$, with $B(0)$ defined to be 1. In other words,

$$B(n) = S(n,0) + S(n,1) + S(n,2) + \cdots + S(n,n).$$

For example, $B(4) = 15$:

$$\begin{gathered}
\{\{1\},\{2\},\{3\},\{4\}\}, \\
\{\{1,2\},\{3\},\{4\}\}, \{\{1,3\},\{2\},\{4\}\}, \{\{1,4\},\{2\},\{3\}\}, \\
\{\{1\},\{2,3\},\{4\}\}, \{\{1\},\{2,4\},\{3\}\}, \{\{1\},\{2\},\{3,4\}\}, \\
\{\{1,2\},\{3,4\}\}, \{\{1,3\},\{2,4\}\}, \{\{1,4\},\{2,3\}\}, \\
\{\{1,2,3\},\{4\}\}, \{\{1,3,4\},\{2\}\}, \{\{1,2,4\},\{3\}\}, \{\{2,3,4\},\{2\}\}, \\
\{\{1,2,3,4\}\}
\end{gathered}$$

Prove that

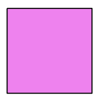
$$B(n) = 1 + \binom{n-1}{1} B(1) + \binom{n-1}{2} B(2) + \cdots + \binom{n-1}{n-1} B(n-1)$$

for $n \geq 1$.

Exercise 4.5. A *partition* of a positive integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. For example, $(4, 3, 1, 1)$ and $(3, 3, 3)$ are both partitions of 9. The entries of a partition are called *parts*. Let $p_k(n)$ denote the number of partitions of n with exactly k parts. For example, $p_3(5) = 2$ since the only partitions of 5 are (5), $(4, 1)$, $(3, 2)$, $(3, 1, 1)$, $(2, 2, 1)$, $(2, 1, 1, 1)$, and $(1, 1, 1, 1, 1)$. By convention, $p_0(0)$ is defined to be 1. Prove the recurrence relation

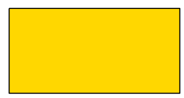
$$p_k(n) = p_{k-1}(n-1) + p_k(n-k).$$

Exercise 4.6. Recall the nth Fibonacci number F_n is the number of ways to tile a line of n adjacent boxes with the tiles



1×1 tile

and



1×2 tile

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Earlier, we proved the recurrence relation $F_n = F_{n-1} + F_{n-2}$. Using similar reasoning about tilings, prove the recurrence relation

$$F_n - 1 = F_1 + F_2 + \cdots + F_{n-2}.$$

Exercise 4.7. We proved the Catalan recurrence by working with triangulations of polygons, which were counted by Catalan numbers. What about all the other things counted by Catalan numbers? Try to prove the recurrence again using one of the other objects counted by Catalan numbers (for example Dyck paths, ballot sequences, parenthesizations…).

References

[1] R. P. Stanley. *Enumerative Combinatorics*, volume 1 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, second edition, 2011.