Mathematics 4530

Ordinal numbers and the well-ordering theorem

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The ordinal numbers form an extension of the natural numbers. Here are the first few of them:

$$0, 1, 2, \dots, \omega, \omega + 1, \omega + 2, \dots, \omega + \omega =: \omega 2, \omega 2 + 1, \dots, \omega 3, \dots, \omega^2, \dots .$$

They go on forever. As soon as some initial segment of ordinals has been constructed, a new one is adjoined that is bigger than all of them.

The theory of ordinals is closely related to the theory of well-ordered sets (see Section 10 of Munkres). Recall that a simply ordered set X is said to be *well-ordered* if every nonempty subset Y has a smallest element, i.e., an element y_0 such that $y_0 \leq y$ for all $y \in Y$. For example, the following sets are well-ordered:

$$\emptyset,\ \{1\}\,,\ \{1,2\}\,,\ \{1,2,\dots,n\}\,,\ \{1,2,\dots\}\,,\ \{1,2,\dots,\omega\}\,,\ \{1,2,\dots,\omega,\omega+1\}\,.$$

On the other hand, $\mathbb{Z}$ is not well-ordered in its natural ordering.

As the list of examples suggests, constructing arbitrarily large well-ordered sets is essentially the same as constructing the system of ordinal numbers. Rather than taking the time to develop the theory of ordinals, we will concentrate on well-ordered sets in what follows.

Notation. In dealing with ordered sets in what follows we will often use notation such as

$$X_{<x} := \{y \in X \mid y < x\}\,.$$

A set of the form $X_{<x}$ is called a *section* of X.

Well-orderings are useful because they allow proofs by induction:

Induction principle. *Let X be a well-ordered set and let Y be a subset. Suppose that for all $x \in X$,*

$$(*) \qquad X_{<x} \subseteq Y \implies x \in Y.$$

Then $Y = X$.

Proof. Suppose $Y \neq X$. Then there is a smallest $x \in X \setminus Y$. This means that every $y < x$ is in Y, contradicting $(*)$. □

Remark. The way this is used in practice is that one has a statement $\mathcal{S}_x$ involving a variable $x \in X$, and Y is the set of $x \in X$ for which the statement is true. The hypothesis $(*)$ then becomes

$$\mathcal{S}_y \text{ for all } y < x \implies \mathcal{S}_x,$$

and the conclusion is that $\mathcal{S}_x$ is true for all $x \in X$.

The main result of this handout is the *well-ordering theorem*:

Theorem. *Every set X admits a well-ordering.*

The idea behind the construction of a well-ordering is the following: Choose an arbitrary $x_0 \in X$ to be the first element in the desired ordering. Now choose $x_1 \in X \setminus \{x_0\}$ to be next in the ordering. Continuing in this way, suppose we have already constructed some initial segment A of the desired ordering; we then choose

$x \in X \setminus A$ and make it bigger than everything in A. (This is reminiscent of the informal description of ordinals above.) Keep going until all of X is exhausted.

The last sentence is where most people find that their intuition has broken down, and one needs to find a way to turn the idea into a rigorous proof. Let's start by dealing with the word "choose" that was used so glibly. A *choice function* for a set X is a function c that associates to every nonempty subset $Y \subseteq X$ an element $c(Y) \in Y$. The *axiom of choice* of set theory asserts that choice functions exist. This seemingly harmless axiom has been controversial historically because it allows one to prove the existence of things that cannot be constructed concretely. In particular, it allows us to prove the existence of well-orderings without being able to say concretely what the ordering is. Here's how the proof goes:[1]

Proof of the well-ordering theorem (sketch). Let c be a choice function for X. By a *c-ordered* subset of X we mean a pair $(A, <)$, where $A \subseteq X$ and "$<$" is a well-ordering on A, such that for every $a \in A$,

$$a = c(X \setminus A_{<a}).$$

This expresses the idea that the ordering on A has been constructed as in the intuitive discussion above. Thus we expect that A will turn out to be a section of X with respect to the ordering on X whose existence we're trying to prove. Given two c-ordered subsets A, B, we write $A \preceq B$ and we say B *extends* A if

(1) $A \subseteq B$.
(2) The two orderings agree on A.
(3) A is an order ideal in B, i.e., if $b < a$ in B and $a \in A$, then $b \in A$.

This definition is supposed to capture the idea that in the course of constructing our well-ordering, A was reached at some stage and then B was reached later.

The heart of the proof is the following *claim*: Given any two c-ordered subsets A, B of X, one of them is an extension of the other. To prove this, we assume $B \not\preceq A$, and we then prove $A \preceq B$.

Step 1. We show that $A_{\leq a} \preceq B$ for all $a \in A$. Arguing by induction with respect to the given well-ordering on A, we may assume that $A_{\leq a'} \preceq B$ for every $a' < a$ in A. It follows that

$$A_{<a} = \bigcup_{a'<a} A_{\leq a'} \preceq B.$$

Now $A_{<a} \neq B$, since otherwise we would have $B \preceq A$. So B contains a first element b (relative to the given well-ordering on B) such that $b \notin A_{<a}$. Using the fact that $A_{<a}$ is an order ideal in B, one deduces that $A_{<a} = B_{<b}$. [You will be asked to supply the details in a problem on Assignment 4, which asserts that every proper order ideal in a well-ordered set is a section.] But then

$$b = c(X \setminus B_{<b}) = c(X \setminus A_{<a}) = a.$$

So $a \in B$ and $A_{\leq a} = B_{\leq a}$ as ordered sets. This implies that $A_{\leq a} \preceq B$ and completes the inductive proof.

[1]Munkres (bottom of p. 65) says that the proof is primarily of interest to logicians. I disagree. The well-ordering theorem is a theorem of mathematics, whose proof is of interest to anyone who wants to understand why the theorem is true.

Step 2. Taking the union over all $a \in A$, one concludes from Step 1 that

$$A = \bigcup_{a \in A} A_{\leq a} \preceq B.$$

This completes the proof of the claim.

Now let Y be the union of all c-ordered subsets of X. In view of the claim, Y has a well-defined c-ordering and hence is the largest c-ordered subset. But then $Y = X$, because otherwise we could get a bigger c-ordered set by adjoining $c(X \smallsetminus Y)$. So X is well-ordered. □

Remarks.

(1) Forming the union of all c-ordered subsets replaced the vague statement, "Keep going until all of X is exhausted" that occurred in the intuitive discussion preceding the proof. What made this work was the claim, which enabled us to put a well-defined ordering on the union.

(2) Given the well-ordering theorem, it is not hard to rigorously construct the ordinal numbers. Alternatively, it is possible to first develop the theory of ordinals and then use ordinals to prove the well-ordering theorem. We will not pursue this connection further since, in practice, having big well-ordered sets is just as good as having ordinal numbers. But you can find the details, if you're interested, in any book on set theory. See, for instance, *Naïve Set Theory* by P. R. Halmos. See also the handout on ordinal numbers that I wrote for Math 6310 a few years ago.

(3) As we will see below, one thing we can do with the well-ordering theorem is to construct the well-ordered set that Munkres calls S_Ω (p. 66). This is an uncountable well-ordered set X in which all the sections $X_{<x}$ are countable. In terms of ordinals, we get S_Ω by constructing ordinals until we've accumulated uncountably many of them, and then we stop. (The symbol "Ω" stands for the next ordinal that we would have adjoined if we hadn't stopped.) As Munkres points out, it is possible to prove the existence of S_Ω without using the axiom of choice. See Exercise 8 on p. 74, which involves ideas similar to those used in the proof of the main claim above.

Corollary. *There exists an uncountable well-ordered set S in which all sections $S_{<\alpha}$ are countable.*

Proof. Take any uncountable set X and well-order it. It may have the required property. If not, there is a smallest $x \in X$ such that $X_{<x}$ is uncountable. Take $S = X_{<x}$. □

Notation. Choose such an S once and for all and call it S_Ω.

Proposition. *Any countable subset $A \subseteq S_\Omega$ has an upper bound in S_Ω.*

Proof. The union of all the sections $(S_\Omega)_{<a}$ $(a \in A)$ is countable, so it cannot be all of S_Ω. Any element not in this union is then an upper bound for A. □