Mathematics 6310

Assignment 1, due September 5, 2011

Review your undergraduate group theory as needed. To begin with, you need a good grasp of normal subgroups, quotients, and the isomorphism laws. (See Section 3.3 for the latter.) Read 3.4 and the portion of 6.1 about solvable groups. Supplementary reading: Isaacs 10AB, Jacobson II.3.3, Lang I.3. [Note: Supplementary reading is optional and refers to books that have been placed on reserve. See the course web page for the list of books I've put on reserve.] Then do:

- 3.4 (p. 106): 8, 11, 12. Suggestion for 8: Instead of following the hint, you can use the results of 6.1.
- 6.1 (pp. 198–201): 31, 32

Additional problems:

1. Let G be an Ω-group with an Ω-composition series. If H is a normal Ω-subgroup of G, show that G has an Ω-composition series in which H is one of the terms. Deduce that H and G/H have Ω-composition series.

2. If G is an Ω-group with a composition series, the length of some (every) composition series is called the *length* of G and is denoted $l(G)$. With the notation of the previous exercise, show that
$$l(G) = l(H) + l(G/H).$$
If H_1, H_2 are two normal Ω-subgroups of G, show that
$$l(H_1H_2) = l(H_1) + l(H_2) - l(H_1 \cap H_2).$$
Deduce from this a familiar result from linear algebra about dimensions of subspaces.

3. Let G have two composition series $(G_i)_{0\le i\le n}$ and $(H_j)_{0\le j\le n}$. Show that the proof given in class of the Jordan–Hölder theorem yields an explicit permutation π of $\{0, \dots, n-1\}$ such that
$$G_i/G_{i+1} \cong H_j/H_{j+1}$$
if $j = \pi(i)$. Spell out the definition of π concretely, without referring to the proof.

4. Recall that a poset (partially ordered set) is said to satisfy the *maximal condition* if every nonempty subset has a maximal element. This is equivalent to the *ascending chain condition* (ACC), which asserts that there cannot exist an infinite chain $x_1 < x_2 < \cdots$. The *minimal condition* and equivalent *descending chain condition* (DCC) are defined similarly. An Ω-group is said to satisfy the maximal or minimal condition (or ACC or DCC) if the set of Ω-subgroups, ordered by inclusion, satisfies the corresponding condition. For example, finite groups satisfy both the ACC and the DCC. Assume now that G is any Ω-group satisfying both chain conditions.
   (a) Show that any Ω-series in G can be refined to an Ω-composition series.
   (b) Let $f\colon G \to G$ be an endomorphism. Prove that, for n sufficiently large, the Ω-subgroups $N := \ker(f^n)$ and $H := \operatorname{im}(f^n)$ satisfy $G = NH$ and $N \cap H = \{1\}$, so that G is the semidirect product of N and H. Moreover, f induces an automorphism of H.

(c) Deduce that if G cannot be decomposed as a semidirect product in a non-trivial way, then every endomorphism either is nilpotent (i.e., some power of it is trivial) or is an automorphism.