Mathematics 6310

Assignment 3, due September 19, 2011

Finish reading 6.1 if you didn't already finish it last week. Continue reviewing undergraduate group theory as needed, especially group actions and Sylow theory. (You can use 4.5 or any other source for the latter.) Start reading the handout on category theory. Then do:

- 4.5 (pp. 146-149): 38, 50
- 6.1 (pp. 198–201): 5

Additional problems:

Some of the problems below use the language of category theory. There is nothing deep here, and I think you can get everything you need from the first few pages of the handout on category theory. But feel free to ask in class for clarification if anything is unclear.

1. (a) If X and Y are G-sets, define an action of G on the Cartesian product $X \times Y$ so that the resulting G-set is the product of X and Y in the category of G-sets. (See Section 5 of the handout on category theory.) Generalize to an arbitrary family of G-sets.

   (b) If X and Y are G-sets, define an action of G on the disjoint union $X \amalg Y$ so that the resulting G-set is the sum of X and Y in the category of G-sets. Generalize to an arbitrary family of G-sets.

2. (a) Given $H \leq G$, state and prove a universal mapping property that characterizes G-set maps from G/H to any other G-set. (In fancier language, I'm asking for a concrete description of the representable functor $\mathrm{Hom}_G(G/H, -)$ from G-sets to sets.)

   (b) Given $H, K \leq G$, describe $\mathrm{Hom}_G(G/H, G/K)$. Be as explicit and concrete as you can.

3. Given $H, K \leq G$, show that the orbits of the G-set $G/H \times G/K$ are in 1–1 correspondence with the H-K double cosets. More precisely, show that the "difference" function $G/H \times G/K \to H\backslash G/K$, given by $(xH, yK) \mapsto (xH)^{-1}(yK) = Hx^{-1}yK$, induces a bijection from the set of orbits to the set of double cosets.

4. Let X be a transitive G-set. Show that X is primitive (in the sense of Exercise 7 on p. 117) if and only if it has the following mapping property: Every G-set map from X to any other G-set is either constant or injective. [Thus primitivity in the category of G-sets is analogous to simplicity in the category of groups.]

5. Let X be a set, and let G and H be transitive subgroups of S_X. Suppose $\phi\colon G \to H$ is an isomorphism such that $\phi(G_x) = H_y$ for some $x, y \in X$. Show that ϕ extends to an inner automorphism of S_X. [Hint: Consider the G-set whose underlying set is X, with the G-action given by the composite $\iota\phi\colon G \to S_X$, where $\iota\colon H \hookrightarrow S_X$ is the inclusion.]

6. If a group G is generated by two nilpotent normal subgroups, show that G is nilpotent. Note that G is not assumed to be finite. But if you can't do the general case, you can get partial credit by proving the result for finite groups.