Mathematics 6310

Assignment 5, due October 3, 2011

Finish reading 6.3. Continue your reading in the handout on category theory, going at least through the first few paragraphs of Section 7. Read what you need from 7.1–7.4 of Dummit and Foote in order to review the basic definitions from elementary ring theory. Then do:

- 7.3 (pp. 247–251): 14, 34(c)
- 7.4 (pp. 256–260): 6, 19

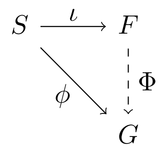
Additional problems:

1. The purpose of this exercise is to construct sums in the category of groups. For historical reasons, the sum in the category of groups has the confusing name "free product", but it is nothing at all like the ordinary direct product. For simplicity, we'll deal with the sum (or free product) of only two groups, but one could treat an arbitrary collection of summands in a similar way. The sum (free product) of two groups A, B is usually denoted $A * B$. You will construct it by following the outline (1)–(3) described in additional problem 1 of Assignment 4.
   (a) State explicitly the UMP that should be satisfied by $A * B$ for it to be the categorical sum of A and B.
   (b) Prove that $A * B$ exists. [Suggestion: Choose arbitrary presentations for A and B, and then construct $A * B$ by combining the presentations.]
   (c) Show that every element of $A * B$ is uniquely expressible as a word consisting of factors coming alternately from $A \setminus \{1\}$ and $B \setminus \{1\}$. [Suggestion: Follow the van der Waerden method used in class for free groups, i.e., use a suitable action of $A * B$ on the set of words.]
   (d) Here is a concrete consequence of (c) that would be difficult to prove using only abstract ideas: If $g \in A * B$ is an element of finite order, prove that g is conjugate to an element of either A or B. [Cryptic hint: You can use conjugation to avoid having to think about cancellation.]
   (e) Explain how Exercise 4 on p. 220 can be quickly deduced as a corollary of (d).

2. Recall that the finite dihedral group D_{2n} of order $2n$ has the presentation
$$\langle r, s \,;\, r^n = s^2 = 1,\, srs^{-1} = r^{-1} \rangle .$$
   This is stated very early in the text, before there is even a rigorous definition of what a presentation is (see pp. 25–27).
   (a) Now that you do have a rigorous definition, prove rigorously that the group defined by the presentation above is isomorphic to D_{2n}. [One proof is sketched on p. 219. If you want to follow this outline, fill in the missing details.]
   (b) We define the *infinite dihedral group*, denoted D_∞, to be the semidirect product $\mathbb{Z} \rtimes \mathbb{Z}_2$, where the nontrivial element of $\mathbb{Z}_2$ acts on $\mathbb{Z}$ as multiplication by -1. Prove that it admits the presentation
$$\langle r, s \,;\, s^2 = 1,\, srs^{-1} = r^{-1} \rangle .$$
   (c) Prove that D_∞ is generated by two elements of order 2.
   (d) Prove that D_∞ is isomorphic to the free product of two groups of order 2.

(e) Spell out explicitly a description of the elements of D_∞ based on (d) and 2(c).

3. Let F be a group and S a set. Suppose there is a set map $\iota\colon S \to F$ such that the following UMP holds: Given any group G and any set map $\phi\colon S \to G$, there is a unique group homomorphism $\Phi\colon F \to G$ such that $\Phi\iota = \phi$:



In other words, F satisfies the UMP for a free group as explained in class and in your text, except that ι is not assumed to be injective and $\iota(S)$ is not assumed to generate F. Prove that ι is injective and that $\iota(S)$ generates F.

4. Let $F = F(a, b)$ be the free group on two generators. The UMP for F gives, for any group G, a bijection of sets

$$\phi\colon \operatorname{Hom}(F, G) \xrightarrow{\sim} G \times G.$$

(a) Describe ϕ explicitly.

(b) Prove that ϕ is natural when both sides are viewed as functors of G. [Suggestion: This can be done by direct verification or by appealing to Yoneda's lemma. It's instructive to do it both ways, but you only have to write up one solution.]