Mathematics 6310

Assignment 7, due October 24, 2011

Read 8.3, and start 9.3. Then do the following:

- 8.3 (pp. 292–294): 11 [If you want to use results from Exercise 8.2.7, please prove them.]
- 9.3 (pp. 306–307): 4

Additional problems:

1. This problem generalizes Euclid's theorem that $\sqrt{2}$ is irrational. Let R be a UFD with field of fractions F. Let f be a monic polynomial in $R[x]$. Show that every root of f in F lies in R. [This has a straightforward direct proof, but you might find it instructive to deduce it from Gauss's lemma.]
2. Let R be a UFD with field of fractions F. Prove that the quotient group $F^\times/R^\times$ is a free abelian group; in other words, it is isomorphic to a direct sum of copies of $\mathbb{Z}$.
3. (a) I stated in class, and almost proved, that an integral domain is a UFD if and only if it satisfies:
   (1) Every irreducible element is prime.
   (2) The principal ideals satisfy the ACC.
   The only part of this that I didn't prove is that every UFD satisfies (2). Prove this now.
   (b) Exercise 9.3.4 gives an example of an integral domain that satisfies (1) but is not a UFD. Therefore (2) must be false. Give an explicit example of an infinite ascending chain of principal ideals in this ring.
4. This last one is just for fun; you don't have to turn it in. Our study of unique factorization made heavy (and sometimes implicit) use of the following fact about integral domains: a and b are associates if and only if $(a) = (b)$. Give an example to show that this can fail in general commutative rings.