Mathematics 6310

Assignment 12, due never

Read Section 10.4. Optionally finish the handout on category theory (or at least read about adjoint functors). Then do the following:

- 10.4 (pp. 375–378): 11, 12, 14, 16

Additional problems:

1. Let M be a right R-module, N a left R-module, and L a $\mathbb{Z}$-module (i.e., an abelian group).
   (a) Explain how the left action of R on N induces induces a *right* action of R on the abelian group $\operatorname{Hom}_{\mathbb{Z}}(N, L)$, making the latter a right R-module. (Just give the main points; you don't have to write down every detail.)
   (b) Show that $\mathbb{Z}$-bilinear R-balanced maps $M \times N \to L$ are in 1–1 correspondence with right R-module maps $M \to \operatorname{Hom}_{\mathbb{Z}}(N, L)$. Deduce that there is an abelian group isomorphism
   $$\operatorname{Hom}_{\mathbb{Z}}(M \otimes_R N, L) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_{\mathbb{Z}}(N, L)).$$
   (c) The result of (b) says (except for one missing detail) that for fixed N, the functor $- \otimes_R N$ from right R-modules to abelian groups is left adjoint to the functor $\operatorname{Hom}_{\mathbb{Z}}(N, -)$ from abelian groups to right R-modules. What is the missing detail?
   (d) If R is commutative, state a result similar to (b) with all Homs being over R.
2. Given a ring inclusion $R \subseteq S$ (or, more generally, a ring homomorphism $R \to S$), extension of scalars "enlarges" a (left) R-module M to the (left) S-module $S \otimes_R M$, which receives an "inclusion" map $\iota \colon M \to S \otimes_R M$ that is universal for R-maps from M to an S-module. [The quotation marks are necessary because ι is not necessarily injective; but it helps the intuition to pretend that it is.] As I stated in class, this implies that extension of scalars is left adjoint to restriction of scalars. In this exercise you will carry out a dual construction, called *co-extension of scalars*, that uses Hom instead of $\otimes$ and is right adjoint to restriction of scalars. Intuitively, it "enlarges" M to an S-module N with an R-map $\pi \colon N \to M$ that is universal for R-maps from an S-module to M. Here one should think intuitively that M is being "enlarged" in the sense that it is a quotient of N, though in fact π need not be surjective. I've formulated the exercise for the case of a ring inclusion $R \subseteq S$ to simplify notation, but it works equally well for a ring homomorphism $R \to S$.
   (a) Given a left R-module M, consider the abelian group $\operatorname{Hom}_R(S, M)$, where S is viewed as a left R-module. Explain how to give this abelian group the structure of left S-module.
   (b) Define, in the only sensible way you can think of, an R-module homomorphism $\pi \colon \operatorname{Hom}_R(S, M) \to M$.
   (c) Formulate and prove a universal property of this map as alluded to above. You'll know you've got it right (i.e., correct) if you can summarize this universal property as an adjunction formula exhibiting co-extension of scalars as right (i.e., the opposite of left) adjoint to restriction of scalars.