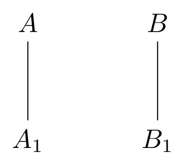
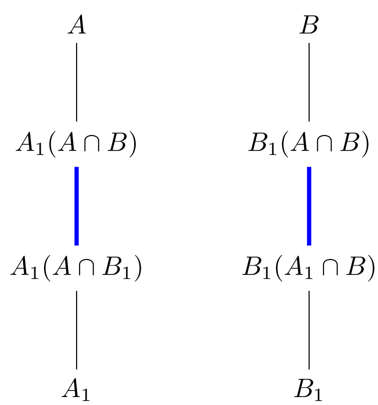
Mathematics 6310
The butterfly lemma
Ken Brown, Cornell University, August 2011

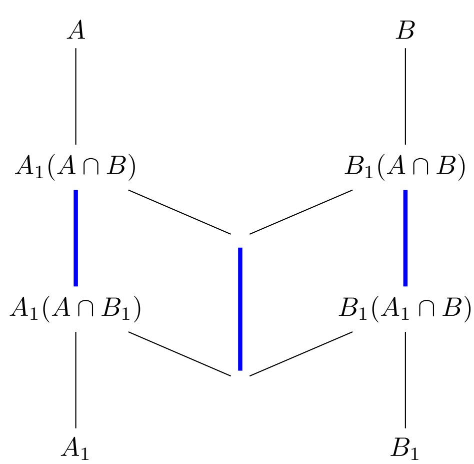
Given $A_1 \trianglelefteq A \leq G$ and $B_1 \trianglelefteq B \leq G$, we can use each of the chains



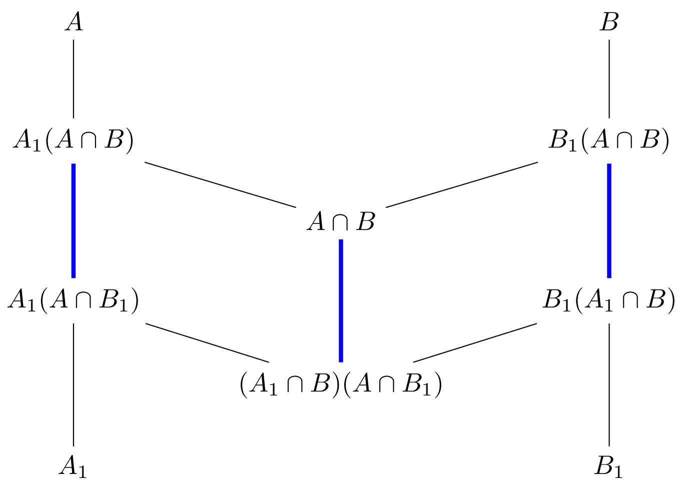
to refine the other. This yields



The content of the butterfly lemma is that the two quotients indicated by heavy blue lines are isomorphic. It is not even obvious a priori how to map either quotient to the other. But we will do this by introducing a third quotient that maps to both of them via inclusion maps:



What groups should appear at the top and bottom of the middle blue line? At the top we need something that is contained in both $A_1(A \cap B)$ and $B_1(A \cap B)$; an obvious candidate is $A \cap B$. This forces what we put at the bottom. Indeed, if we want to prove the lemma by applying the second isomorphism law, the group at the bottom should be $A_1(A \cap B_1) \cap (A \cap B)$, and it should also be $B_1(A_1 \cap B) \cap (A \cap B)$. Fortunately these two intersections are equal; in fact, we will show in class that they are both equal to $(A_1 \cap B)(A \cap B_1)$, and the proof will follow easily:



Although the proof is done, we can add two more groups to make the picture look a little more like a butterfly: