Mathematics 6310
Solution to Exercise 50 on p. 148
Ken Brown, Cornell University, September 2011

We are given a finite group G, a Sylow p-subgroup P of G, and two normal subsets $U, V \subseteq P$. If U and V are conjugate in G, we must show that they are conjugate in $N_G(P)$. [I think this result is called Burnside's fusion theorem, but I'm not positive.]

Choose $g \in G$ so that $gUg^{-1} = V$. Then $Q := gPg^{-1}$ is a Sylow p-subgroup of G containing V, and V is normal in Q. [Reason: Conjugation by g is an automorphism taking P to Q and U to V; the assertion follows from the fact that U is normal in P.] So P and Q are both contained in $N_G(V)$ and are therefore Sylow p-subgroups of the latter. By Sylow's theorem, there is an element $h \in N_G(V)$ such that $hQh^{-1} = P$. Then hg normalizes P and conjugates U to V.

The following diagram summarizes the proof:

$$
\begin{tikzcd}
N_G(U) \arrow[rr, dashed, bend left, blue] \arrow[dr, dash] & & N_G(V) \arrow[dl, dash] \arrow[dr, dash] & \\
& P \arrow[rr, dashed, bend left, blue] \arrow[dl, dash] \arrow[dr, dash] & & Q \arrow[ll, dashed, bend left, red] \arrow[dl, dash] \\
U \arrow[rr, dashed, bend left, blue] & & V &
\end{tikzcd}
$$

Here the dashed blue lines represent conjugation by g, and the dashed red line represents conjugation by h.