Mathematics 6310   Take-home prelim   Due November 29, 2011

What follows is an introduction to the theory of I-adic filtrations. Your task is to fill in the proofs of all results in Sections 2–7 that are labeled as Theorems, Propositions, Lemmas, or Corollaries. [Section 1 is general topology rather than algebra, and I have included sketches of proofs of all results. Section 8 is optional; you may do it for extra credit if you want.] I have written "*** Your proof goes here. ***" in all places where you are required to give a proof.

   You may use any result proved in class, in the assigned reading, or in your homework. You may not use any sources other than your textbook and class notes. Anna and I will be glad to clear up any ambiguities; **please do not discuss the exam with anyone else**.

   All rings in what follows will be assumed commutative.

1. Preliminaries on topological abelian groups

   Recall that a topological (abelian) group is a group G that is also a topological space, such that $(x, y) \mapsto x + y$ and $x \mapsto -x$ are continuous. Notice that we are writing the group law additively because we are only considering the abelian case. Note also that, in contrast to some treatments of topological groups, we do *not* require G to be Hausdorff.

   The theory is slightly simpler if we confine attention to topologies that arise from (pseudo)norms. A *pseudonorm* on an abelian group G is a function $x \mapsto |x|$ from G to the nonnegative reals, satisfying:

   - $|x + y| \le |x| + |y|$
   - $|-x| = |x|$
   - $|0| = 0$

We will drop the prefix "pseudo" if, in addition,

   - $|x| = 0 \implies x = 0$

A pseudonorm defines a pseudometric

$$d(x, y) := |x - y|$$

and hence a topology, making G a topological group. [A pseudometric is like a metric, except that we allow the possibility that $d(x, y) = 0$ for $x \ne y$.] A neighborhood base at 0 is given by the open balls $\{x : |x| < \epsilon\}$. A topological abelian group that arises in this way will be called *(pseudo)metrizable*. For brevity, we will call a pseudometrizable topological abelian group a PTAG. The following observation clarifies the difference between metrizability and pseudometrizability:

Proposition 1. *Let G be a PTAG, and choose any pseudonorm giving its topology. Then the following conditions are equivalent:*

   (i) G is Hausdorff.
   (ii) $\{0\}$ is closed.
   (iii) The intersection of all neighborhoods of 0 is $\{0\}$.
   (iv) The chosen pseudonorm is a norm.
   (v) G is metrizable.

In general, $G_0 := \{x : |x| = 0\}$ is a closed subgroup and G/G_0, with the quotient topology, is a metrizable topological abelian group.

Sketch of proof. It is easy to check that G_0 is the closure of $\{0\}$ and is the intersection of all neighborhoods of 0. The equivalence of (ii)–(iv) follows at once. Clearly (i) $\implies$ (ii). Conversely, if (ii)–(iv) hold, then the diagonal in $G \times G$ is $\{(x, y) \in G \times G : x - y = 0\}$ and is a closed set; this implies (i). So we now have the equivalence of (i)–(iv). The implications (iv) $\implies$ (v) $\implies$ (i) are trivial, so (i)–(v) are equivalent. Finally, G/G_0 inherits a norm from the pseudonorm on G, and the algebraic quotient map $G \twoheadrightarrow G/G_0$ maps open balls to open balls. This implies that it is an open map and hence a topological quotient map. □

The quotient map $G \to G/G_0$ is called the *Hausdorffification* of G; it is universal for continuous homomorphisms from G to a Hausdorff PTAG.

We close this section with some remarks about subgroups and quotient groups. Note first that if G is a PTAG, then every subgroup of G (with the subspace topology) is also a PTAG. The following lemma treats quotients.

Lemma 1. *Let G be a PTAG, let H be a subgroup, and let $K := G/H$ be the quotient group. Then K, with the quotient topology, is a PTAG.*

Sketch of proof. Choose a pseudonorm giving the topology on G, let $\pi \colon G \twoheadrightarrow K$ be the quotient map, and set

$$|y| := \inf_{\pi(x)=y} |x|$$

for $y \in K$. [Thinking of y as a coset of H in G, this is just the distance from that coset to 0.] This is a pseudonorm on K. Note that $|\pi(x)| \leq |x|$ for $x \in G$, so π is continuous if we topologize K via this pseudonorm. And π maps every open ball in G centered at 0 onto the corresponding open ball in K, so π is an open map (again when we topologize K via the pseudonorm). It follows that the pseudonorm topology is the quotient topology. □

2. Filtrations

From now on we restrict our attention to a particular type of topology that arises in commutative algebra. Let G be an abelian group with a *filtration*, i.e., a descending chain of subgroups

$$G = G_0 \geq G_1 \geq G_2 \geq \cdots.$$

For example, $\mathbb{Z}$ is filtered by the subgroups $p^n \mathbb{Z}$ for a fixed prime p; this filtration is called the *p-adic filtration*.

Proposition 2. *There is a unique topology on G such that G is a PTAG with the subgroups G_n as a neighborhood base at 0. The pseudonorm defining the topology can be taken to satisfy the following strong form of the triangle inequality:*

- $|x + y| \leq \max\{|x|, |y|\}$.

A basis for the topology on G is given by the inverses images of points under the quotient maps $G \twoheadrightarrow G/G_n$ ($n \geq 0$).

Proof. *** Your proof goes here. *** □

[Hint: Consider the function $v \colon G \to \mathbb{Z} \cup \{\infty\}$ given by $v(x) := \sup\{n : x \in G_n\}$.]

3. I-ADIC TOPOLOGIES

Let A be a (commutative) ring and I an ideal. The *I-adic topology* on A is the topology induced by the filtration $\{I^n\}$ by powers of I. This generalizes the p-adic topology mentioned above.

Lemma 2. *A with the I-adic topology is a topological ring (i.e., it is a topological abelian group, and multiplication $A \times A \to A$ is continuous).*

Proof. *** Your proof goes here. *** $\square$

Similarly, any A-module M has an I-adic topology induced by the filtration $\{I^n M\}$. This makes M a topological A-module (i.e., the action $A \times M \to M$ is continuous).

To prove anything significant about I-adic topologies we will need to assume that A is noetherian. We will do this in Section 5, after some remarks about graded rings.

4. DIGRESSION: GRADED RINGS

A *graded ring* is a ring A that comes equipped with an additive decomposition

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$$

such that $A_i A_j \subseteq A_{i+j}$ for all $i, j \geq 0$. The elements of the subgroup A_n are said to be *homogeneous* of degree n. Thus every $a \in A$ is uniquely expressible as

$$a = \sum_{n \geq 0} a_n$$

with a_n homogeneous of degree n and $a_n = 0$ for almost all n.

Example 1. Let A be a polynomial ring $k[x_1, \ldots, x_m]$. Then A is a graded ring with A_n equal to the set of homogeneous polynomials of degree n in the usual sense; in other words, A_n is the k-span of the set of monomials of degree n.

Starting from this example one can create many more examples by forming quotients by homogeneous ideals. Here an ideal I is said to be *homogeneous* if it has the form $I = \bigoplus_{n \geq 0} I_n$, where I_n is an additive subgroup of A_n; equivalently, I is generated by homogeneous elements. For example, any monomial ideal in a polynomial ring is homogeneous. If I is a homogeneous ideal, then A/I is a graded ring with $(A/I)_n := A_n/I_n$.

Graded rings occur naturally in many subjects. For example, the following construction arises in algebraic geometry in connection with "blowing up". Let A be a ring with a filtration

$$A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$$

by ideals such that $A_i A_j \subseteq A_{i+j}$. [The I-adic filtration has this property for any ideal I in A.] Then we can form the external direct sum

$$A^* := \bigoplus_{n \geq 0} A_n$$

and use the given multiplication maps $A_i \times A_j \to A_{i+j}$ to make A^* a graded ring.

Lemma 3. *Let A be a noetherian ring and I an ideal. Then $A^* := \bigoplus_n I^n$ is noetherian.*

Proof. *** Your proof goes here. *** $\qquad\square$

If A is a graded ring, then a *graded A-module* is an A-module M that comes equipped with an additive decomposition $M = \bigoplus_{n \geq 0} M_n$ such that $A_i M_j \subseteq M_{i+j}$ for all $i, j \geq 0$. The "blowing-up" construction above generalizes to modules as follows. Let A be a filtered ring and M a filtered A-module. This means, by definition, that we are given a decreasing sequence of A-submodules M_n such that $A_i M_j \subseteq M_{i+j}$ for all $i, j \geq 0$. Then the given multiplication maps $A_i \times M_j \to M_{i+j}$ induce an A^*-action on

$$M^* := \bigoplus_{n \geq 0} M_n,$$

making the latter a graded A^*-module.

Consider, for example, the I-adic filtration on a ring A. Then a filtered A-module is simply a module M with a filtration by submodules M_n such that $I M_n \subseteq M_{n+1}$ for all $n \geq 0$. The following lemma will be needed in Section 5.

Lemma 4. *Let A have the I-adic filtration and let M be a filtered A-module as in the previous paragraph. If M^* is a finitely generated A^*-module, then $I M_n = M_{n+1}$ for all sufficiently large n. Consequently, the I-adic topology on M coincides with the topology induced by the filtration $\{M_n\}$.*

Proof. *** Your proof goes here. *** $\qquad\square$

[Hint: For a fixed $N \geq 0$, what does the A^*-submodule generated by $\bigoplus_{n=0}^{N} M_n$ look like?]

5. The Artin–Rees lemma

Let A be a noetherian ring and I an ideal. The following result is one version of the *Artin–Rees lemma.*

Theorem 1. *Let M be a finitely generated A-module and M' a submodule. Then the I-adic topology on M' is the same as the subspace topology that M' inherits from the I-adic topology on M.*

Proof. *** Your proof goes here. *** $\qquad\square$

[Hint: The two topologies on M' are induced by two filtrations, one finer than the other. Use Lemma 4.]

6. Digression: The Jacobson radical and Nakayama's lemma

Recall that the *Jacobson radical* of a ring A, denoted $\operatorname{rad} A$, is the intersection of all maximal ideals. It is the ring-theoretic analogue of the Fitting subgroup of a group. It is the largest ideal J such that $1 - x$ is invertible in A for all $x \in J$. This is proved in Dummit and Foote on p. 751, which also contains a proof of *Nakayama's lemma.* This asserts that if M is a finitely generated A-module such that $JM = M$ (where $J = \operatorname{rad} A$), then $M = \{0\}$. The following result generalizes Nakayama's lemma.

Proposition 3. *Let I be an arbitrary ideal of A, and let M be a finitely generated A-module such that $IM = M$. Then M is annihilated by $1 - x$ for some $x \in I$.*

Proof. *** Your proof goes here. *** $\qquad\square$

[Hint: Use localization to reduce this to Nakayama's lemma. Namely, let S be the multiplicative set $1 + I$. Then $S^{-1}I \subseteq \operatorname{rad}(S^{-1}A)$. Now consider the $S^{-1}A$-module $S^{-1}M$.]

7. Krull's theorem

We return to the setup of Section 5. Thus A is noetherian and I is an arbitrary ideal. As a consequence of the Artin–Rees lemma, we can determine the intersection of all I-adic neighborhoods of 0 in any finitely generated A-module. The following result is known as *Krull's theorem*.

Theorem 2. *Let M be a finitely generated A-module. Then*

$$\bigcap_{n \geq 0} I^n M = \{m \in M : (1 - x)m = 0 \text{ for some } x \in I\}.$$

Proof. *** Your proof goes here. *** □

[Hint: Let M' be the module on the left side of the equation. Apply Theorem 1 and Proposition 3.]

Remark. This is false without the hypothesis that A is noetherian. For example, let A be the ring of infinitely differentiable functions of one real variable, and let I be the ideal consisting of functions that vanish at 0. Then the intersection of the powers of I consists of the functions that vanish along with all their derivatives at 0, whereas any function annihilated by $1 - f$ for some $f \in I$ must vanish in a neighborhood of 0.

Corollary 1. *A is I-adically Hausdorff if (a) A is an integral domain and I is proper or (b) if $I \subseteq \operatorname{rad} A$.*

Proof. *** Your proof goes here. *** □

Corollary 2. *If $I \subseteq \operatorname{rad} A$, then every finitely generated A-module is I-adically Hausdorff, and every submodule is I-adically closed.*

Proof. *** Your proof goes here. *** □

Remark. An important special case of the last corollary is the case where A is a local ring and I is its maximal ideal.

The exam is finished. You may continue to the next section for extra credit, but do it only if you are interested; the effect on your grade will be minimal.

8. Completions

In classical algebraic geometry (over $\mathbb{C}$), one can use analytic methods by viewing polynomials as holomorphic functions and using power series expansions. In abstract algebraic geometry, we can try to mimic this by using formal power series. Passing from polynomials to formal power series is an example of *completion*.

Completion is also used in number theory. Suppose we are trying to solve an equation $f(x) = 0$ for an unknown integer x. This may be too hard, so we try to solve it mod p for some prime p, then mod p^2, then mod p^3, and so on. If we are successful, we get "approximate" solutions x_i, which may converge to a solution in the ring of p-adic integers, which is a completion of $\mathbb{Z}$. Finding such p-adic solutions for all primes p can be the starting point for finding a solution in $\mathbb{Z}$.

8.1. **Generalities.** This subsection is a continuation of Section 1, and I am again including sketches of proofs. There is nothing for you to do but read it.

Let G be a PTAG. A *Cauchy sequence* in G is a sequence (x_n) such that every neighborhood of 0 contains $x_m - x_n$ for sufficiently large m, n. This is the same as a Cauchy sequence in the usual sense with respect to some (any) pseudonorm defining the topology on G. We say that G is *complete* if it is Hausdorff and every Cauchy sequence converges. [We insist on G being Hausdorff here so that a convergent sequence has a *unique* limit.] A *completion* of G is a continuous homomorphism $\iota\colon G \to \hat{G}$ such that $\hat{G}$ is complete and ι is universal for continuous homomorphisms from G to a complete metrizable topological abelian group.

Standard arguments show that the completion, if it exists, is unique up to canonical isomorphism, so we will often say "the" completion, once we have proved that completions always exist.

Lemma 5. *Let $\iota\colon G \to \hat{G}$ be a continuous homomorphism with $\hat{G}$ complete. Assume that $\ker \iota = G_0$, where the latter is the intersection of all neighborhoods of 0 as in Proposition 1. Assume further that $\iota(G)$ is dense in $\hat{G}$ and that ι induces an isomorphism $G/G_0 \xrightarrow{\sim} \iota(G)$ of topological groups. Then ι is a completion.*

Sketch of proof. Let $f\colon G \to L$ be a continuous homomorphism with L complete. Since L is Hausdorff, there is an induced continuous homomorphism $\bar{f}\colon G/G_0 \to L$. Then $\bar{f}$ is uniformly continuous and hence takes Cauchy sequences to Cauchy sequences. Identifying G/G_0 with a dense subgroup of $\hat{G}$, we may now apply the following lemma to conclude that $\bar{f}$ extends to a continuous map $\hat{G} \to L$ and that the extension is a group homomorphism. $\square$

Lemma 6. *Let X and Y be metric spaces, let A be a dense subset of X, and let $f\colon A \to Y$ be a continuous map with the following property: For every sequence (a_n) in A that converges in X, the image sequence $f(a_n)$ converges in Y. Then f extends (uniquely) to a continuous map $F\colon X \to Y$.*

Sketch of proof. For any $x \in X$, choose a sequence (a_n) in A that converges to x. By assumption, $f(a_n) \to y$ for some $y \in Y$. The limit is independent of the choice of sequence, since any two such sequences are subsequences of a third. We can therefore set $F(x) := y$. To prove continuity, consider any $x_0 \in X$ and let $y_0 := F(x_0)$. If F is not continuous at x_0, then there is an $\epsilon > 0$ such that every neighborhood U of x_0 contains a point x with $d(F(x), y_0) > \epsilon$. It follows from the definition of F that we can take x to be in A. But this means that we can find a sequence $a_n \to x_0$ with $a_n \in A$, such that $f(a_n) \nrightarrow y_0$, contradicting the definition of F. $\square$

We can now prove the existence of completions.

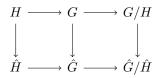**Proposition 4.** *Every PTAG has a completion.*

Sketch of proof. Let G' be the Hausdorffification of G, and let $\hat{G}$ be the metric space completion of G' (relative to some norm defining the topology on G'). The addition map $G' \times G' \to G'$ is a homomorphism because G' is abelian, so it is uniformly continuous. It therefore extends to a continuous map (called *addition*) $\hat{G} \times \hat{G} \to \hat{G}$ by Lemma 6. Similarly, the inversion map $G' \to G'$ extends to a continuous map $\hat{G} \to \hat{G}$. The group axioms are satisfied by $\hat{G}$ because they hold

on the dense subgroup G'. The composite $G \twoheadrightarrow G' \hookrightarrow \hat{G}$ is now a completion of G by Lemma 5. $\qquad\square$

Proposition 5. *Let G be a PTAG, let H be a closed subgroup, and set $K := G/H$. If G is complete, then H and K are also complete.*

Sketch of proof. The result for H is clear. Turning now to K, note first that it is Hausdorff by Proposition 1 since H is closed. Let (y_n) be a Cauchy sequence in K. Passing to a subsequence if necessary, we may assume that $|y_n - y_{n+1}| < 2^{-n}$ for all n. Now lift y_n to $x_n \in G$ with the same property. [Assuming inductively that x_n has been defined, set $x_{n+1} = x_n + z$, where $\pi(z) = y_{n+1} - y_n$ and $|z| < 2^{-n}$.] Then (x_n) is a Cauchy sequence in G, so it converges; hence the image sequence (y_n) converges in K. $\qquad\square$

Finally, we prove that completion is an exact functor:

Proposition 6. *Let G be a PTAG, and let H be a subgroup. Let $\iota\colon G \to \hat{G}$ be the completion of G, and let $\hat{H}$ be the closure of $\iota(H)$ in $\hat{G}$. Then $\hat{H}$ is the completion of H, and $\hat{G}/\hat{H}$ is the completion of G/H. More precisely, there is a commutative diagram*

$$
\begin{array}{ccccc}
H & \longrightarrow & G & \longrightarrow & G/H \\
\downarrow & & \downarrow & & \downarrow \\
\hat{H} & \longrightarrow & \hat{G} & \longrightarrow & \hat{G}/\hat{H}
\end{array}
$$

where the horizontal arrows are the canonical maps and the vertical arrows are completions.

Sketch of proof. Choose a pseudonorm giving the topology on G. Then $\hat{G}$ has a norm such that ι is an isometry (i.e., it is distance preserving) by the proof of Proposition 4. Then ι maps H isometrically onto a dense subgroup of $\hat{H}$, which is complete, so the latter can be identified with the completion of H by the same proof. Let $j\colon G/H \to \hat{G}/\hat{H}$ be the continuous homomorphism induced by ι. The image is dense, and $\hat{G}/\hat{H}$ is complete by Proposition 5. So we will be done if we can show that j is an isometry when we give the quotients the pseudonorms defined in the proof of Lemma 1. Given $y = \pi(x)$ in G/H, we have

$$
\begin{aligned}
|j(y)| &= \inf_{\hat{h} \in \hat{H}} |\iota(x) + \hat{h}| \\
&= \inf_{h \in H} |\iota(x) + \iota(h)| \\
&= \inf_{h \in H} |x + h| \\
&= |y|,
\end{aligned}
$$

as required. $\qquad\square$

8.2. **The completion of a filtered group.** Let G be a PTAG whose topology comes from a filtration $\{G_n\}$ as in Section 2

The last sentence of Proposition 2 suggests that we consider the tower of groups

$$
\cdots \twoheadrightarrow G/G_2 \twoheadrightarrow G/G_1 \twoheadrightarrow G/G_0.
$$

We have a compatible family of maps $G \to G/G_n$, and hence a map

$$\iota \colon G \to \hat{G} := \varprojlim_n G/G_n$$

from G to the inverse limit of the tower. [If you are not familiar with inverse limits, you can find the definition in Dummit and Foote.] We topologize $\hat{G}$ by viewing it as a subspace of the direct product $\prod_n G/G_n$, where each G/G_n is given the discrete topology.

Proposition 7. *(1) The topology on $\hat{G}$ is induced by the filtration by the subgroups $\hat{G}_n := \ker\left\{\hat{G} \twoheadrightarrow G/G_n\right\}$.*

(2) $\iota \colon G \to \hat{G}$ is the completion of G.

(3) $\hat{G}_n$ is the closure of $\iota(G_n)$ and is the completion of G_n.

(4) The canonical map $G/G_n \to \hat{G}/\hat{G}_n$ is an isomorphism for each n.

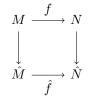*(5) An infinite series in $\hat{G}$ converges if and only if its terms tend to 0.*

Proof. *** Your proof goes here. *** □

8.3. *I-adic completions.* Let A be a ring endowed with the I-adic filtration for some ideal I. The completion $\hat{A}$ then admits a unique structure of topological ring such that $\iota \colon A \to \hat{A}$ is a continuous homomorphism.

Example 2. If A is the polynomial ring $k[x]$ (k a field) and $I = (x)$, then $\hat{A}$ is the ring $k[[x]]$ of formal power series.

Example 3. If $A = \mathbb{Z}$ with the p-adic filtration, then $\hat{A}$ is called the ring of p-*adic integers.* Its elements are uniquely expressible as convergent infinite series $\sum_{n \geq 0} a_n p^n$ with $0 \leq a_n < p$.

Similarly, we can form the I-adic completion $\hat{M}$ of any A-module M; it is a topological $\hat{A}$-module. Any module homomorphism $f \colon M \to N$ is continuous when both modules are given the I-adic topology, and there is an induced continuous homomorphism $\hat{f} \colon \hat{M} \to \hat{N}$ such that the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\ f\ } & N \\
\downarrow & & \downarrow \\
\hat{M} & \xrightarrow{\ \hat{f}\ } & \hat{N}
\end{array}
$$

commutes.

Proposition 8. *(1) If $f \colon M \to N$ is surjective, then so is $\hat{f} \colon \hat{M} \to \hat{N}$.*

(2) $(M_1 \oplus M_2)\hat{\ } = \hat{M}_1 \oplus \hat{M}_2$. More precisely,

$$\iota_1 \oplus \iota_2 \colon M_1 \oplus M_2 \to \hat{M}_1 \oplus \hat{M}_2$$

is a completion.

(3) If M is finitely generated, then $\hat{M} = \hat{A}M$, where the right hand side is an abbreviation for $\hat{A} \cdot \iota(M)$. [Hint: First check this for $M = A$, then deduce it for M free of finite rank, then deduce it for arbitrary finitely generated M.]

(4) *The A-module inclusion $I^n \hookrightarrow A$ induces a monomorphism $\widehat{I^n} \hookrightarrow \hat{A}$ on I-adic completions.*

Proof. *** Your proof goes here. *** □

Remark. In view of (4), we will identify $\widehat{I^n}$ with its image in $\hat{A}$. Suppose now that I is finitely generated. Then this image is $\hat{A}I^n$ by (3), and we have

$$\hat{A}I^n = (\hat{A}I)^n = \hat{I}^n.$$

It follows that the topology on $\hat{A}$ is the $\hat{I}$-adic topology. Similarly (still assuming I is finitely generated), the topology on $\hat{M}$ is the $\hat{I}$-adic topology for any finitely generated A-module M. Indeed, $\hat{M}$ has a neighborhood base at 0 given by the ideals

$$(I^n M)\hat{} = \hat{A}(I^n M) = (\hat{A}I^n)M = \hat{I}^n M.$$

Remark. We still have $\hat{I}^n \subseteq \widehat{I^n}$ even if I is not finitely generated; this simply says that elements of $\hat{I}^n$ can be approximated by elements of I^n. So the content of the first part of Remark 8.3 is that $\hat{I}^n$ is closed in $\hat{A}$ if I is finitely generated.

Proposition 9. *$\hat{I}$ is contained in the Jacobson radical* rad $\hat{A}$.

Proof. *** Your proof goes here. *** □

[Hint: Geometric series.]

Corollary 3. *If I is a maximal ideal in A, then $\hat{A}$ is a local ring with maximal ideal $\hat{I}$.*

Proof. *** Your proof goes here. *** □

8.4. **The noetherian case.** We finish with some consequences of the Artin–Rees lemma. Assume that A is noetherian.

Proposition 10. *I-adic completion is an exact functor on the category of finitely generated A-modules, i.e., a short exact sequence*

$$M' \hookrightarrow M \twoheadrightarrow M''$$

yields a short exact sequence

$$\widehat{M'} \hookrightarrow \widehat{M} \twoheadrightarrow \widehat{M''}.$$

Proof. *** Your proof goes here. *** □

Corollary 4. *For any finitely generated A-module M, the canonical map*

$$\hat{A} \otimes_A M \to \hat{M}$$

is an isomorphism.

Proof. *** Your proof goes here. *** □

Corollary 5. *$\hat{A}$ is a flat A-module.*

Proof. *** Your proof goes here. *** □

8.5. **What's next?** I will stop here because it would take much too long to prove the next interesting result. To whet your appetite, here is what I would do next, stated in geometric language. Let x be a smooth point of an n-dimensional algebraic variety X, and let A be the local ring of X at x. Then the completion of A with respect to its maximal ideal is isomorphic to the ring of formal power series in n variables.