Mathematics 6310
Brief solutions to take-home prelim

1. Preliminaries on topological abelian groups

Recall that a topological (abelian) group is a group G that is also a topological space, such that $(x, y) \mapsto x + y$ and $x \mapsto -x$ are continuous. Notice that we are writing the group law additively because we are only considering the abelian case. Note also that, in contrast to some treatments of topological groups, we do *not* require G to be Hausdorff.

The theory is slightly simpler if we confine attention to topologies that arise from (pseudo)norms. A *pseudonorm* on an abelian group G is a function $x \mapsto |x|$ from G to the nonnegative reals, satisfying:

- $|x + y| \leq |x| + |y|$
- $|-x| = |x|$
- $|0| = 0$

We will drop the prefix "pseudo" if, in addition,

- $|x| = 0 \implies x = 0$

A pseudonorm defines a pseudometric

$$d(x, y) := |x - y|$$

and hence a topology, making G a topological group. [A pseudometric is like a metric, except that we allow the possibility that $d(x, y) = 0$ for $x \neq y$.] A neighborhood base at 0 is given by the open balls $\{x : |x| < \epsilon\}$. A topological abelian group that arises in this way will be called *(pseudo)metrizable*. For brevity, we will call a pseudometrizable topological abelian group a PTAG. The following observation clarifies the difference between metrizability and pseudometrizability:

Proposition 1. *Let G be a PTAG, and choose any pseudonorm giving its topology. Then the following conditions are equivalent:*

(i) G is Hausdorff.
(ii) $\{0\}$ is closed.
(iii) The intersection of all neighborhoods of 0 is $\{0\}$.
(iv) The chosen pseudonorm is a norm.
(v) G is metrizable.

In general, $G_0 := \{x : |x| = 0\}$ is a closed subgroup and G/G_0, with the quotient topology, is a metrizable topological abelian group.

Sketch of proof. It is easy to check that G_0 is the closure of $\{0\}$ and is the intersection of all neighborhoods of 0. The equivalence of (ii)–(iv) follows at once. Clearly (i) $\implies$ (ii). Conversely, if (ii)–(iv) hold, then the diagonal in $G \times G$ is $\{(x, y) \in G \times G : x - y = 0\}$ and is a closed set; this implies (i). So we now have the equivalence of (i)–(iv). The implications (iv) $\implies$ (v) $\implies$ (i) are trivial, so (i)–(v) are equivalent. Finally, G/G_0 inherits a norm from the pseudonorm on G, and the algebraic quotient map $G \twoheadrightarrow G/G_0$ maps open balls to open balls. This implies that it is an open map and hence a topological quotient map. $\square$

The quotient map $G \to G/G_0$ is called the *Hausdorffification* of G; it is universal for continuous homomorphisms from G to a Hausdorff PTAG.

We close this section with some remarks about subgroups and quotient groups. Note first that if G is a PTAG, then every subgroup of G (with the subspace topology) is also a PTAG. The following lemma treats quotients.

Lemma 1. *Let G be a PTAG, let H be a subgroup, and let $K := G/H$ be the quotient group. Then K, with the quotient topology, is a PTAG.*

Sketch of proof. Choose a pseudonorm giving the topology on G, let $\pi\colon G \twoheadrightarrow K$ be the quotient map, and set

$$|y| := \inf_{\pi(x)=y} |x|$$

for $y \in K$. [Thinking of y as a coset of H in G, this is just the distance from that coset to 0.] This is a pseudonorm on K. Note that $|\pi(x)| \leq |x|$ for $x \in G$, so π is continuous if we topologize K via this pseudonorm. And π maps every open ball in G centered at 0 onto the corresponding open ball in K, so π is an open map (again when we topologize K via the pseudonorm). It follows that the pseudonorm topology is the quotient topology. $\square$

2. Filtrations

From now on we restrict our attention to a particular type of topology that arises in commutative algebra. Let G be an abelian group with a *filtration*, i.e., a descending chain of subgroups

$$G = G_0 \geq G_1 \geq G_2 \geq \cdots .$$

For example, $\mathbb{Z}$ is filtered by the subgroups $p^n\mathbb{Z}$ for a fixed prime p; this filtration is called the *p-adic filtration*.

Proposition 2. *There is a unique topology on G such that G is a PTAG with the subgroups G_n as a neighborhood base at 0. The pseudonorm defining the topology can be taken to satisfy the following strong form of the triangle inequality:*

- $|x+y| \leq \max\{|x|, |y|\}$.

A basis for the topology on G is given by the inverses images of points under the quotient maps $G \twoheadrightarrow G/G_n$ ($n \geq 0$).

Sketch of proof. A topology on a set X is completely determined if one knows, for each $x \in X$, a neighborhood base at x. By *neighborhood* here I mean a set containing x in its interior. And a *neighborhood base* at x is a family of neighborhoods of x such that every neighborhood of x contains one of them. In the case of a topological group, a neighborhood base at the identity yields, by translation, a neighborhood base at any other point. The topology is therefore determined if one knows a neighborhood base at the identity. This proves the uniqueness assertion. For existence, define a "valuation" $v\colon G \to \mathbb{Z} \cup \{\infty\}$ by

$$v(x) := \sup\{n : x \in G_n\}.$$

Then $v(x+y) \geq \min\{v(x), v(y)\}$; we now get the desired pseudonorm by setting

$$|x| := c^{-v(x)}$$

for any fixed $c > 1$. The rest of the proof consists of routine verifications. $\square$

3. I-adic topologies

Let A be a (commutative) ring and I an ideal. The *I-adic topology* on A is the topology induced by the filtration $\{I^n\}$ by powers of I. This generalizes the p-adic topology mentioned above.

Lemma 2. *A with the I-adic topology is a topological ring (i.e., it is a topological abelian group, and multiplication $A \times A \to A$ is continuous).*

Sketch of proof. Imitate any proof you have ever seen that multiplication $\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous. $\square$

Similarly, any A-module M has an I-adic topology induced by the filtration $\{I^n M\}$. This makes M a topological A-module (i.e., the action $A \times M \to M$ is continuous).

To prove anything significant about I-adic topologies we will need to assume that A is noetherian. We will do this in Section 5, after some remarks about graded rings.

4. Digression: Graded rings

A *graded ring* is a ring A that comes equipped with an additive decomposition

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$$

such that $A_i A_j \subseteq A_{i+j}$ for all $i, j \geq 0$. The elements of the subgroup A_n are said to be *homogeneous* of degree n. Thus every $a \in A$ is uniquely expressible as

$$a = \sum_{n \geq 0} a_n$$

with a_n homogeneous of degree n and $a_n = 0$ for almost all n.

Example 1. Let A be a polynomial ring $k[x_1, \ldots, x_m]$. Then A is a graded ring with A_n equal to the set of homogeneous polynomials of degree n in the usual sense; in other words, A_n is the k-span of the set of monomials of degree n.

Starting from this example one can create many more examples by forming quotients by homogeneous ideals. Here an ideal I is said to be *homogeneous* if it has the form $I = \bigoplus_{n \geq 0} I_n$, where I_n is an additive subgroup of A_n; equivalently, I is generated by homogeneous elements. For example, any monomial ideal in a polynomial ring is homogeneous. If I is a homogeneous ideal, then A/I is a graded ring with $(A/I)_n := A_n/I_n$.

Graded rings occur naturally in many subjects. For example, the following construction arises in algebraic geometry in connection with "blowing up". Let A be a ring with a filtration

$$A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$$

by ideals such that $A_i A_j \subseteq A_{i+j}$. [The I-adic filtration has this property for any ideal I in A.] Then we can form the external direct sum

$$A^* := \bigoplus_{n \geq 0} A_n$$

and use the given multiplication maps $A_i \times A_j \to A_{i+j}$ to make A^* a graded ring.

Lemma 3. *Let A be a noetherian ring and I an ideal. Then $A^* := \bigoplus_n I^n$ is noetherian.*

Sketch of proof. Identify A with the subring of A^* consisting of the elements that are homogeneous of degree 0. Then A^* is generated, as an A-algebra, by any set of generators of I as an ideal (where I is identified with the set of elements in A^* that are homogeneous of degree 1). In particular, A^* is a finitely generated A-algebra and hence is noetherian by Hilbert's basis theorem (together with the fact that a quotient of a noetherian ring is noetherian). $\square$

If A is a graded ring, then a *graded A-module* is an A-module M that comes equipped with an additive decomposition $M = \bigoplus_{n\geq 0} M_n$ such that $A_i M_j \subseteq M_{i+j}$ for all $i, j \geq 0$. The "blowing-up" construction above generalizes to modules as follows. Let A be a filtered ring and M a filtered A-module. This means, by definition, that we are given a decreasing sequence of A-submodules M_n such that $A_i M_j \subseteq M_{i+j}$ for all $i, j \geq 0$. Then the given multiplication maps $A_i \times M_j \to M_{i+j}$ induce an A^*-action on

$$M^* := \bigoplus_{n\geq 0} M_n,$$

making the latter a graded A^*-module.

Consider, for example, the I-adic filtration on a ring A. Then a filtered A-module is simply a module M with a filtration by submodules M_n such that $IM_n \subseteq M_{n+1}$ for all $n \geq 0$. The following lemma will be needed in Section 5.

Lemma 4. *Let A have the I-adic filtration and let M be a filtered A-module as in the previous paragraph. If M^* is a finitely generated A^*-module, then $IM_n = M_{n+1}$ for all sufficiently large n. Consequently, the I-adic topology on M coincides with the topology induced by the filtration $\{M_n\}$.*

Sketch of proof. For a fixed $N \geq 0$, the A^*-submodule generated by $\bigoplus_{n=0}^N M_n$ is

$$M_0 \oplus \cdots \oplus M_N \oplus IM_N \oplus I^2 M_N \oplus \cdots .$$

If M^* is a finitely generated A^*-module, then this submodule must be all of M^* for some $N \geq 0$, i.e., $I^k M_N = M_{N+k}$ for all $k \geq 0$. This proves the first assertion. For the second, we must show that every neighborhood of 0 in one topology is also a neighborhood of 0 in the other. It suffices to consider neighborhoods in some neighborhood base. By definition, we have $M_n \supseteq I^n M_0 = I^n M$. So every basic neighborhood of 0 in the topology induced by $\{M_n\}$ is a neighborhood of 0 in the I-adic topology. Conversely, consider a basic I-adic neighborhood $I^n M$; we may assume $n \geq N$, with N as above. Then $I^n M \supseteq I^n M_N = M_{N+n}$, so $I^n M$ is a neighborhood of 0 in the topology induced by $\{M_n\}$. $\square$

5. The Artin–Rees lemma

Let A be a noetherian ring and I an ideal. The following result is one version of the *Artin–Rees lemma*.

Theorem 1. *Let M be a finitely generated A-module and M' a submodule. Then the I-adic topology on M' is the same as the subspace topology that M' inherits from the I-adic topology on M.*

Sketch of proof. The subspace topology on M' is induced by the filtration $\{M'_n\}$ with $M'_n := M' \cap I^n M$. In view of Lemma 4, it suffices to show that the corresponding M'^* is a finitely generated A^*-module. Now M^* is easily seen to be a finitely generated A^*-module, since M is a finitely generated A-module. And M'^* is an A^*-submodule of M^*. So the result follows from the fact that A^* is noetherian (Lemma 3). □

6. Digression: The Jacobson radical and Nakayama's lemma

Recall that the *Jacobson radical* of a ring A, denoted $\operatorname{rad} A$, is the intersection of all maximal ideals. It is the ring-theoretic analogue of the Fitting subgroup of a group. It is the largest ideal J such that $1-x$ is invertible in A for all $x \in J$. This is proved in Dummit and Foote on p. 751, which also contains a proof of *Nakayama's lemma.* This asserts that if M is a finitely generated A-module such that $JM = M$ (where $J = \operatorname{rad} A$), then $M = \{0\}$. The following result generalizes Nakayama's lemma.

Proposition 3. *Let I be an arbitrary ideal of A, and let M be a finitely generated A-module such that $IM = M$. Then M is annihilated by $1 - x$ for some $x \in I$.*

Sketch of proof. Let S be the multiplicative set $1+I$. Then any $x = a/s$ in the ideal $S^{-1}I$ of $S^{-1}A$ ($a \in I$, $s \in S$) satisfies $1 - x = (s-a)/s$, which is invertible because the numerator is in $S = 1 + I$. Hence $S^{-1}I \subseteq \operatorname{rad}(S^{-1}A)$. Now the hypothesis $IM = M$ implies that $(S^{-1}I)(S^{-1}M) = S^{-1}M$, so $S^{-1}M = 0$ by Nakayama's lemma. In other words, every element of M is annihilated by some element of S. Since M is finitely generated, it follows that M itself is annihilated by some element of S. □

7. Krull's theorem

We return to the setup of Section 5. Thus A is noetherian and I is an arbitrary ideal. As a consequence of the Artin–Rees lemma, we can determine the intersection of all I-adic neighborhoods of 0 in any finitely generated A-module. The following result is known as *Krull's theorem.*

Theorem 2. *Let M be a finitely generated A-module. Then*

$$\bigcap_{n\geq 0} I^n M = \{m \in M : (1-x)m = 0 \text{ for some } x \in I\}.$$

Sketch of proof. Let M' be the module on the left side of the equation. It is contained in every I-adic neighborhood of 0 in M, so M' is the only neigborhood of 0 in the subspace topology. Theorem 1 now implies that $IM = M$. Hence M' is contained in the right side of the equation by Proposition 3. For the opposite inclusion, every element m in the module on the right satisfies $xm = m$ for some $x \in I$; so trivially $m \in I^n M$ for all n. □

Remark. This is false without the hypothesis that A is noetherian. For example, let A be the ring of infinitely differentiable functions of one real variable, and let I be the ideal consisting of functions that vanish at 0. Then the intersection of the powers of I consists of the functions that vanish along with all their derivatives at 0, whereas any function annihilated by $1 - f$ for some $f \in I$ must vanish in a neighborhood of 0.

Corollary 1. *A is I-adically Hausdorff if (a) A is an integral domain and I is proper or (b) if $I \subseteq \operatorname{rad} A$.*

Sketch of proof. In both cases, no element of the form $1 - x$ with $x \in I$ can be a zero divisor. So Theorem 2 implies that $\bigcap_{n\geq 0} I^n = \{0\}$, and hence A is Hausdorff by Proposition 1. $\square$

Corollary 2. *If $I \subseteq \operatorname{rad} A$, then every finitely generated A-module is I-adically Hausdorff, and every submodule is I-adically closed.*

Sketch of proof. Every element of A of the form $1 - x$ with $x \in I$ is invertible, so the right side of the equation in Theorem 2 is $\{0\}$. The theorem therefore implies that every finitely generated A-module is Hausdorff. Given such a module M and a submodule M', we know that M/M' is Hausdorff in its I-adic topology. Since the quotient map $M \twoheadrightarrow M/M'$ is I-adically continuous, it follows that M', which is the inverse image of $\{0\}$, is closed. $\square$

Remark. An important special case of the last corollary is the case where A is a local ring and I is its maximal ideal.