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★**Buildings.**
Theory and applications.
Graduate Texts in Mathematics, 248.
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There are many excellent introductions to the theory of Tits buildings such as [J. Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Math., 386, Springer, Berlin, 1974; MR0470099 (57 #9866); K. S. Brown, *Buildings*, Springer, New York, 1989; MR0969123 (90e:20001); M. A. Ronan, *Lectures on buildings*, Academic Press, Boston, MA, 1989; MR1005533 (90j:20001); R. M. Weiss, *The structure of spherical buildings*, Princeton Univ. Press, Princeton, NJ, 2003; MR2034361 (2005b:51027); *The structure of affine buildings*, Ann. of Math. Stud., 168, Princeton Univ. Press, Princeton, NJ, 2009; MR2468338]. The book under review, however, is the first encyclopedic treatment of buildings made available in the literature, so that with writing this book the authors manage to close a serious gap in the mathematical literature: the non-existence of an easily accessible reference for buildings beyond an introduction.

In their introduction the authors state that it is their "goal in this book to treat buildings from all three [. . .] points of view [via the simplicial, the combinatorial, and the metric approach]. The various approaches complement one another and are useful. On the other hand, [the authors] recognize that some readers may prefer one particular viewpoint. [The authors] have therefore tried to create more than one path through the book so that, for example, the reader interested only in the combinatorial approach can learn the basics without having to spend too much time studying buildings as simplicial complexes." In the reviewer's opinion, the authors perfectly reach this goal and, moreover, manage to show that the theory of buildings and their applications are active, thriving, exciting and beautiful areas of mathematics. This book is very well written. The reviewer is sure that it will become a standard reference for buildings and remain one for a very long time.

The first five chapters of this book treat "Finite reflection groups", "Coxeter groups", "Coxeter complexes", "Buildings as chamber complexes" (with the simplicial definition of a building in Definition 4.1), and "Buildings as W-metric spaces" (with the combinatorial definition of a building in Definition 5.1 and a variation in Definition 5.21 and Proposition 5.23). A significant advantage of the simplicial approach to buildings, which is used in [J. Tits, op. cit.], is that it helps to intuitively understand buildings as topological objects with a canonical notion of dimension. For instance, the simplicial approach allows one to prove the Solomon-Tits Theorem [L. Solomon, in *Theory of Finite Groups (Symposium, Harvard Univ., Cambridge, Mass., 1968)*, 213–221, Benjamin, New York, 1969; MR0246951 (40 #220) (Theorems 4.73 and 4.127)] without difficulties, while in this context the combinatorial approach would only be suitable for establishing simple connectedness. The latter approach has been introduced in [J. Tits, in *The geometric vein*, 519–547, Springer, New

York, 1981; MR0661801 (83k:51014)], and is used in the variation alluded to above in [M. A. Ronan, op. cit.] and in [R. M. Weiss, op. cit.; MR2034361 (2005b:51027)]. The main advantages of the combinatorial approach to buildings are that it is elementary and abstract and, hence, easily accessible. Since the simplicial and combinatorial approaches are equivalent (see Proposition 4.84 and Theorem 5.91), one can always work with the one which is more suitable for one's own problems or more appealing to one's own taste. It is indeed one of the great achievements of this book to introduce both concepts simultaneously and to combine them in a natural way.

In Chapters 6, 7 and 8 the authors carefully explain why the theory of buildings has such powerful applications in group theory. Many very important and prominent classes of groups such as Chevalley groups, connected semisimple algebraic/Lie groups, finite groups of Lie type, Kac-Moody groups, etc. admit a BN-pair (see Sections 6.5 through 6.12, Section 7.9, and Section 8.11) and, hence, act on a building in a natural way. The theory of buildings provides the means to study structural properties of these groups combinatorially and independently of the underlying field. Recent achievements in this direction include the generalisation of the Curtis-Tits theorem to Kac-Moody groups (see [P. Abramenko and B. Mühlherr, C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), no. 7, 701–706; MR1483702 (98h:20043)] and page 497 of the book under review) and the solution of the isomorphism problem for Kac-Moody groups [see P.-E. Caprace and B. Mühlherr, Invent. Math. **161** (2005), no. 2, 361–388; MR2180452 (2006k:20095); Adv. Math. **206** (2006), no. 1, 250–278; MR2261755 (2007k:20060); P.-E. Caprace, Mem. Amer. Math. Soc. **198** (2009), no. 924, 84 pp.].

By Theorem 7.59 (see also page 274 of [J. Tits, op. cit.; MR0470099 (57 #9866)] and [J. Tits, Invent. Math. **43** (1977), no. 3, 283–295; MR0460485 (57 #478)]) any thick irreducible spherical building of rank at least three is Moufang and therefore admits a strongly transitive group of automorphisms, which thus admits a BN-pair (Theorem 7.10). Since thick spherical Moufang buildings of rank at least two are classifiable (see [J. Tits and R. M. Weiss, *Moufang polygons*, Springer, Berlin, 2002; MR1938841 (2003m:51008)] and also Chapter 9 of the book under review), group theory and the theory of buildings allow for a very fruitful interaction. The concept of a root group datum resp. an RGD system introduced in Definition 7.82 resp. Section 8.6 is well suited for deciding whether a given (twin) building is Moufang. Highlights in this context are Theorem 8.27 (any thick, irreducible, 2-spherical twin building of rank at least three that satisfies property (co) from Section 5.11 is Moufang) and Theorems 8.80 and 8.81 (on how to construct a twin BN-pair and a twin building from an RGD system).

Chapter 8 is extremely valuable for students and researchers interested in the theory of twin buildings, as it makes the theory introduced in [J. Tits, in *Groups, combinatorics & geometry (Durham, 1990)*, 249–286, Cambridge Univ. Press, Cambridge, 1992; MR1200265 (94d:20030)] easily accessible. It is particularly important for the mathematical community that the folklore Theorem 8.27 has finally been recorded. Chapter 9 gives a statement of the classification of spherical buildings. Since proofs of this classification are well documented and easily accessible (see [J. Tits, op. cit.; MR0470099 (57 #9866)] for the original proof and [J. Tits and R. M. Weiss, op. cit.] for a revised proof based on the classification of Moufang polygons), this book does not contain a proof of this classification.

Chapters 10, 11 and 12 concentrate on the metric approach to buildings. The flavour of these three chapters is comparable to [M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer, Berlin, 1999; MR1744486 (2000k:53038)] and [M. W. Davis, *The geometry and topology of Coxeter groups*, Princeton Univ. Press, Princeton, NJ, 2008; MR2360474 (2008k:20091)]. A reader interested in this approach is in the wonderful situation of being able to study all three books simultaneously.

In Chapter 13 the authors describe a fascinating area of mathematics in which buildings are applicable: the cohomology of S-arithmetic groups. In particular, the global function field case currently is a very active field of research with many extremely interesting recent results such as [K.-U. Bux and K. Wortman, Invent. Math. **167** (2007), no. 2, 355–378; MR2270455 (2007k:11082); "Connectivity properties of horospheres in Euclidean buildings and applications to finiteness properties of discrete groups", preprint, arxiv.org/abs/0808.2087]. The reviewer would also be interested in the following sister of Question 13.20: Let Γ be a non-uniform lattice of the completion of a Kac-Moody group G over a finite field, the latter endowed with the topology of compact convergence on one half of its twin building. Is Γ of type F_{n-1} if and only if G is n-spherical? The book concludes with a survey of further interesting applications in Chapter 14.

Altogether, the book under review is a wonderful piece of work that will have many enthusiastic readers.

Reviewed by *Ralf Gramlich*