Mathematics 4340

A nonabelian finite simple group has a proper nonabelian subgroup

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Abelian groups have the property that every subgroup is normal. Simple groups, by contrast, have *no* normal subgroups, except the obvious ones. Intuitively, then, nonabelian simple groups are about as different from abelian groups as possible. I will support that intuition by proving that a nonabelian finite simple group cannot have the property that all of its proper subgroups are abelian.

Suppose G is a nonabelian finite simple group in which every proper subgroup is abelian. We will see that this leads to a contradiction.

Step 1. *If H is any maximal subgroup, then H equals its own normalizer.*

To see this, recall from an easy homework problem that either $N(H) = H$ or $N(H) = G$. The second possibility is ruled out by the simplicity of G, since H is clearly nontrivial.

Step 2. *If H and K are two distinct maximal subgroups, then $H \cap K = \{1\}$.*

Otherwise, their intersection L would be a nontrivial proper subgroup normalized by both H and K, hence by G. [H and K are abelian, so they normalize all of their subgroups. And they generate G, since they are distinct maximal subgroups.]

Step 3. *Any two maximal subgroups are conjugate.*

Suppose that H and K are nonconjugate maximal subgroups. By Step 1, H has $|G : H|$ conjugates and K has $|G : K|$ conjugates. This gives us $|G : H| + |G : K|$ distinct maximal subgroups, containing a total of

$$n := |G : H| \cdot (|H| - 1) + |G : K| \cdot (|K| - 1)$$

nontrivial elements by Step 2. Carrying out the multiplications, we find

$$\begin{aligned} n &= 2|G| - |G : H| - |G : K| \\ &= 2|G| - \frac{|G|}{|H|} - \frac{|G|}{|K|} \\ &\geq 2|G| - \frac{|G|}{2} - \frac{|G|}{2} \\ &= |G|. \end{aligned}$$

But then G has at least $|G|$ nontrivial elements, which is absurd.

Step 4. *G is the union of its maximal subgroups.*

This is almost obvious; for if x is any element of G, then the cyclic subgroup $\langle x \rangle$ must be proper [because G is nonabelian], so it can be enlarged to a maximal subgroup.

We now have the desired contradiction: Steps 3 and 4 establish that G is the union of the conjugates of a proper subgroup, but you proved in Exercise 4.3.24 that this is impossible. Recalling the proof as specialized to the present context, the number of nontrivial elements in the union of the conjugates of a single maximal H is

$$|G : H| \cdot (|H| - 1) = |G| - |G : H| < |G| - 1.$$

So this union can't possibly be all of G.