REMARKS ON ASSOCIATED PRIMES

All rings are commutative in what follows. Recall the following consequence of the theory of primary decomposition: For any proper ideal $I$ of a noetherian ring $A$, there is a finite set of “associated primes” $\mathfrak{p}$ with the following two properties:

(a) A prime $\mathfrak{p}$ is associated to $I$ if and only if $\mathfrak{p}$ is the annihilator of some element of $A/I$.
(b) The union of the associated primes is the set of elements of $A$ that are 0-divisors in $A/I$.

There are similar results with $A/I$ replaced by an arbitrary finitely-generated $A$-module. One can prove this by generalizing the theory of primary decomposition to modules; everything goes through with no essential change. But, for variety, here is a more direct approach, based on Eisenbud, Chapter 3. To get started, we simply take property (a) as a definition.

**Definition.** Let $M$ be an $A$-module. A prime $\mathfrak{p}$ of $A$ is said to be associated to $M$ if $M$ contains an element whose annihilator is $\mathfrak{p}$ or, equivalently, if there is an embedding $A/\mathfrak{p} \hookrightarrow M$. The set of primes associated to $M$ is denoted $\text{Ass}(M)$.

**Theorem.** If $A$ is noetherian and $M$ is a finitely generated nonzero $A$-module, then $\text{Ass}(M)$ is finite and nonempty. The union of the primes in $\text{Ass}(M)$ is the set of elements of $A$ that are 0-divisors in $M$.

This has the following consequence, which is by no means obvious a priori:

**Corollary.** Let $A$ and $M$ be as in the theorem, and let $I$ be an ideal of $A$. If every element of $I$ is a 0-divisor in $M$, then there is a single nonzero element of $M$ that is annihilated by $I$.

**Proof.** $I$ is contained in the union of the associated primes, so it must be contained in one of them.

The proof of the theorem will now be given in a series of lemmas. The first step is to prove the existence of at least one associated prime. To this end we need only choose a maximal annihilator (which is possible because $A$ is noetherian):

**Lemma 1.** Let $A$ be a noetherian ring and $M$ a nonzero $A$-module. Let $\mathfrak{p}$ be maximal among the ideals that occur as annihilators of nonzero elements of $M$. Then $\mathfrak{p}$ is prime and hence is in $\text{Ass}(M)$.

**Proof.** Let $\mathfrak{p}$ be the annihilator of $x \in M$. By maximality, $\mathfrak{p}$ is also the annihilator of every nonzero element of $Ax$. Suppose now that $ab \in \mathfrak{p}$ ($a, b \in A$). Then $abx = 0$. If $bx = 0$, then $b \in \mathfrak{p}$ and we’re done. Otherwise, $a$ annihilates the nonzero element $bx$ of $Ax$, so $a \in \mathfrak{p}$.

Note that we could start with any annihilator and enlarge it to a maximal one. This immediately yields the second assertion of the theorem. It remains to prove that $\text{Ass}(M)$ is finite.

**Lemma 2.** Let $A$ be a noetherian ring and $M$ a finitely-generated $A$-module. Then $M$ has a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that each layer $M_i/M_{i-1}$ ($i = 1, \ldots, n$) is cyclic with prime annihilator, i.e., is isomorphic to $A/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i$. 

Proof. If $M \neq 0$, then Lemma 1 gives us a submodule $M_1 \subseteq M$ isomorphic to $A/p_1$ for some prime $p_1$. If $M/M_1 \neq 0$, then we can apply the same result to $M/M_1$ to get $M_2 \supset M_1$ with $M_2/M_1 \cong A/p_2$. Continuing in this way, we eventually reach $M_n = M$ by the ascending chain condition.

We can now prove the finiteness of $\text{Ass}(M)$ and thereby complete the proof of the theorem.

**Lemma 3.**

(a) *Given a short exact sequence* $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ *of $A$-modules,*

$$\text{Ass}(M) \subseteq \text{Ass}(M') \cup \text{Ass}(M'').$$

(b) *In the situation of Lemma 2,*

$$\text{Ass}(M) \subseteq \{p_1, \ldots, p_n\}.$$  

**Proof.** (a) Given $p \in \text{Ass}(M)$, we have an element $x \in M$ with $A/p \cong Ax \subseteq M$. If the composite $A/p \hookrightarrow M \twoheadrightarrow M''$ is injective, we get $p \in \text{Ass}(M'')$. Otherwise, $Ax \cap M'$ is nonzero and $p$ is the annihilator of each of its nonzero elements; so $p \in \text{Ass}(M')$.

(b) Repeatedly apply (a), noting that $\text{Ass}(A/p) = \{p\}$ if $p$ is prime. \qed