REMARKS ON ASSOCIATED PRIMES

All rings are commutative in what follows. Recall the following consequence of the theory of primary decomposition: For any proper ideal I of a noetherian ring A, there is a finite set of "associated primes" $\mathfrak{p}$ with the following two properties:

(a) A prime $\mathfrak{p}$ is associated to I if and only if $\mathfrak{p}$ is the annihilator of some element of A/I.
(b) The union of the associated primes is the set of elements of A that are 0-divisors in A/I.

There are similar results with A/I replaced by an arbitrary finitely-generated A-module. One can prove this by generalizing the theory of primary decomposition to modules; everything goes through with no essential change. But, for variety, here is a more direct approach, based on Eisenbud, Chapter 3. To get started, we simply take property (a) as a definition.

Definition. Let M be an A-module. A prime $\mathfrak{p}$ of A is said to be *associated* to M if M contains an element whose annihilator is $\mathfrak{p}$ or, equivalently, if there is an embedding $A/\mathfrak{p} \hookrightarrow M$. The set of primes associated to M is denoted $\mathrm{Ass}(M)$.

Theorem. *If A is noetherian and M is a finitely generated nonzero A-module, then $\mathrm{Ass}(M)$ is finite and nonempty. The union of the primes in $\mathrm{Ass}(M)$ is the set of elements of A that are 0-divisors in M.*

This has the following consequence, which is by no means obvious *a priori*:

Corollary. *Let A and M be as in the theorem, and let I be an ideal of A. If every element of I is a 0-divisor in M, then there is a single nonzero element of M that is annihilated by I.*

Proof. I is contained in the union of the associated primes, so it must be contained in one of them. □

The proof of the theorem will now be given in a series of lemmas. The first step is to prove the existence of at least one associated prime. To this end we need only choose a maximal annihilator (which is possible because A is noetherian):

Lemma 1. *Let A be a noetherian ring and M a nonzero A-module. Let $\mathfrak{p}$ be maximal among the ideals that occur as annihilators of nonzero elements of M. Then $\mathfrak{p}$ is prime and hence is in $Ass(M)$.*

Proof. Let $\mathfrak{p}$ be the annihilator of $x \in M$. By maximality, $\mathfrak{p}$ is also the annihilator of every nonzero element of Ax. Suppose now that $ab \in \mathfrak{p}$ $(a, b \in A)$. Then $abx = 0$. If $bx = 0$, then $b \in \mathfrak{p}$ and we're done. Otherwise, a annihilates the nonzero element bx of Ax, so $a \in \mathfrak{p}$. □

Note that we could start with any annihilator and enlarge it to a maximal one. This immediately yields the second assertion of the theorem. It remains to prove that $\mathrm{Ass}(M)$ is finite.

Lemma 2. *Let A be a noetherian ring and M a finitely-generated A-module. Then M has a finite filtration*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that each layer M_i/M_{i-1} $(i = 1, \ldots, n)$ is cyclic with prime annihilator, i.e., is isomorphic to $A/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i$.

Proof. If $M \neq 0$, then Lemma 1 gives us a submodule $M_1 \subseteq M$ isomorphic to $A/\mathfrak{p}_1$ for some prime $\mathfrak{p}_1$. If $M/M_1 \neq 0$, then we can apply the same result to M/M_1 to get $M_2 \supset M_1$ with $M_2/M_1 \cong A/\mathfrak{p}_2$. Continuing in this way, we eventually reach $M_n = M$ by the ascending chain condition. □

We can now prove the finiteness of $\operatorname{Ass}(M)$ and thereby complete the proof of the theorem.

Lemma 3.

(a) *Given a short exact sequence* $0 \to M' \to M \to M'' \to 0$ *of* A*-modules,*

$$\operatorname{Ass}(M) \subseteq \operatorname{Ass}(M') \cup \operatorname{Ass}(M'').$$

(b) *In the situation of Lemma 2,*

$$\operatorname{Ass}(M) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}.$$

Proof. (a) Given $\mathfrak{p} \in \operatorname{Ass}(M)$, we have an element $x \in M$ with $A/\mathfrak{p} \cong Ax \subseteq M$. If the composite $A/\mathfrak{p} \hookrightarrow M \to M''$ is injective, we get $\mathfrak{p} \in \operatorname{Ass}(M'')$. Otherwise, $Ax \cap M'$ is nonzero and $\mathfrak{p}$ is the annihilator of each of its nonzero elements; so $\mathfrak{p} \in \operatorname{Ass}(M')$.

(b) Repeatedly apply (a), noting that $\operatorname{Ass}(A/\mathfrak{p}) = \{\mathfrak{p}\}$ if $\mathfrak{p}$ is prime. □