Lectures on the Cohomology of Groups

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0 Historical introduction

The cohomology theory of groups arose from both topological and algebraic sources. The starting point for the topological aspect of the theory was a 1936 paper by Hurewicz [7], in which he introduced aspherical spaces. These are spaces X such that $\pi_n(X) = 0$ for $n \neq 1$. (Hurewicz had introduced higher homotopy groups just one year earlier, and he was now trying to understand the spaces with the simplest possible higher homotopy groups.) Hurewicz proved that such an X is determined up to homotopy equivalence by its fundamental group $\pi := \pi_1(X)$. Thus homotopy invariants of X can be thought of as invariants of the group π. Examples of such invariants include homology, cohomology, and the Euler characteristic. Thus we can define

$$H_*(\pi) := H_*(X) \tag{0.1}$$

if X is an aspherical space with fundamental group π, and similarly for cohomology and the Euler characteristic. [We will replace (0.1) with an equivalent algebraic definition in the next section.]

For example, we have $H_0(\pi) = \mathbb{Z}$ for all π (so H_0 is a boring invariant of π), and $H_1(\pi) = \pi_{\text{ab}} := \pi/[\pi, \pi]$, the abelianization of π. This is less boring, but still well known. The next group, $H_2(\pi)$ is more interesting. As a sample calculation, note that $H_2(\pi) = \mathbb{Z}$ if π is free abelian of rank 2, since we can take X to be the torus $S^1 \times S^1$.

The next step was taken by Hopf [6] in 1942. One of the results of his paper is that $H_2(\pi)$ can be used to measure the failure of the Hurewicz map to be surjective in dimension 2. More precisely, consider an arbitrary path-connected space X (not necessarily aspherical), and set $\pi := \pi_1(X)$. The Hurewicz map $h_2 \colon \pi_2(X) \to H_2(X)$ is an isomorphism if π is the trivial group (i.e., if X is simply connected). In general, according to Hopf, there is an exact sequence

$$\pi_2(X) \xrightarrow{h_2} H_2(X) \longrightarrow H_2(\pi) \longrightarrow 0.$$

A second result of Hopf's paper is a calculation of $H_2(\pi)$ if π is given as a quotient of a free group. Suppose $\pi = F/R$, where F is a free group and R is a normal subgroup. Then Hopf's formula says

$$H_2(\pi) \cong (R \cap [F,F])\,/[R,F].$$

Thus $H_2(\pi)$ can be viewed as the group of relations among commutators, modulo those relations that hold trivially.

Following Hopf's paper, the subject developed rapidly in the 1940s, due primarily to the work of Eckmann, Eilenberg–MacLane, Freudenthal, and Hopf. By the end of the decade there was a purely algebraic definition of the homology and cohomology of a group. As a consequence of the algebraic definition, one could make connections with algebra going back to the early 1900s. For example, H^1 turned out to be a group of equivalence classes of "derivations" (also called "crossed homomorphisms"). And H^2 turned out to be similarly related to "factor sets", which had been studied by Schur [11], Schreier [9], and Brauer [2], while H_2 coincided with the "Schur multiplier" (also introduced in [11]). Finally, H^3 had appeared in the work of Teichmüller [14]. These are the algebraic sources referred to in the first sentence of this section. Of course, none of this algebra had suggested that derivations, factor sets, and so on were part of a coherent "cohomology theory". This had to wait for the impetus from topology.

We now proceed to the six lectures, which correspond to the six sections that follow. They give a very brief introduction to the homology and cohomology theory of groups, with an emphasis on infinite groups and finiteness properties. The lectures are based on my book [3] and are organized as follows:

1. In the first lecture we will redefine $H_*(G)$ for an arbitrary group G, taking the algebraic point of view (homological algebra) that had evolved by the end of the 1940s. Although we are now thinking algebraically, we will always keep the topological interpretation in mind and will use it when convenient.
2. In the second lecture we take a first look at finiteness properties of infinite groups. These properties may be defined either algebraically or topologically.
3. The third lecture is devoted to $H_*(G, M)$ and $H^*(G, M)$, homology and cohomology with coefficients in a G-module M. These generalize $H_*(G)$, which is $H_*(G, \mathbb{Z})$ (with trivial G-action on the coefficient

module $\mathbb{Z}$). They are important for both algebraic and topological applications, even if one is primarily interested in the special case $H_*(G)$.

4. With the material of Lecture 3 available, we can treat finiteness properties more seriously in Lecture 4. In particular, we will discuss homological duality. This includes Poincaré duality as a special case, but it is much more general.
5. Lecture 5 is technical. I will attempt to give an introduction to equivariant homology theory and the associated spectral sequences.
6. The final lecture treats the special features of cohomology theory when the group G is finite. Although we are mainly interested in infinite groups in these lectures, anyone learning the cohomology theory of groups for the first time should know the basic facts about the finite case.

1 The homology of a group

A reference for this section is [3, Chapter II]. We will give an algebraic approach to homology theory, based on free resolutions. These generalize presentations of modules by generators and relations. If R is a ring and M is an R-module with n generators, then we have an exact sequence

$$R^n \longrightarrow M \longrightarrow 0. \tag{1.1}$$

Here n is a cardinal number, possibly infinite. The surjection $R^n \twoheadrightarrow M$ has a kernel K, whose elements represent relations among the given generators. If K admits m generators as an R-module (so that M is defined by n generators and m relations), then we can map R^m onto K, thereby obtaining a continuation of (1.1) to a diagram

$$\begin{array}{ccccccc} R^m & \longrightarrow & R^n & \longrightarrow & M & \longrightarrow & 0 \\ & \searrow & \nearrow & & & & \\ & & K & & & & \end{array} \tag{1.2}$$

with the top row exact. The surjection $R^m \twoheadrightarrow K$ has a kernel L (consisting of "relations among the relations"). Choosing a free module that maps onto L and continuing *ad infinitum*, we obtain an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0,$$

where each F_i is a free R-module. Such an exact sequence is called a *free resolution* of M.

We will mainly be interested in the case where R is the integral group ring $\mathbb{Z}G$ of a group G and $M = \mathbb{Z}$, with trivial G-action. This situation arises naturally in topology, as follows. Suppose X is a G-CW-complex, by which we mean a CW-complex with a G-action that permutes the cells. Then G acts on the cellular chain complex $C_*(X)$, which therefore becomes a chain complex of $\mathbb{Z}G$-modules. This complex is naturally augmented over $\mathbb{Z}$, so we have a diagram

$$\cdots \to C_2(X) \to C_1(X) \to C_0(X) \to \mathbb{Z} \to 0 \tag{1.3}$$

of $\mathbb{Z}G$-modules. If G acts freely on X (i.e., all cell stabilizers are trivial), then each module $C_n(X)$ is a direct sum of copies of $\mathbb{Z}G$, with one copy for each G-orbit of n-cells. In particular, $C_n(X)$ is a free $\mathbb{Z}G$-module. If, in addition, X is contractible, then (1.3) is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

We can reformulate this as follows: Introduce the quotient complex $Y := G\backslash X$. If G acts freely on X, then we can view X as a regular covering space of Y, with G as the group of deck transformations. If X is simply connected, then X is the universal cover of Y, and G can be identified with $\pi_1(Y)$. If, in addition, X is contractible, then elementary homotopy theory implies that the higher homotopy groups $\pi_i(Y)$ are trivial, since these do not change when one passes to a covering space. Thus Y is an aspherical space in the sense of Section 0. Conversely, if we start with an aspherical CW-complex Y, then its universal cover X has trivial homotopy groups in all dimensions and hence is contractible by a theorem of Whitehead. So the situation we described in the previous paragraph can be summarized as follows: If Y is an aspherical CW-complex with fundamental group G, then its universal cover X is a contractible, free G-CW-complex, whose cellular chain complex gives rise to a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ as in (1.3). There is some standard notation and terminology associated with this situation:

Definition 1.1. Let Y be a CW-complex with fundamental group G. We say that Y is an *Eilenberg–MacLane complex* of type $K(G, 1)$ if it is aspherical or, equivalently, if its universal cover is contractible.

In some contexts one also writes $Y = BG$ and calls Y a *classifying space* for G, but we will stick to the $K(G, 1)$ notation here.

It is a fact, which is not difficult to prove, that every group G admits a $K(G, 1)$-complex Y; moreover, Y is unique up to homotopy equivalence. (The uniqueness part is a theorem of Hurewicz that we quoted in Section 0.) This fact has an algebraic analogue, whose proof is even easier, which is sometimes called the *fundamental lemma of homological algebra*: Given any module M (over an arbitrary ring), free resolutions of M exist and are unique up to chain homotopy equivalence. We already proved existence above. And a proof of uniqueness can be found in any book on homological algebra. It consists of constructing the desired chain maps and homotopies step by step. At each step, one has to solve a mapping problem of the following form:

$$\begin{array}{ccccc} & & F & & \\ & \swarrow & \downarrow & \overset{0}{\searrow} & \\ M' & \longrightarrow & M & \longrightarrow & M'' \end{array}$$

where F is free and the row is exact. The solid arrows represent given maps, with the composite $F \to M \to M''$ equal to the zero map, and the dotted arrow represents a map we want to construct. Note that the image of $F \to M$ lies in the kernel of $M \to M''$ and hence in the image of $M' \to M$; so we can construct the desired map by lifting the images of basis elements of the free module F.

Free modules are not the only modules for which such mapping problems can be solved. If P is a module such that all mapping problems as above can be solved (with F replaced by P), then P is said to be a *projective* module. There are many equivalent characterizations of projective modules. For example, a module P is projective if and only if it is a direct summand of a free module, i.e., there is a module Q such that $P \oplus Q$ is free.

The upshot of the discussion above is that the fundamental lemma of homological algebra is applicable to *projective resolutions*, not just to free ones. This gives us more flexibility in constructing resolutions of a module, all of which are homotopy equivalent. Even in the setting where we can find a free resolution via topology as above, it might be more convenient to consider projective resolutions that are not necessarily free (and hence, in particular, do not necessarily come from topology).

We proceed now to two examples, where we do in fact use topology to get the desired resolutions.

Example 1.2. Let $G = F(S)$, the free group generated by a set S. Let X be the Cayley graph of G with respect to S. It is a tree with vertex set G, with an edge from g to gs for each $g \in G$ and $s \in S$. It is a free, contractible, G-CW-complex. There is a single G-orbit of vertices, and there is one G-orbit of edges for each $s \in S$. Thus the quotient complex Y is a bouquet of circles indexed by S. The resulting free resolution is

$$0 \longrightarrow \mathbb{Z}G^{(S)} \xrightarrow{\partial} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0, \tag{1.4}$$

where $\mathbb{Z}G^{(S)}$ is a free $\mathbb{Z}G$-module with a basis $(e_s)_{s\in S}$. Here $\varepsilon\colon \mathbb{Z}G \to \mathbb{Z}$ is the *canonical augmentation*, given by $\varepsilon(g) = 1$ for all $g \in G$, and $\partial\colon \mathbb{Z}G^{(S)} \to \mathbb{Z}G$ is given by $\partial(e_s) = s - 1$ for $s \in S$. (This is because the circle corresponding to s lifts to the edge in X from the vertex 1 to the vertex s.) Although we have proved the exactness of (1.4) via topology, it is possible to give a purely algebraic proof; see [3, Section IV.4.2, Exercise 3].

In the special case where S is a singleton $\{t\}$ (i.e., G is an infinite cyclic group generated by t), we can identify $\mathbb{Z}G$ with the ring $\mathbb{Z}[t, t^{-1}]$ of Laurent polynomials, and (1.4) becomes

$$0 \longrightarrow \mathbb{Z}[t, t^{-1}] \xrightarrow{t-1} \mathbb{Z}[t, t^{-1}] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0. \tag{1.5}$$

The arrow labeled $t-1$ is given by multiplication by $t-1$, and ε can be described as the evaluation map $f \mapsto f(1)$ for $f = f(t) \in \mathbb{Z}[t, t^{-1}]$. In this special case, it is quite easy to verify the exactness by pure algebra.

Example 1.3. Suppose X is a free G-CW-complex homeomorphic to an odd-dimensional sphere S^{2k-1}. Then X is of course not contractible, but we can still use its cellular chain complex $C_* := C_*(X)$ to get a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Indeed, X has homology only in dimensions 0 and $2k-1$, where it is $\mathbb{Z}$ (with trivial G-action by the Lefschetz fixed-point theorem). So we have an exact sequence of $\mathbb{Z}G$-modules

$$0 \to \mathbb{Z} \to C_{2k-1} \to \cdots \to C_1 \to C_0 \to \mathbb{Z} \to 0 \tag{1.6}$$

with each C_i free. We can now obtain the desired free resolution by splicing together infinitely many copies of (1.6):

$$\begin{array}{ccccccc} \cdots \longrightarrow C_1 \longrightarrow & C_0 & & \longrightarrow & & C_{2k-1} & \longrightarrow \cdots \longrightarrow C_0 \longrightarrow \mathbb{Z} \longrightarrow 0 \\ & & \searrow & & \nearrow & & \\ & & & \mathbb{Z} & & & \end{array} \tag{1.7}$$

Note that, in contrast, to our previous examples, this resolution continues forever to the left.

As a simple special case, consider $G = \langle\, t \,;\, t^n = 1 \,\rangle$, the finite cyclic group of order n, acting on the circle by rotation. We can triangulate the circle by cutting it into n arcs, so that there is a single orbit of 1-cells and a single orbit of 0-cells. It is then easy to check that (1.6) becomes

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\eta} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0, \tag{1.8}$$

where ε is again the canonical augmentation and

$$\eta(1) = N := \sum_{i=0}^{n-1} t^i.$$

(Note that N is simply the sum of all the group elements. Such an element N exists for any finite group G and is called the *norm element* of $\mathbb{Z}G$.) The resulting resolution (1.7) is then

$$\cdots \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0, \tag{1.9}$$

the maps being alternately multiplication by N and by $t-1$. As in (1.5), it is not hard to verify exactness by pure algebra. (Note that $\mathbb{Z}G = \mathbb{Z}[t]/(t^n-1)$ and that $t^n - 1 = (t-1)N$ in the polynomial ring $\mathbb{Z}[t]$.)

We are now ready to give the (algebraic) definition of the homology of a group G. It makes use of the *coinvariants* functor $M \mapsto M_G$, which is the algebraic analogue of forming the quotient by a G-action. Here M is a G-module (i.e., a $\mathbb{Z}G$-module), and

$$M_G := M/\langle gm - m \mid g \in G,\, m \in M\rangle,$$

where the angle brackets denote the subgroup generated by the given elements. Thus M_G is the largest quotient of M on which G acts trivially. (Compare this to the set of *invariants*

$$M^G := \{m \in M \mid gm = m \text{ for all } g \in G\},$$

which is the largest submodule of M on which G acts trivially.)

Definition 1.4. Given a group G, choose a projective resolution $P = (P_i)_{i\geq 0}$ of $\mathbb{Z}$ over $\mathbb{Z}G$ and set

$$H_*(G) := H_*(P_G).$$

In words, we apply the coinvariants functor to the acyclic chain complex P and then take homology. The fundamental lemma of homological algebra guarantees that $H_*(G)$ is well-defined (independent of the choice of P) up to canonical isomorphism. We also immediately get

$$H_*(G) = H_*(Y)$$

if Y is a $K(G,1)$-complex. Indeed, if X is the universal cover of Y, then we can take P to be $C_*(X)$, and then $P_G = C_*(Y)$.

From the algebraic point of view, we can think of $H_*(G)$ as measuring the failure of the coinvariants functor to be exact. (If it were exact, then P_G would again be acyclic and hence would have trivial homology in positive dimensions.) But this algebraic point of view would seem very artificial without the motivation from topology. The latter dictated the choice of $\mathbb{Z}G$ as the ring, the choice of $\mathbb{Z}$ as the module to resolve, and the choice of coinvariants as the functor to apply.

Exercise 1.5. Use the resolutions in Examples 1.2 and 1.3 to calculate $H_*(G)$ if G is free or cyclic.

2 Finiteness properties (introduction)

There is a canonical free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, sometimes called the bar resolution, that one can use in principle to compute the homology of any group G. It comes from a canonical $K(G,1)$-complex. See [3, Section I.5]. The bar resolution and canonical $K(G,1)$ are useful for theoretical purposes, but they are very big. If one wants to actually compute the homology, or at least discover qualitative properties of it (such as finite generation or vanishing in high dimensions), it is desirable to have a small resolution P or a small $K(G,1)$-complex Y. Here "small" might mean that $P_n = 0$ for large n (or that Y is finite dimensional). Alternatively, it might mean that each module P_n is finitely generated (or that Y has only finitely many cells in each dimension). Examples 1.2 and 1.3 illustrate this. We will treat these two notions of smallness in the two subsections that follow. Our treatment is based on [3, Chapter VIII].

2.1 Dimension

There are two natural definitions of the dimension of a group, depending on whether we think topologically or algebraically.

Definition 2.1. The *geometric dimension* of G, denoted $\operatorname{gd} G$, is the smallest non-negative integer n such that there exists an n-dimensional $K(G,1)$-complex. (Or, if no such n exists, then we set $\operatorname{gd} G = \infty$.)

Definition 2.2. The *cohomological dimension* of G, denoted $\operatorname{cd} G$, is the smallest non-negative integer n such that there exists a projective resolution $P = (P_i)_{i\geq 0}$ of $\mathbb{Z}$ over $\mathbb{Z}G$ of length $\leq n$, i.e., satisfying $P_i = 0$ for $i > n$. (Or, if no such n exists, then we set $\operatorname{cd} G = \infty$.)

Remark 2.3. It is not clear at this stage why we call $\operatorname{cd}(-)$ "cohomological dimension" instead of, for example, "projective dimension". The name comes from a characterization of $\operatorname{cd} G$ that we will explain in Section 4.

Since a $K(G,1)$-complex Y yields a free resolution of length equal to the dimension of Y, it is clear that

$$\operatorname{cd} G \leq \operatorname{gd} G. \tag{2.1}$$

Another simple observation is that dimension can only go down if one passes to a subgroup. In other words, if $\dim(-)$ denotes either cohomological dimension or geometric dimension, then

$$\dim(H) \leq \dim(G) \quad \text{if } H \leq G. \tag{2.2}$$

For cd this follows from the fact that $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module, which implies that a projective $\mathbb{Z}G$-module is also projective as a $\mathbb{Z}H$-module. Hence any projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ is still a projective resolution of the same length when viewed as a complex of $\mathbb{Z}H$-modules. For gd one instead argues using covering spaces. [Any $K(G,1)$-complex has a covering space that is a $K(H,1)$-complex of the same dimension.]

Examples 2.4. (a) A 0-dimensional $K(G,1)$ is necessarily a point, so it exists if and only if G is the trivial group. The algebraic analogue of this (which requires a little thought but is still easy) is that $\mathbb{Z}$ is a projective $\mathbb{Z}G$-module if and only if G is the trivial group. Hence

$$\operatorname{cd} G = 0 \iff \operatorname{gd} G = 0 \iff G = \{1\}\,.$$

(b) A 1-dimensional $K(G,1)$ is necessarily homotopy equivalent to a bouquet of circles, so there exists one if and only if G is free. (For the "if" part, see Example 1.2.) It is also true, but much more difficult, that $\operatorname{cd} G = 1$ only if G is free. This is a deep theorem of Stallings [12] and Swan [13]. Hence

$$\operatorname{cd} G = 1 \iff \operatorname{gd} G = 1 \iff G \text{ is free and nontrivial.}$$

(c) If G is the fundamental group of a closed surface Y other than the sphere or projective plane, then $\operatorname{cd} G = \operatorname{gd} G = 2$. Indeed, Y is a $K(G,1)$ since its universal cover is homeomorphic to $\mathbb{R}^2$, so $\operatorname{cd} G \le \operatorname{gd} G \le 2$; equality holds in the orientable case because $H_2(G) = H_2(Y) \ne 0$, implying $\operatorname{cd} G \ge 2$. In the non-orientable case one can give a similar argument based on mod 2 homology, or one can pass to the orientable double cover and apply (2.2).

(d) The free abelian group $G = \mathbb{Z}^n$ has $\operatorname{cd} G = \operatorname{gd} G = n$. This follows as in (c) from the fact that the n-torus is a $K(G,1)$ and has nontrivial homology in dimension n. More generally, if G is the fundamental group of a closed aspherical n-manifold, then $\operatorname{cd} G = \operatorname{gd} G = n$.

(e) If G is a nontrivial finite cyclic group, then $\operatorname{cd} G = \operatorname{gd} G = \infty$ since G has nontrivial homology in arbitrarily high dimensions (see Exercise 1.5). In view of (2.2), it follows that $\operatorname{cd} G = \operatorname{gd} G = \infty$ for any group G with torsion. Equivalently,

$$\operatorname{cd} G < \infty \implies G \text{ is torsion-free.}$$

This can be used to prove, for instance, that knot groups are torsion-free (since knot complements are aspherical).

(f) Torsion-free arithmetic groups have finite cohomological dimension, which can be calculated explicitly by methods that we will explain at the end of Section 4.

It is not known whether equality always holds in (2.1). We have seen in (a) and (b) above that equality holds if $\operatorname{cd} G \le 1$. It also holds if $\operatorname{cd} G \ge 3$. In fact, for any $n \ge 3$ one can show by fairly straightforward homotopy theory that the existence of a projective resolution of length n implies the existence of an n-dimensional $K(G,1)$-complex. The argument breaks down if $n = 2$, however, so it is conceivable that there is a group G with $\operatorname{cd} G = 2$ but $\operatorname{gd} G = 3$. The famous *Eilenberg–Ganea problem* asks whether or not this can happen. In summary, we have:

Theorem 2.5. *For any group G,*

$$\operatorname{cd} G \leq \operatorname{gd} G,$$

with equality except possibly if $\operatorname{cd} G = 2$ *but* $\operatorname{gd} G = 3$.

In view of the close connection between $\operatorname{cd} G$ and $\operatorname{gd} G$, we will often simply write $\dim G$ in cases where they are known to be equal or where it does not matter which one we use. For example, we can speak of *finite-dimensional groups* and write $\dim G < \infty$.

2.2 The F_n and FP_n conditions

Definition 2.6. We say that a group G is of *type F_n* $(0 \leq n < \infty)$ if there is a $K(G,1)$-complex with a finite n-skeleton, i.e., with only finitely many cells in dimensions $\leq n$. We say that G is of *type F_∞* if there is a $K(G,1)$ with all of its skeleta finite, and that G is of *type F* if there is a finite $K(G,1)$.

One can always build a $K(G,1)$-complex by starting with a single vertex, then attaching 1-cells corresponding to generators of G, then attaching 2-cells corresponding to relators, and then attaching 3-cells, 4-cells, and so on, to kill the higher homotopy groups π_2, $\pi_3, \dots$. This leads to the following interpretation of the F_n property for small n:

- Every group is of type F_0.
- G is of type F_1 if and only if it is finitely generated.
- G is of type F_2 if and only if it is finitely presented.

The successively stronger higher finiteness properties F_3, F_4, $\dots$, F_∞, and F are more subtle and do not have simple group-theoretic interpretations. Groups of type F, of course, are finite dimensional; in particular, only torsion-free groups can have this property.

Remarks 2.7. (a) All of the examples mentioned in our discussion of dimension are of type F_∞ (provided, in the case of free groups, that we require the group to be finitely generated). And all of the torsion-free groups (with the same proviso) are even of type F.

(b) In view of (a), one might wonder whether all torsion-free groups of type F_∞ are in fact of type F. This is not the case. The first counterexample was given by Brown and Geoghegan [4]. That example, however, is infinite dimensional. So one can still ask whether every *finite-dimensional* group of type F_∞ is of type F. In other words, if there is a $K(G,1)$-complex with only finitely many cells in each dimension, and if there is also a $K(G,1)$-complex that is finite-dimensional, is there a $K(G,1)$-complex that satisfies both of these conditions simultaneously? This question is still open. We will understand the difficulty better shortly, after we discuss the FP_n conditions.

Definition 2.8. We say that a group G is of *type* FP_n $(0 \leq n < \infty)$ if there is a projective resolution P of $\mathbb{Z}$ over $\mathbb{Z}G$ such that P_i is finitely generated for $i \leq n$. We say that G is of *type* FP_∞ if there is a projective resolution P of $\mathbb{Z}$ over $\mathbb{Z}G$ with P_i finitely generated for all i, and that G is of *type FP* if there is a finite projective resolution, i.e., a projective resolution such that P_i is finitely generated for all i and is 0 for sufficiently large i.

It is obvious that $\mathrm{F}_n \implies \mathrm{FP}_n$. Moreover, one can show:

- Every group is of type FP_0.
- G is of type FP_1 if and only if it is finitely generated (so FP_1 is equivalent to F_1).
- G is of type FP_2 if and only if $G \cong \widetilde{G}/N$, where $\widetilde{G}$ is finitely presented and N is a perfect normal subgroup.

If $G \cong \widetilde{G}/N$ as above, then G is finitely presented if and only if N is finitely generated as a normal subgroup. Examples due to Bestvina and Brady [1] show that this need not be the case. In other words, FP_2 is definitely weaker than F_2. For higher n, however, there is no further difference between FP_n and F_n beyond finite presentability. In other words, G is of type F_n $(2 \leq n \leq \infty)$ if and only if it is finitely presented and of type FP_n. The analogous statement for property F is not known:

Question 2.9. If G is finitely presented and of type FP, is G of type F?

To understand this better, we introduce one more finiteness condition, which is algebraic in nature but is motivated by topology:

Definition 2.10. G is of *type FL* if there is a finite free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, i.e., a free resolution F such that F_i is finitely generated for all i and is 0 for sufficiently large i.

(The “L” in “FL” stands for “libre”. Some authors write “FF”, for “finite free” instead of “FL”.)

Now it is not difficult to show that G is of type F if and only if it is finitely presented and of type FL. So the real question underlying Question 2.9 is:

Question 2.11. If G is of type FP, is it of type FL?

Remarks 2.12. (a) A group G is of type FP if and only if it is finite dimensional and of type FP_∞. (Thus algebra is easier than topology in this setting, cf. Remark 2.7(b).) In fact, if G is of type FP_∞ and finite dimensional, then one can construct a projective resolution step by step using finitely generated modules, and the kernel will eventually be projective. One can therefore stop at that stage and have a finite projective resolution.

(b) In the step-by-step construction just described, we can use finitely-generated free modules at each stage. The kernels will all be finitely generated, so there is no obstruction to continuing the process. But when we reach the point where the kernel K is projective, we cannot be sure that it will be free. It is therefore not clear that we can stop and have a finite free resolution. In fact, one can show that there exists a finite *free* resolution if and only if K is stably free. So Questions 2.9 and 2.11 would have affirmative answers if we knew that finitely-generated projective $\mathbb{Z}G$-modules are always stably free if G is torsion free. A famous conjecture asserts that this is the indeed the case; see Lück's lectures in this volume.

(c) It can be shown that a group G is finitely presented and of type FP if and only if $K(G,1)$ is finitely dominated (i.e., is a retract up to homotopy of a finite complex). This is a topological analogue of the fact that a module is projective if and only if it is free. So the question we have been discussing is whether a finitely-dominated $K(G,1)$ is homotopy equivalent to a finite complex. Now there are in fact plenty of examples of finitely-dominated spaces that are not homotopy equivalent to finite complexes; but no known examples have torsion-free fundamental groups. In particular, they cannot be $K(G,1)$s.

3 Homology with coefficients

This section is based on [3, Chapter III].

3.1 Definitions

We defined the homology of a group G by applying the coinvariants functor $(-)_G$ to a projective resolution P of $\mathbb{Z}$ over $\mathbb{Z}G$. There are other functors we could apply.

First, fix a G-module M and consider the tensor-product functor $-\otimes_G M$. Applying this to a projective resolution P yields a nonnegative chain complex

$$\cdots \to P_1 \otimes_G M \to P_0 \otimes_G M,$$

and taking homology gives the *homology of G with coefficients in M*:

$$H_*(G, M) := H_*(P \otimes_G M).$$

A word of explanation is in order concerning the tensor product. Normally, a tensor product $N \otimes_R M$ is defined when N is a right R-module and M is a left R-module. It is obtained by forming the tensor product $N \otimes M := N \otimes_{\mathbb{Z}} M$ and introducing the relations $nr \otimes m = n \otimes rm$ for $n \in N$, $r \in R$, and $m \in M$. In case $R = \mathbb{Z}G$, however, we work entirely with left modules and, in order to form the tensor product $N \otimes_G M := N \otimes_{\mathbb{Z}G} M$, we convert the left G-module N to a right G-module by setting $ng := g^{-1}n$ for $g \in G$ and $n \in N$. Thus $N \otimes_G M$ is obtained from $N \otimes M$ by introducing the relations $g^{-1}n \otimes m = n \otimes gm$. If we replace n by gn, this becomes

$$n \otimes m = gn \otimes gm. \tag{3.1}$$

So we can also describe the tensor product by

$$N \otimes_G M = (N \otimes M)_G\,, \tag{3.2}$$

where the coinvariants on the right are formed with respect to the diagonal G-action on $N \otimes M$ (i.e., $g(n \otimes m) := gn \otimes gm$). If we take $M = \mathbb{Z}$, for example, then $N \otimes \mathbb{Z} = N$, so (3.1) gives $N \otimes_G \mathbb{Z} = N_G$. Thus

$$H_*(G, \mathbb{Z}) = H_*(G),$$

and we see that the homology defined in Section 1 is a special case of the homology that we are considering now.

As in that case, there is a topological interpretation, provided one knows about homology with local coefficients. Namely, if Y is a $K(G,1)$-complex, then the G-module M may be viewed as a local coefficient system on Y, and

$$H_*(G, M) = H_*(Y, M). \tag{3.3}$$

If G acts trivially on M, the right side is just the ordinary homology of Y with coefficients in the abelian group M.

Next, we consider the (contravariant) Hom functor $\mathrm{Hom}_G(-, M)$ for a fixed G-module M. Applying this to a projective resolution P yields a nonnegative cochain complex

$$\mathrm{Hom}_G(P_0, M) \to \mathrm{Hom}_G(P_1, M) \to \cdots\,,$$

and taking cohomology gives the *cohomology of G with coefficients in M*:

$$H^*(G, M) := H^*(\mathrm{Hom}_G(P, M)).$$

To better see the analogy with homology, note that, for any G-module N, there is a natural "diagonal" G-action on $\text{Hom}(N, M) := \text{Hom}_{\mathbb{Z}}(N, M)$ by functoriality, simply because G acts on both N and M. A moment's thought (taking account of the contravariance of Hom in the first variable), shows that this action should be defined by

$$(gf)(n) := gf(g^{-1}n)$$

for $g \in G$, $f \in \text{Hom}(N, M)$, and $n \in N$. This immediately yields the following analogue of (3.2):

$$\text{Hom}_G(N, M) = \text{Hom}(N, M)^G, \tag{3.4}$$

where the invariants on the right are formed with respect to the G-action on $\text{Hom}(N, M)$ that we have just defined. And, as in (3.3), cohomology with coefficients admits a topological interpretation as the cohomology of $Y = K(G, 1)$ with local coefficients in M.

In dimension 0, our homology and cohomology functors (viewed as functors of the coefficient module M) are familiar ones. Indeed, the tensor-product functor $- \otimes_G M$ is right exact, so we have an exact sequence

$$P_1 \otimes_G M \to P_0 \otimes_G M \to \mathbb{Z} \otimes_G M \to 0;$$

thus we can make the identification

$$H_0(G, M) = \mathbb{Z} \otimes_G M = M_G\,.$$

Similarly, the contravariant functor $\text{Hom}_G(-, M)$ takes right exact sequences to left exact sequences, so we have an exact sequence

$$0 \to \text{Hom}(\mathbb{Z}, M) \to \text{Hom}(P_0, M) \to \text{Hom}(P_1, M);$$

thus we can make the identification

$$H^0(G, M) = \text{Hom}_G(\mathbb{Z}, M) = M^G.$$

As we will see, the existence of nontrivial higher homology and cohomology functors reflects the failure of the invariants and coinvariants functors to be exact. We will also see that homology and cohomology with coefficients are fundamental technical tools, even if we only care about $H_*(G) = H_*(G, \mathbb{Z})$.

3.2 Examples

Example 3.1. Let $G = \langle t \rangle$ be infinite cyclic, generated by t. Recall from Example 1.2 that we have a free resolution

$$0 \longrightarrow \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Applying $- \otimes_G M$ as above (for a fixed G-module M), we obtain the chain complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow M \xrightarrow{t-1} M$$

for computing $H_*(G, M)$. And applying $\operatorname{Hom}_G(-, M)$, we obtain the cochain complex

$$M \xrightarrow{t-1} M \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

for computing $H^*(G, M)$. (In both cases, the two copies of M are in dimensions 0 and 1.) Thus

$$H_0(G, M) = H^1(G, M) = M_G\,,$$

and

$$H^0(G, M) = H_1(G, M) = M^G,$$

and all higher homology and cohomology functors vanish. The vanishing is of course to be expected, since G is a 1-dimensional group. And the calculations above are perhaps also to be expected. Indeed, they suggest Poincaré duality, which one would predict since the standard $K(G, 1)$ (the circle) is a closed 1-manifold. We will return to this in the next section.

Example 3.2. Now let $G = \langle\, t\ ;\ t^n = 1 \,\rangle$, the finite cyclic group of order n. Recall from Example 1.3 that we have a free resolution

$$\cdots \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0.$$

As in the previous example, this yields the chain complex

$$\cdots \xrightarrow{N} M \xrightarrow{t-1} M \xrightarrow{N} M \xrightarrow{t-1} M$$

for computing homology and the cochain complex

$$M \xrightarrow{t-1} M \xrightarrow{N} M \xrightarrow{t-1} M \xrightarrow{N} \cdots$$

for computing cohomology. We therefore obtain, for $i \geq 1$,

$$H^i(G, M) = \begin{cases} M^G/N \cdot M & \text{if } i \text{ is even} \\ \ker N/(t-1) \cdot M & \text{if } i \text{ is odd.} \end{cases} \tag{3.5}$$

The result for homology is similar, with the roles of even and odd indices reversed. To get a neater statement, note that the norm operator (i.e., multiplication by N) induces a map

$$\overline{N} \colon M_G \to M^G,$$

and the quotients in (3.5) are simply the cokernel and kernel of this map.

3.3 Properties

Topology motivates thinking of homology and cohomology as functors of a space, and hence of the group G in our setting. Algebra, on the other hand, suggests that we think of homology and cohomology as functors of the coefficient module M. Both points of view are useful. Here we explore the second. Thus we fix G and let the module vary.

The first observation is that a short exact sequence of G-modules

$$0 \to M' \to M \to M'' \to 0$$

yields a **long exact sequence**

$$\cdots \to H_1(G, M') \to H_1(G, M) \to H_1(G, M'') \to H_0(G, M') \\ \to H_0(G, M) \to H_0(G, M'') \to 0.$$

Thus we can think of H_1 as measuring the failure of the coinvariants functor H_0 to be left exact. Similarly, H_2 measures the failure of H_1 to be left exact, and so on. If $\dim G < \infty$, we eventually reach a left-exact functor and the remaining functors are 0.

The derivation of the long exact sequence is straightforward. One simply tensors the given short exact sequence with a projective resolution. This yields a short exact sequence of chain complexes (because projective modules are flat, i.e., tensoring with a projective module is an exact functor). A standard argument therefore yields a long exact sequence in homology.

Similar remarks apply to cohomology. This time we use the fact that the functor $\mathrm{Hom}_G(-, -)$ is exact (and covariant) in the second variable if we fix a projective module in the first variable. Our exact sequence of coefficient modules therefore yields a long exact sequence

$$0 \to H^0(G, M') \to H^0(G, M) \to H^0(G, M'') \to H^1(G, M') \\ \to H^1(G, M) \to H^1(G, M'') \to \cdots .$$

Thus H^1 measures the failure of the invariants functor to be right exact, H^2 measures the failure of H^1 to be right exact, and so on.

The next property is sometimes called **acyclicity**: If M is a projective $\mathbb{Z}G$-module, then

$$H_n(G, M) = 0$$

for $n > 0$. This is again a consequence of the flatness of projectives. [Tensoring a projective resolution with M preserves the exactness of the resolution.] Readers familiar with injective modules will note that, in the same way, injectives are acyclic for cohomology, i.e.,

$$H^n(G, M) = 0$$

for $n > 0$ if M is injective [because then $\mathrm{Hom}_G(-, M)$ is an exact functor].

The significance of acyclicity is that it leads to **effaceability**: Every module is a quotient of a module that is acyclic for homology. Similarly, every module can be embedded in a module that is acyclic for cohomology. This makes it possible to use a technique known as **dimension shifting** for reducing high-dimensional homology and cohomology to low-dimensional homology and cohomology, provided one is willing to change the coefficient module. See [3, Section III.7] for details.

The next property we wish to discuss is often called Shapiro's lemma (or, more accurately, the Eckmann–Shapiro lemma). In order to explain it, we need to digress and talk about induction and coinduction, which are special cases of extension and co-extension of scalars.

Consider an arbitrary inclusion $R \subseteq S$ of rings. Then any S-module may be viewed as an R-module by restriction of scalars. We wish to go in the other direction. In other words, given an R-module, we want to enlarge it so as to obtain an S-module. There are two ways to do this, depending on how one interprets "enlarge". Many readers will have seen the first method when R and S are fields. For example, the complexification of a real vector space is an example of the construction that follows.

Method 1: Extension of scalars. Given an R-module M, form the tensor product $S \otimes_R M$. Here we view S as a right R-module in order to make sense out of the tensor product, and we then use the left action of S on itself to make the tensor product an S-module:

$$s \cdot (s' \otimes m) := ss' \otimes m$$

for $s, s' \in S$ and $m \in M$. This is legitimate because the left action of S on itself commutes with the right action of R on S that was used in forming the tensor product. We say that $S \otimes_R M$ is the S-module obtained from M by *extension of scalars* from R to S. It is an "enlargement" of M in the sense that there is a canonical R-module map

$$i\colon M \to S \otimes_R M$$

given by $m \mapsto 1 \otimes m$. (This is often injective in concrete examples, so that M is embedded in $S \otimes_R M$.) The map i is universal for R-maps of M to an S-module, in the following sense: Given an S-module N and an R-module map $f\colon M \to N$, there is a unique S-module map $g\colon S \otimes_R M \to N$ such that $gi = f$, as illustrated in Figure 3.1. More concisely,

$$\mathrm{Hom}_R(M, N) = \mathrm{Hom}_S(S \otimes_R M, N) \tag{3.6}$$

for any R-module M and S-module N. This says that extension of scalars is left adjoint to restriction of scalars.

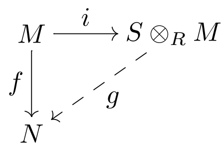


Figure 3.1: The universal property of extension of scalars

Method 2: Co-extension of scalars. Given an R-module M, consider the abelian group $\mathrm{Hom}_R(S, M)$. Here we view S as a left R-module in order to make sense out of the Hom, and we then use the right action of S on itself to make $\mathrm{Hom}_R(S, M)$ an S-module:

$$(sf)(s') := f(s's)$$

for $s, s' \in S$ and $f \in \mathrm{Hom}_R(S, M)$. We say that $\mathrm{Hom}_R(S, M)$ is the S-module obtained from M by *co-extension of scalars* from R to S. It is an "enlargement" of M in the sense that there is a canonical R-module map

$$p\colon \mathrm{Hom}_R(S, M) \to M$$

given by $f \mapsto f(1)$. (This is often surjective in concrete examples, so that M is a quotient of $\mathrm{Hom}_R(S, M)$.) The map p is universal for R-maps of an S-module to M, in the following sense: Given an S-module N and an R-module map $f\colon N \to M$, there is a unique S-module map $g\colon N \to \mathrm{Hom}_R(S, M)$ such that $pg = f$, as illustrated in Figure 3.2. More concisely,

$$\mathrm{Hom}_R(N, M) = \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)) \tag{3.7}$$

for any R-module M and S-module N. This says that co-extension of scalars is right adjoint to restriction of scalars.

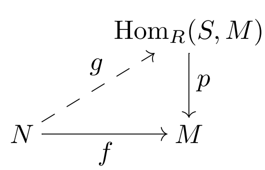


Figure 3.2: The universal property of co-extension of scalars

We will be interested in the case where R and S are group rings, say $R = \mathbb{Z}H$ and $S = \mathbb{Z}G$, where $H \leq G$. In this case extension of scalars is called *induction* and co-extension of scalars is called *coinduction*. Given a G-module M, we set

$$\operatorname{Ind}_H^G := \mathbb{Z}G \otimes_{\mathbb{Z}H} M$$

and

$$\operatorname{Coind}_H^G := \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M).$$

We will talk mainly about induction, but everything we say has an analogue for coinduction.

The first observation is that the canonical H-map $i\colon M \to \operatorname{Ind}_H^G M$ is injective. We may therefore identify M with its image $i(M)$, and M then becomes an H-invariant additive subgroup of the G-module $\operatorname{Ind}_H^G M$. The transform gM for $g \in G$ therefore depends only on the class of g in G/H, and one checks easily that there is an abelian group decomposition

$$\operatorname{Ind}_H^G M = \bigoplus_{g \in G/H} gM. \tag{3.8}$$

This property in fact characterizes induced modules. More precisely, suppose N is a G-module which is a direct sum of additive subgroups that are permuted transitively by the G-action. If M is one of the summands and H is its stabilizer in G, then N is canonically isomorphic to $\operatorname{Ind}_H^G M$. Thus (3.8) really captures the essence of the induction construction.

Similar remarks apply to coinduction, but this time we have a direct product decomposition such that the G-action permutes the factors. In case H has finite index in G, there is no difference between a direct sum decomposition and a direct product decomposition, and one concludes that

$$\operatorname{Ind}_H^G M \cong \operatorname{Coind}_H^G M \tag{3.9}$$

for any H-module M in this case.

We now return to our list of properties of homology and cohomology. The **Eckmann–Shapiro lemma** is the following result:

Proposition 3.3. *Given $H \leq G$ and an H-module M, there are canonical isomorphisms*

$$H_*(H, M) \cong H_*(G, \operatorname{Ind}_H^G M)$$

and

$$H^*(H, M) \cong H^*(G, \operatorname{Coind}_H^G M).$$

Proof. Take a projective resolution P of $\mathbb{Z}$ over $\mathbb{Z}G$. If we restrict operators from G to H, P is still a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$, and we have

$$\begin{aligned} H^*(H,M) &= H^*(\mathrm{Hom}_H(P,M)) \\ &= H^*(\mathrm{Hom}_G(P,\mathrm{Coind}_H^G M)) \\ &= H^*(G,\mathrm{Coind}_H^G M), \end{aligned}$$

where the second equality comes from the universal property of coinduction, as formulated in (3.7). For homology, we instead use properties of tensor products:

$$\begin{aligned} H_*(H,M) &= H_*(P \otimes_H M) \\ &= H_*(P \otimes_G (\mathbb{Z}G \otimes_H M)) \\ &= H_*(G,\mathrm{Ind}_H^G M). \end{aligned}$$

□

Taking $M = \mathbb{Z}$, for example, the induced module $\mathrm{Ind}_H^G \mathbb{Z}$ can be identified, in view of (3.8), with the permutation module $\mathbb{Z}[G/H]$. So we have

$$H_*(H) \cong H_*(G,\mathbb{Z}[G/H]).$$

Thus the ordinary homology of a group can be computed as the homology of any supergroup, provided we are willing to introduce coefficients. And if H has finite index in G, then we also have

$$H^*(H,\mathbb{Z}) \cong H^*(G,\mathbb{Z}[G/H])$$

in view of (3.9).

The final property we wish to state is the existence of **transfer maps**. We continue to assume that we are given $H \le G$. One always has the expected covariance of homology with respect to group homomorphisms, giving a map

$$H_*(H,M) \to H_*(G,M)$$

for any G-module M. Algebraically, this comes from the canonical surjection

$$P \otimes_H M \twoheadrightarrow P \otimes_G M$$

if P is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Topologically, it comes from the canonical map $K(H,1) \to K(G,1)$. Similarly, cohomology is contravariant with respect to the group: We have a map

$$H^*(G,M) \to H^*(H,M)$$

for any G-module M. But if $[G : H] < \infty$, we can also define so-called "transfer" maps that go in the opposite direction.

Consider cohomology for example, where we seek a map

$$H^*(H, M) \to H^*(G, M)$$

for any G-module M. If we use Proposition 3.3 and the isomorphism (3.9) to identify $H^*(H, M)$ with $H^*(G, \operatorname{Ind}_H^G M)$, the desired map has the form

$$H^*(G, \operatorname{Ind}_H^G M) \to H^*(G, M). \tag{3.10}$$

To define it, simply observe that there is a canonical G-module map

$$\operatorname{Ind}_H^G M \to M. \tag{3.11}$$

Indeed, since we started with a G-module M, restricted it to H, and then induced back up, we can use the original G-action to define a G-map

$$\mathbb{Z}G \otimes_{\mathbb{Z}H} M \to M$$

by $r \otimes m \mapsto rm$ for $r \in \mathbb{Z}G$ and $m \in M$. Formally, this is just an instance of the universal mapping property of induction. [Apply (3.6), and consider the identity map on M, viewed as an H-map to a G-module.] The map (3.10) that we are seeking is now obtained from (3.11) by functoriality of cohomology with respect to the coefficient module. Similar remarks apply to homology.

See [3, Section III.9] for other ways of explaining the existence of the transfer map. The name "transfer" comes from the special case $H_1(G) \to H_1(H)$, which is a map on abelianizations $G_{\text{ab}} \to H_{\text{ab}}$ that goes back to Schur [10], who called it the transfer ("Verlagerung"). The extension of Schur's transfer to homology and cohomology is due to Eckmann [5].

Finally, we remark that the ordinary functorial map

$$H^*(G, M) \to H^*(H, M)$$

can also be explained in terms of the Eckmann–Shapiro lemma and maps of coefficient modules (and similarly for homology). This does not require finite index, but, if we assume finite index for simplicity, the relevant map $M \to \mathbb{Z}G \otimes_{\mathbb{Z}H} M$ turns out to be given by

$$m \mapsto \sum_{g \in G/H} g \otimes g^{-1} m.$$

In particular, this shows that the composite

$$M \to \operatorname{Ind}_H^G M \to M$$

is simply multiplication by the index $[G : H]$, so the same is true of the composite

$$H^*(G, M) \to H^*(H, M) \to H^*(G, M). \tag{3.12}$$

Here the first map is the ordinary functorial map (sometimes called "restriction" in the case of cohomology), and the second map is the transfer. Suppose, for example, that G is finite and H is the trivial subgroup. Then this argument shows that multiplication by $[G : H] = |G|$ in positive-dimensional cohomology is the zero map, and similarly for homology. Consequently:

Corollary 3.4. *If G is finite, then $H_n(G, M)$ and $H^n(G, M)$ are annihilated by $|G|$ for every G-module M and all $n > 0$.* □

Remark 3.5. We noted above that the functorial map $H^*(G) \to H^*(H)$ is sometimes called *restriction*; here we have suppressed the coefficient module from the notation for simplicity. The transfer map $H^*(H) \to H^*(G)$ may then be called *corestriction*. In homology, on the other hand, it is the transfer map $H_*(G) \to H_*(H)$ that is called *restriction*, while the functorial map $H_*(H) \to H_*(G)$ is called *corestriction*. In both cases, then, "restriction" refers to the map $H(G) \to H(H)$, where $H(-)$ denotes either homology or cohomology. This uniform terminology makes it easier to remember certain formulas. For example, restriction followed by corestriction as in (3.12) is multiplication by the index in both homology and cohomology.

4 Finiteness properties revisited

This is a continuation of Section 2. The reference is still [3, Chapter VIII]. We begin with dimension.

4.1 Dimension

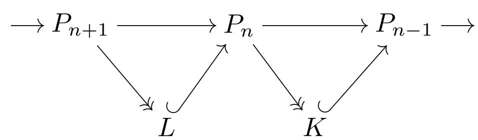
Recall that we defined the cohomological dimension $\operatorname{cd} G$ in terms of projective resolutions. We are now in a position to explain why this is called cohomological dimension. Namely, $\operatorname{cd} G$ is the largest n (if any) for which the cohomology functor $H^n(G, -)$ is not identically 0:

Proposition 4.1. *For any group G,*

$$\operatorname{cd} G = \sup \{n \mid H^n(G, -) \neq 0\}.$$

In other words, $\operatorname{cd} G \leq m \iff H^k(G, M) = 0$ *for all $k > m$ and all G-modules M.*

Proof. Let's temporarily write $d(G)$ for the supremum on the right side of the equation to be proved. Clearly $d(G) \leq \operatorname{cd} G$. For the opposite inequality, we may assume $d(G) < \infty$. Set $n := d(G)$, and consider an arbitrary projective resolution P of $\mathbb{Z}$ over $\mathbb{Z}G$. Let L and K be the images of the boundary maps $P_{n+1} \to P_n$ and $P_n \to P_{n-1}$:



I claim that K is projective and hence $\operatorname{cd} G \le n$. We are given that $H^{n+1}(G, M) = 0$ for every G-module M. Now an $(n+1)$-cocycle of $\operatorname{Hom}_G(P, M)$ is a homomorphism $P_{n+1} \to M$ that becomes 0 when composed with the boundary map $P_{n+2} \to P_{n+1}$. Thus the group of n-cocycles can be identified with $\operatorname{Hom}_G(L, M)$. Saying that every such cocycle is a coboundary means that every map $L \to M$ extends to a map $P_n \to M$. This being true for every G-module M, it follows that L is a direct summand of P_n. Hence $P_n \cong L \oplus K$, so K is a direct summand of a projective module and is therefore projective, as claimed. □

Note that every P_n above could be taken to be free. But the proof would still only show that K is projective. That is why we need to work with projective resolutions in developing the theory, in spite of the fact that in all known examples the modules can be taken to be free.

Recall from (2.2) that $\operatorname{cd} H \le \operatorname{cd} G$ if $H \le G$. We proved this using projective resolutions, but it also follows immediately from Propositions 3.3 and 4.1. Our next goal is to show that equality holds if $[G : H] < \infty$ and G is finite dimensional. (But trivial examples show that one can have $\operatorname{cd} H < \infty$ and $\operatorname{cd} G = \infty$ if G has torsion.)

Proposition 4.2. *If $\operatorname{cd} G < \infty$, then $\operatorname{cd} H = \operatorname{cd} G$ for every subgroup $H \le G$ of finite index.*

The proof will use the first part of the following lemma:

Lemma 4.3. (a) *If $\operatorname{cd} G < \infty$, then*

$$\operatorname{cd} G = \max\{n \mid H^n(G, F) \neq 0 \text{ for some free } \mathbb{Z}G\text{-module } F\}.$$

(b) *If G is of type* FP*, then*

$$\operatorname{cd} G = \max\{n \mid H^n(G, \mathbb{Z}G) \neq 0\}.$$

The proposition now follows immediately from the Eckmann–Shapiro lemma, since every free $\mathbb{Z}G$-module is induced from the free $\mathbb{Z}H$-module of the same rank. It remains to prove the lemma.

Proof of the lemma. (a) If $\operatorname{cd} G = n < \infty$, then the functor $H^n(G, -)$ is right exact (see Section 3.3). Since this functor is nonzero and every module is a quotient of a free module, it follows that the functor is nonzero on some free module.

(b) It is easy to check that $H^n(G, -)$ preserves direct sums if G is of type FP. So the nonvanishing of this functor on a direct sum of copies of $\mathbb{Z}G$ implies that it is nonvanishing on $\mathbb{Z}G$. □

We noted above that one can have $\operatorname{cd} H \neq \operatorname{cd} G$ if G has torsion. Our next result, which is due to Serre, says that this is the *only* way the equality can fail for subgroups of finite index.

Theorem 4.4. *If G is torsion free, then $\operatorname{cd} H = \operatorname{cd} G$ for every subgroup $H \leq G$ of finite index.*

Sketch of the proof. In view of Proposition 4.2, it suffices to show that $\operatorname{cd} H < \infty \implies \operatorname{cd} G < \infty$. To this end, one starts with a finite-dimensional, free, contractible H-CW-complex X. One can then use a topological analogue of coinduction to produce a G-CW-complex which, as a CW-complex, is a product of $[G : H]$ copies of X, with the G-action permuting the factors. This complex is still finite dimensional and contractible, and the H-action on it is still free. It follows that the G-action has finite stabilizers and hence is free (because G is torsion free). □

We close by using Lemma 4.3(b) to give an extremely useful topological characterization of cohomological dimension for groups of type F. Theorem 2.5, of course, gives a topological characterization for more-or-less arbitrary groups. But we have in mind something quite different. Suppose G is of type F, and let X be a free, contractible G-CW-complex with $G\backslash X$ finite. Then $H^*(G, \mathbb{Z}G)$, which occurred in Lemma 4.3, has the following interpretation:

$$H^*(G, \mathbb{Z}G) = H_c^*(X, \mathbb{Z}), \tag{4.1}$$

where H_c^* denotes cohomology with compact supports (based on cellular cochains that vanish on all but finitely many cells of X). To prove (4.1), one need only stare at $\operatorname{Hom}_G(C_*(X), \mathbb{Z}G)$ and observe that it can be identified with the group of $\mathbb{Z}$-valued cochains of X with compact supports. Thus Lemma 4.3(b) immediately yields:

Corollary 4.5. *Let G be a group of type* F*, and let X be a free, contractible G-CW-complex with finite quotient. Then*

$$\operatorname{cd} G = \max \{n \mid H_c^n(X) \neq 0\}.$$

□

4.2 Duality

In the topology of manifolds, one proves duality (classically) by using dual cell decompositions; this makes a cochain complex look like a chain complex. From the algebraic point of view, one instead uses duality theory for finitely generated projective modules to achieve the same effect. We begin by reviewing this duality theory. It is probably familiar to many readers for vector spaces, and the general case is no more difficult.

Let R be a ring and P a finitely-generated projective (left) R-module. The *dual* of P is defined by

$$P^* := \operatorname{Hom}_R(P, R).$$

Here R is viewed as a left R-module in forming the Hom. But there is also a right action of R on itself that commutes with the left action, and this makes P^* a right R-module. One easily checks that it is again finitely generated and projective. Dualizing again, one gets back a left R-module P^{**} and a canonical isomorphism

$$P \xrightarrow{\simeq} P^{**}$$

defined as in duality theory for finite-dimensional vector spaces. Duality is useful for us because it can convert Hom to tensor product and vice versa: If P is a finitely-generated projective and M is arbitrary, then

$$\operatorname{Hom}_R(P, M) \cong P^* \otimes_R M \quad \text{and} \quad P \otimes_R M \cong \operatorname{Hom}_R(P^*, M). \tag{4.2}$$

(To be precise here, one needs to distinguish between left modules and right modules; I leave this as an exercise for the reader.)

Suppose now that G is a group of type FP with $\operatorname{cd} G = n$, and take a finite projective resolution

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

of $\mathbb{Z}$ over $\mathbb{Z}G$. Form the cochain complex dual to P, i.e., apply the functor $\operatorname{Hom}_G(-, \mathbb{Z}G)$, to get

$$P_0^* \to P_1^* \to \cdots \to P_n^* .$$

Note that this is what one uses to compute $H^*(G, \mathbb{Z}G)$. If we re-index by setting $Q_i := P_{n-i}^*$, then we have a chain complex

$$Q_n \to Q_{n-1} \to \cdots \to Q_0 ,$$

and (4.2) yields a canonical identification

$$\operatorname{Hom}_G(P_i, M) = Q_{n-i} \otimes_G M$$

for each i. Hence

$$H^i(G, M) = H_{n-i}(Q \otimes_G M). \tag{4.3}$$

It might happen that the dual complex Q is again a resolution of $\mathbb{Z}$. This happens if and only if

$$H^i(G, \mathbb{Z}G) \cong \begin{cases} \mathbb{Z} & \text{for } i = n \\ 0 & \text{for } i \neq n. \end{cases} \tag{4.4}$$

[A technical point: $H^n(G, \mathbb{Z}G)$ is naturally a right G-module via the right action of G on $\mathbb{Z}G$, which commutes with the left action that is used in defining $H^n(G, \mathbb{Z}G)$. What we mean in (4.4) is that $H^n(G, \mathbb{Z}G)$ is infinite cyclic with trivial G-action.] In this case (4.3) yields a result that looks like Poincaré duality for a closed orientable manifold:

$$H^i(G, M) = H_{n-i}(G, M).$$

More generally, suppose Q is a resolution of $\mathbb{Z}$ with a possibly nontrivial action of G. We write $\widetilde{\mathbb{Z}}$ for this G-module, and for any G-module M we set $\widetilde{M} := M \otimes \widetilde{\mathbb{Z}}$, with the diagonal G-action. Thus $\widetilde{M}$ is the same abelian group as M, with the G-action twisted by a homomorphism $G \to \{\pm 1\}$. Then (4.3) becomes

$$H^i(G, M) = H_{n-i}(G, \widetilde{M}),$$

which looks like Poincaré duality for a (possibly non-orientable) closed manifold. The justification for this is that Q and $\widetilde{P} = P \otimes \widetilde{\mathbb{Z}}$ are both projective resolutions of $\widetilde{\mathbb{Z}}$, so they are homotopy equivalent. The homology $H_*(Q \otimes_G M)$ that occurs in (4.3) is therefore canonically isomorphic to $H_*((P \otimes \widetilde{\mathbb{Z}} \otimes M)_G) = H_*(P \otimes_G \widetilde{M}) = H_*(G, \widetilde{M})$.

Still more generally, suppose Q is a resolution of *some* G-module D, not necessarily infinite cyclic. In other words, we are simply assuming that

$$H^i(G, \mathbb{Z}G) = 0 \quad \text{for } i \neq n,$$

and we are setting

$$D := H^n(G, \mathbb{Z}G).$$

Assume for simplicity that D is free as an abelian group. Then we can argue as above to obtain

$$H^i(G, M) \cong H_{n-i}(G, D \otimes M), \tag{4.5}$$

where $D \otimes M$ is the tensor product over $\mathbb{Z}$, with the diagonal G-action. This is known as *Bieri–Eckmann duality*, and G is said to be a *duality group* if it holds. The G-module D is called the *dualizing module*.

Borel and Serre proved that all torsion-free arithmetic groups are duality groups, so there is no shortage of examples.

Incidentally, there are also "inverse duality" isomorphisms under the hypotheses above (including the assumption that $D := H^n(G, \mathbb{Z}G)$ is free abelian):

$$H_i(G, M) \cong H^{n-i}(G, \mathrm{Hom}(D, M)),$$

where the Hom is taken over $\mathbb{Z}$ and has the diagonal G-action. To see this, note that there is a homotopy equivalence $Q \simeq P \otimes D$ (which is already needed in the proof of (4.5)), whence

$$\begin{aligned} P \otimes_G M &= \mathrm{Hom}_G(P^*, M) \\ &= \mathrm{Hom}_G(Q, M) \\ &\simeq \mathrm{Hom}_G(P \otimes D, M) \\ &= \mathrm{Hom}_G(P, \mathrm{Hom}(D, M)). \end{aligned}$$

4.3 Topological interpretation

Suppose G is a group of type F. As usual, let X be a free, contractible G-CW-complex with finite quotient. Then the hypothesis (4.4) that led to duality above can be written, in view of (4.1), as

$$H^i_c(X) = \begin{cases} 0 & i \neq n \\ D & i = n \end{cases}$$

with D free abelian. Here the G-action on the dualizing module D is induced by the G-action on X. Suppose, for example, that X is an m-manifold, possibly with nonempty boundary ∂X. [Necessarily $m \geq n = \mathrm{cd}\, G$.] Then we can use Poincaré–Lefschetz duality to compute $H^i_c(X)$. This yields

$$H^i_c(X) \cong H_{m-i}(X, \partial X) \cong \widetilde{H}_{m-i-1}(\partial X), \tag{4.6}$$

where the second isomorphism comes from the contractibility of X, and $\widetilde{H}$ denotes reduced homology. Suppose further that ∂X has the homotopy type of a bouquet of k-spheres for some k. (This holds with $k = -1$ if $\partial X = \emptyset$.) We then conclude from (4.6) that $H^i_c(X) = 0$ for $i \neq m-k-1$ and is free abelian when $i = m-k-1$. Thus G is a duality group of dimension $n = m - k - 1$.

It is precisely in this way that Borel and Serre proved duality for torsion-free arithmetic groups (and calculated the cohomological dimension).

5 Equivariant homology

This section gives a very brief introduction to a technical subject that is treated in more detail in [3, Chapter VII]. Soulé's lectures in this volume give some interesting applications.

5.1 Introduction to spectral sequences

The only spectral sequences we will consider here are those that arise from double complexes. A double complex is a commutative diagram of abelian groups as in Figure 5.1 (extending forever in all four directions) in which every row and every column is a chain complex. Thus $\partial'^2 = 0$, $\partial''^2 = 0$, and $\partial''\partial' = \partial'\partial''$. In practice, double complexes are often concentrated in the first quadrant, i.e., $C_{pq} = 0$ unless $p, q \geq 0$.

$$
\begin{array}{ccccccc}
 & & \downarrow{\scriptstyle \partial''} & & \downarrow{\scriptstyle \partial''} & & \\
\xleftarrow{\partial'} & & C_{p-1,q} & \xleftarrow{\partial'} & C_{pq} & \xleftarrow{\partial'} & \\
 & & \downarrow{\scriptstyle \partial''} & & \downarrow{\scriptstyle \partial''} & & \\
\xleftarrow{\partial'} & & C_{p-1,q-1} & \xleftarrow{\partial'} & C_{p,q-1} & \xleftarrow{\partial'} & \\
 & & \downarrow{\scriptstyle \partial''} & & \downarrow{\scriptstyle \partial''} & &
\end{array}
$$

Figure 5.1: A double complex

A simple example of a double complex is provided by the tensor product of two chain complexes. Thus if D and E are chain complexes, then there is a double complex with $C_{pq} = D_p \otimes E_q$. The horizontal boundary operator ∂' is induced by the boundary operator in D, and the vertical boundary operator ∂'' is induced by the boundary operator in E.

There are three ways to think about a double complex:

(a) Each (vertical) column $C_{p,*}$ for fixed p is a chain complex, and the horizontal arrows are chain maps such that the composite of two consecutive ones is 0. Thus we have a "chain complex in the category of chain complexes".

(b) Each (horizontal) row $C_{*,q}$ for fixed q is a chain complex, and the vertical arrows are chain maps such that the composite of two consecutive ones is 0. This is a second way of viewing C as a chain complex in the category of chain complexes.

(c) We can simplify the whole double complex to an ordinary chain complex TC (the *total complex*) by setting

$$(TC)_n := \bigoplus_{p+q=n} C_{pq} .$$

It has a boundary operator ∂, whose restriction to the summand C_{pq} is given by $\partial = \partial' + (-1)^p \partial''$ as in the familiar case of the tensor product of two chain complexes.

The theory that we wish to sketch here relates all three of these viewpoints.

The basic fact is that there is a machine called a *spectral sequence* for computing $H_*(TC)$ by successive approximation. The spectral sequence is an infinite sequence of bigraded abelian groups

$$E^0_{pq},\, E^1_{pq},\, E^2_{pq},\, \ldots .$$

Under mild hypotheses (which are satisfied in the first-quadrant case), the sequence E^r_{pq} stabilizes for each fixed p, q as $r \to \infty$. Let E^∞_{pq} denote E^r_{pq} for large r. It turns out to be the "pth layer" of $H_{p+q}(TC)$, in the following sense.

There is an increasing filtration of TC by subcomplexes F_pTC, where F_pTC is given in dimension n by

$$(F_pTC)_n := \bigoplus_{i\le p} C_{i,n-i} .$$

Thus we draw a vertical line at the horizontal position p, and everything to the left of it is in F_pTC. As the line moves to the right, we get bigger and bigger subcomplexes. There is an induced filtration on homology, given by

$$F_pH_*(TC) := \operatorname{Im}\{H_*(F_pTC) \to H_*(TC)\} .$$

Thus $F_pH_n(TC)$ consists of homology classes represented by cycles in $\bigoplus_{i\le p} C_{i,n-i}$. We call the subquotient

$$F_pH_n(TC)/F_{p-1}H_n(TC)$$

of $H_n(TC)$ its *pth layer*. Thus our claim at the end of the previous paragraph is that

$$E^\infty_{pq} \cong F_pH_{p+q}(TC)/F_{p-1}H_{p+q}(TC).$$

Informally, then, one has to combine the groups E^∞_{pq} along a diagonal line $p + q = n$ to compute $H_n(TC)$.

The final ingredient of the spectral sequence is that there is a procedure for passing from each approximation E^r to the next one by taking homology. More precisely, there is a boundary operator d^r on E^r, induced by the boundary operator on TC, such that $E^{r+1} = H(E^r, d^r)$. This is easy to explain for $r = 0$ and 1. For $r = 0$, we have $E^0_{pq} = C_{pq}$ and $d^0 = \partial''$. Thus (E^0, d^0) is simply the collection of columns of C, each viewed as a chain complex as in (a) above, and hence E^1 is the vertical homology. Next, d^1 is the map induced on the vertical homology by ∂', viewed as a chain map as in (a). So E^2 can be described as the horizontal homology of the vertical homology. The differential d^2 is more subtle, and all I will say about it is that it maps E^2_{pq} to $E^2_{p-2,q+1}$. So one can visualize it as going 2 steps to the left and 1 step up (decreasing the total degree $p+q$ by 1, as do d^0 and d^1). In general, d^r goes r steps to the left and $r-1$ steps up.

Note that one can transpose the double complex C to get a new double complex C' with $C'_{pq} = C_{qp}$. The total complex TC' is canonically isomorphic to TC, so we obtain a second spectral sequence for computing $H_*(TC)$ with the roles of ∂' and ∂'' reversed.

In summary, there are two spectral sequences converging to $H_*(TC)$, based on the two viewpoints (a) and (b). [Warning: The statement that the two spectral sequences both converge to $H_*(TC)$ is somewhat misleading, since there are two different filtrations on $H_*(TC)$ and hence two different families of "layers" that occur as E^∞.] We have introduced these spectral sequences because they are useful in connection with equivariant homology, to which we turn next.

5.2 Equivariant homology

Equivariant homology is the same as what Lück calls "Borel homology" in his lectures in this volume, but I will describe an algebraic approach. For simplicity I will stick to homology, but everything I say has an analogue for cohomology.

Definition 5.1. If X is a G-CW-complex and M is a G-module, then we set

$$H^G_*(X, M) := H_*(P \otimes_G C(X, M)),$$

where P is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and $C(X, M)$ is the cellular chain complex of X with coefficients in M (with diagonal G-action). We call $H^G_*(X, M)$ the *equivariant homology* of (G, X) with coefficients in M.

Heuristically, equivariant homology is a mixture of homology of groups and homology of spaces. For us, the main point is that it is a tool for getting information about $H_*(G)$ from *any* action (not necessarily free) on *any* space (not necessarily contractible).

Examples 5.2. (a) If X is a point, then $H_*^G(X,M) = H_*(G,M)$. More generally, this holds if X is contractible. Thus equivariant homology is the same as the homology of the group if the space is trivial. [Sketch of proof: There is a weak equivalence $C(X,M) \to M$, where M is viewed as a chain complex concentrated in dimension 0, and a weak equivalence is a map that induces an isomorphism in homology. Tensoring over G with the complex of projectives P, we obtain a weak equivalence $P \otimes_G C(X,M) \to P \otimes_G M$.]

(b) At the other extreme, if G is the trivial group, then $H_*^G(X,M) = H_*(X,M)$.

(c) If G acts freely on X, and $Y := G\backslash X$, then

$$H_*^G(X,M) = H_*(Y,M);$$

here M on the right side is viewed as a $\pi_1(Y)$-module (and hence a local coefficient system on Y) via the canonical map $\pi_1(Y) \to G$ provided by the theory of covering spaces. [Sketch of proof: If P is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, then there is a weak equivalence $P \otimes M \to M$. Tensoring over G with the complex of free $\mathbb{Z}G$-modules $C(X)$, we obtain a weak equivalence $P \otimes_G C(X,M) \to C(X) \otimes_G M = C(Y,M)$.]

If X is contractible and the G-action is free, then (a) and (c) give back the familiar fact that $H_*(G) = H_*(K(G,1))$.

Note that $H_*^G(X,M)$ is, by definition, the homology of the total complex associated to the double complex $P_p \otimes_G C_q(X,M)$ or, alternatively, the double complex $P_q \otimes_G C_p(X,M)$. The theory sketched in Section 5.1 therefore gives us two spectral sequences for computing $H_*^G(X)$. The E^2 term in each case is the horizontal homology of the vertical homology. Let's start with the second spectral sequence, where the double complex is viewed as $P_q \otimes_G C_p(X,M)$. The vertical homology is obtained by fixing p and taking the homology with respect to q; thus

$$E^1_{pq} = H_q(G, C_p(X,M)). \tag{5.1}$$

To analyze this further, recall that $C_p(X)$ is the direct sum of infinite cyclic groups $\mathbb{Z}_\sigma$, one for each p-cell σ, where the two generators of $\mathbb{Z}_\sigma$ correspond to the two possible orientations of σ. Tensoring with M (over $\mathbb{Z}$) and grouping the summands into G-orbits, we recognize $C_p(X,M)$ as a direct sum of induced modules:

$$C_p(X,M) = \bigoplus_\sigma \operatorname{Ind}_{G_\sigma}^G M_\sigma\,. \tag{5.2}$$

Here σ ranges over a set of representatives for the G-orbits of p-cells, G_σ is the stabilizer of σ, and $M_\sigma := \mathbb{Z}_\sigma \otimes M$ with the diagonal G-action.

Thus M_σ is M, with the action restricted to G_σ and twisted by the "orientation homomorphism" $G_\sigma \to \{\pm 1\}$. [Each element of G_σ either preserves the orientation of σ or reverses it.] Combining (5.1) and (5.2), and using the Eckmann–Shapiro lemma, we obtain

$$E^1_{pq} = \bigoplus_\sigma H_q(G_\sigma, M_\sigma)$$

where, again, σ ranges over representatives for the p-cells of X mod G. It is customary to summarize what we have done so far by writing

$$E^1_{pq} = \bigoplus_\sigma H_q(G_\sigma, M_\sigma) \implies H^G_{p+q}(X; M). \tag{5.3}$$

This is a short-hand way of saying that we have constructed a spectral sequence with the given E^1 term, which converges to $H^G_*(X; M)$ (i.e., E^∞_{pq} is the pth layer of $H^G_{p+q}(X; M)$ with respect to some filtration on the latter).

Remark 5.3. Intuitively, E^1_{pq} in (5.3) is the group of p-chains of $Y := G\backslash X$ with coefficients in the "system of coefficients" $\sigma \mapsto \{H_q(G_\sigma, M_\sigma)\}$. This suggests that

$$E^2_{pq} = H_p(Y, \{H_q(G_\sigma, M_\sigma)\}). \tag{5.4}$$

We will not attempt to formulate this precisely.

In case G acts freely on X, all the stabilizers are trivial. The spectral sequence is then concentrated on the horizontal line $q = 0$, and $E^2 = E^\infty$. One easily recovers the result of Example 5.2(c) in this case. For a more interesting application, suppose that the action is not necessarily free but that X is contractible. Then the equivariant homology can be identified with $H_*(G, M)$, so the spectral sequence converges to the latter. Thus (5.3) becomes

$$E^1_{pq} = \bigoplus_\sigma H_q(G_\sigma, M_\sigma) \implies H_{p+q}(G, M)$$

in this case. See Soulé's lectures in this volume for some illustrations of how this spectral sequence can be used to obtain concrete results.

We turn now to the other spectral sequence, arising from the double complex $P_p \otimes_G C_q(X, M)$. This time if we take the vertical homology (fixing p and taking the homology with respect to q), we get $E^1_{pq} = P_p \otimes_G H_q(X, M)$ by the flatness of P_p. Taking homology with respect to p now gives $H_p(G, H_q(X, M))$. Thus our spectral sequence has the form

$$E^2_{pq} = H_p(G, H_q(X)) \implies H^G_{p+q}(X)$$

where, to simplify the notation, I have suppressed the coefficient module M. If X is contractible, this gives back the result of Example 5.2(a). For a more interesting application, suppose that X is not necessarily contractible but that the G-action is free. Then we know that $H_*^G(X) = H_*(G\backslash X)$, so the spectral sequence becomes

$$E^2_{pq} = H_p(G, H_q(X)) \implies H_{p+q}(G\backslash X). \tag{5.5}$$

This is called the *Cartan–Leray* spectral sequence.

As a special case of (5.5) we can derive a spectral sequence associated to a group extension. Suppose we have a normal subgroup $N \lhd G$ and we set $Q := G/N$. Start with a free, contractible G-CW-complex X. Then we can form the quotient $X_G := G\backslash X$ in two steps, in which we first divide out by the N-action to get a Q-CW-complex X_N, and then go mod Q:

$$X_G = (X_N)_Q\,.$$

Now X_N is a $K(N,1)$-complex and a free Q-complex, and the quotient $(X_N)_Q = X_G$ is a $K(G,1)$-complex. The Cartan–Leray spectral sequence for the Q-complex X_N therefore becomes

$$E^2_{pq} = H_p(Q, H_q(N)) \implies H_{p+q}(G). \tag{5.6}$$

This is the *Hochschild–Serre* spectral sequence.

Remarks 5.4. (a) The description of the E^2 term in (5.6) suggests that there is an action of Q on $H_*(N)$. To see where this comes from, observe that the conjugation action of G on N induces an action of G on $H_*(N)$. One can show that the action of N on its own homology is trivial, so we get an action of $Q = G/N$ on $H_*(N)$.

(b) Even if the suppressed coefficient module M is $\mathbb{Z}$ (with trivial G-action), the action of Q on $H_*(N)$ is generally nontrivial. Thus, once again, homology with coefficients arises naturally when one tries to compute ordinary integral homology.

6 The cohomology theory of finite groups

In this final lecture we point out some special features of the homology and cohomology theory of finite groups. Even if one is primarily interested in infinite groups, the homology of finite groups will arise whenever one tries to apply the spectral sequence (5.3) to a proper action (where the stabilizers G_σ are finite). A reference for this section is [3, Chapter VI]. Assume from now on that G is a finite group.

6.1 Functoriality

When there is no need to distinguish between homology and cohomology, we will often write $H(G, M)$ to denote the homology or cohomology of a group G with coefficients in M. For brevity, we may even suppress M from the notation and simply write $H(G)$. Recall that, for arbitrary groups, we often have restriction and corestriction maps between $H(G)$ and $H(H)$ if H is a subgroup of G; see Remark 3.5. The qualifier "often" refers to the fact that we have to assume H has finite index in G in order to have a restriction map in homology and a corestriction map in cohomology. All of this simplifies if G is finite, since every subgroup has finite index. Thus we always have maps in both directions, and formal differences between homology and cohomology disappear.

A similar (and related) phenomenon is that there is no need to distinguish between induced modules and coinduced modules; see (3.9). Thus we can give a unified statement of the Eckmann–Shapiro lemma (Proposition 3.3):

$$H(H, M) \cong H(G, \operatorname{Ind}_H^G M),$$

with no distinction between homology and cohomology.

6.2 Local computation of homology and cohomology

We continue to write $H(G)$ for homology or cohomology with an arbitrary coefficient module. Our starting point is Corollary 3.4, which states that $H(G)$ is annihilated by $|G|$ in positive dimensions. (This is of course false for H_0 and H^0.) We therefore have a decomposition

$$H(G) = \bigoplus_p H(G)_{(p)} \tag{6.1}$$

in positive dimensions, where p ranges over the primes dividing $|G|$, and $(-)_{(p)}$ denotes the p-primary component. This simple observation can be quite useful in practice, since it allows one to localize the computation of $H(G)$ by focusing on one prime at a time. For example, one might be trying to compute $H(G)$ with the aid of a seemingly complicated spectral sequence. If one is lucky, the complications will disappear (or at least become more manageable) after localizing at one prime at a time.

To take (6.1) one step further, we show that the p-primary component $H(G)_{(p)}$ can be described in terms of the p-subgroups of G. There are two versions of the result. For the first, fix a prime p, and choose a Sylow p-subgroup $S \le G$. Then $H(S)$ consists entirely of p-torsion by Corollary 3.4. The result that we will state below says, roughly speaking, that $H(G)$ can be identified with the subgroup of $H(S)$ invariant under conjugation. This is easy to state precisely if S is normal in G. Recall from Remark 5.4(a) that, if $S \lhd G$, there is an action of G [and even of G/S] on $H(S)$. So we can certainly talk about the invariants under this action. In general, we have to work a little harder.

Given $g \in G$, conjugation by g induces an isomorphism between $H(S)$ and $H(gSg^{-1})$, which we denote by $z \mapsto gz$ for $z \in H(S)$. Now it does not make sense to ask whether $gz = z$ unless g normalizes S. But we can ask whether gz and z restrict to the same class in $H(S \cap gSg^{-1})$. If z has the property that this is true for all $g \in G$, then we will say that z is invariant under conjugation. We can now state:

Theorem 6.1. *If S is a Sylow p-subgroup of G, then the restriction map $H(G) \to H(S)$ is injective in positive dimensions. Its image is the set of elements of $H(S)$ invariant under conjugation.*

The second version of the result is more concise. It simply says that the canonical map

$$H(G)_{(p)} \to \lim_{P} H(P)$$

is an isomorphism in positive dimensions, where the limit (or inverse limit) is taken over the category whose objects are the p-subgroups P of G and whose morphisms are the maps $P_1 \to P_2$ induced by conjugation by elements of G. It is straightforward to prove that this version of the result is equivalent to the one in Theorem 6.1.

The proof of the theorem is based on formal properties of the restriction and corestriction maps between $H(G)$ and $H(S)$. Recall first that the composite

$$H(G) \to H(S) \to H(G)$$

(restriction followed by corestriction) is simply multiplication by the index $[G : S]$, which is relatively prime to p. This composite therefore induces an automorphism of the p-primary component $H(G)_{(p)}$; in particular, the restriction map is injective on this component. Next, one checks that the image of the monomorphism $H(G)_{(p)} \hookrightarrow H(S)$ is contained in the conjugation invariants in $H(S)$. Finally, to prove that the image is the set of all invariants, one considers the other composite:

$$H(S) \to H(G)_{(p)} \to H(S)$$

(corestriction followed by restriction). This composite is not as easy to describe as the other one, but there is in fact a formula for it involving conjugation, and the formula allows one to conclude that the composite, when restricted to the invariants, is again multiplication by $[G : S]$ and hence an automorphism. Theorem 6.1 follows at once.

6.3 Complete cohomology (Tate)

We have seen in Sections 6.1 and 6.2 that, for finite groups, homology and cohomology have similar properties and can be treated simultaneously. Carrying this idea to the extreme, one finds that there is a unified cohomology theory that combines all of the homology and cohomology functors (after H_0 and H^0 are modified so that they behave like the others). The resulting theory, called *complete cohomology* or *Tate cohomology*, involves a doubly-infinite family of functors $\widehat{H}^n$ $(n \in \mathbb{Z})$, which are related to the usual functors as follows:

$$\begin{array}{cccccccc} \cdots & H_2 & H_1 & H_0 & H^0 & H^1 & H^2 & \cdots \\ & \Vert & \Vert & \uparrow\!\!\!\!\!\!\!\!\!\!\!\!\!\!\hookrightarrow & \downarrow\!\!\downarrow & \Vert & \Vert & \\ \cdots & \widehat{H}^{-3} & \widehat{H}^{-2} & \widehat{H}^{-1} & \widehat{H}^{0} & \widehat{H}^{1} & \widehat{H}^{2} & \cdots \end{array}$$

The modifications of H_0 and H^0 that produce $\widehat{H}^{-1}$ and $\widehat{H}^0$ are easy to explain. Recall that, for any G-module M, there is a *norm operator* $N\colon M \to M$ given by $m \mapsto \sum_{g\in G} gm$ for $m \in M$. It is obvious that the image of this operator is contained in $M^G = H^0(G, M)$. And it is equally easy to check that the norm operator induces a map on the quotient $M_G = H_0(G, M)$ of M. Thus we have a map

$$\overline{N}\colon H_0(G, M) \to H^0(G, M),$$

which already arose naturally in Example 3.2. The functors $\widehat{H}^{-1}$ and $\widehat{H}^0$ are simply the kernel and cokernel of $\overline{N}$. Taking $M = \mathbb{Z}$, for example, we have $H_0(G, \mathbb{Z}) = H^0(G, \mathbb{Z}) = \mathbb{Z}$, and $\overline{N}$ is multiplication by $m := |G|$. So $\widehat{H}^{-1}(G, \mathbb{Z}) = 0$, and

$$\widehat{H}^0(G, \mathbb{Z}) = \mathbb{Z}_m\,, \tag{6.2}$$

where the right side is the group of integers mod m.

As a first illustration of the usefulness of the complete theory, let's return to Example 3.2, where G is a finite cyclic group. The calculation of homology and cohomology in that example can be restated as follows: $\widehat{H}^n$ is periodic of period 2; it is the kernel of $\overline{N}$ if n is odd and the cokernel of $\overline{N}$ if n is even.

6.4 Construction of the complete theory

The definition of the functors $\widehat{H}^*$ above does not explain why they fit together to form a cohomology theory. We will therefore give an alternate definition based on the notion of *complete resolution*. The starting point is the fact that $\mathbb{Z}$ admits a "backwards" projective resolution

$$0 \to \mathbb{Z} \to Q^0 \to Q^1 \to \cdots . \tag{6.3}$$

One can prove this by relative homological algebra [3, Section VI.2] or by duality theory (in the spirit of Section 4.2). The duality proof goes as follows.

Start with an ordinary resolution

$$\cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

by finitely generated projective $\mathbb{Z}G$-modules, and dualize, i.e., set $Q^i := \mathrm{Hom}_G(P_i, \mathbb{Z}G)$. To see that this yields an exact sequence as in (6.3), observe that the cochain complex Q computes $H^*(G, \mathbb{Z}G)$; the latter is the same as the cohomology of the trivial group with $\mathbb{Z}$ coefficients (Eckmann–Shapiro), so it is $\mathbb{Z}$ in dimension 0 and it vanishes elsewhere. [Note that we have used the finiteness of G in this argument: The coefficient module $\mathbb{Z}G$ is an induced module, hence also a coinduced module.]

We now splice together an ordinary projective resolution of $\mathbb{Z}$ with a backwards resolution of $\mathbb{Z}$ to obtain an exact sequence of projectives

$$\cdots \to P_1 \to P_0 \to Q^0 \to Q^1 \to \cdots ,$$

where the map $P_0 \to Q^0$ is the composite

$$P_0 \twoheadrightarrow \mathbb{Z} \hookrightarrow Q^0.$$

Setting $P_i := Q^{-1-i}$ for $i < 0$, we obtain a complete resolution in the sense of the following definition:

Definition 6.2. A *complete resolution* for the finite group G is an acyclic chain complex of projectives

$$\cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$$

such that the map $P_0 \to P_{-1}$ factors as a composite $P_0 \twoheadrightarrow \mathbb{Z} \hookrightarrow P_{-1}$. Here, as usual, $\mathbb{Z}$ is assumed to have trivial G-action.

We have seen that complete resolutions exist. Here is a situation in which they arise naturally.

Example 6.3. Suppose G acts freely on a CW-complex X homeomorphic to the sphere S^{2k-1} as in Example 1.3. Recall that one can construct an ordinary resolution by splicing together infinitely many copies of the chain complex $C_* := C_*(X)$. Similarly, one can construct a complete resolution, which is periodic of period $2k$, by splicing together a doubly-infinite collection of copies of C_*:

$$\cdots \to C_0 \to C_{2k-1} \to \cdots \to C_0 \to C_{2k-1} \to \cdots \to C_0 \to C_{2k-1} \to \cdots$$

One can show that complete resolutions are unique up to homotopy. We use them to give the "right" definition of complete cohomology:

Definition 6.4. For any G-module M,

$$\widehat{H}^*(G,M) := H^*(\mathrm{Hom}_G(P,M)),$$

where P is a complete resolution.

It is not hard to check that this definition is consistent with the *ad hoc* definition given in Section 6.3. The advantage of the present approach is that one can develop all the usual cohomological properties (long exact sequences, restriction and corestriction, cup products, etc.) Here is a nice illustration of cup products: For any integer i, there is a cup product

$$\widehat{H}^i(G,\mathbb{Z}) \otimes \widehat{H}^{-i}(G,\mathbb{Z}) \to \widehat{H}^0(G,\mathbb{Z}) = \mathbb{Z}_m, \tag{6.4}$$

where $m := |G|$; see (6.2). Now the groups $\widehat{H}^*(G,M)$ are always annihilated by m as in Corollary 3.4, i.e., they are $\mathbb{Z}_m$-modules. Moreover, there is a good duality theory for finitely generated $\mathbb{Z}_m$-modules. [The dual is given by $\mathrm{Hom}(-,\mathbb{Z}_m)$. Passage to the dual is an exact contravariant functor. A finitely-generated $\mathbb{Z}_m$-module is always non-canonically isomorphic to its dual and canonically isomorphic to its double dual.]

Proposition 6.5. *The cup product in* (6.4) *is a duality pairing. In other words, the induced map*

$$\widehat{H}^i(G,\mathbb{Z}) \to \mathrm{Hom}(\widehat{H}^{-i}(G,\mathbb{Z}),\mathbb{Z}_m)$$

is an isomorphism.

We omit the proof but simply remark that elementary arguments (involving the universal coefficient theorem) can be used to prove the abstract duality of $\widehat{H}^i(G,\mathbb{Z})$ and $\widehat{H}^{-i}(G,\mathbb{Z})$, based on the definitions of these cohomology groups in Section 6.3. The significance of the proposition, then, is the fact that duality is given by cup product; this takes some work to prove.

6.5 Groups with periodic cohomology

Groups that act freely on a $(2k-1)$-sphere have a periodic complete resolution of period $2k$ by Example 6.3, hence

$$\widehat{H}^n(G,M) \cong \widehat{H}^{n+2k}(G,M)$$

for all G-modules M and all integers n. The complete cohomology theory allows one to formulate this kind of periodicity in several equivalent ways:

Proposition 6.6. *The following conditions on the finite group G are equivalent.*

(i) For some $d \neq 0$ there is an element $u \in \widehat{H}^d(G,\mathbb{Z})$ that is invertible in the ring $\widehat{H}^*(G,\mathbb{Z})$.

(ii) For some $d \neq 0$ there is an element $u \in \widehat{H}^d(G,\mathbb{Z})$ such that multiplication by u gives an isomorphism $\widehat{H}^n(G,M) \xrightarrow{\sim} \widehat{H}^{n+d}(G,M)$ for all $n \in \mathbb{Z}$ and all G-modules M.

(iii) There are integers n and d, with $d \neq 0$, such that $\widehat{H}^n(G,M) \cong \widehat{H}^{n+d}(G,M)$ for all G-modules M.

(iv) There is an integer $d \neq 0$ such that $\widehat{H}^d(G,\mathbb{Z}) \cong \mathbb{Z}_m$ $(m := |G|)$.

(v) There is an integer $d \neq 0$ such that $\widehat{H}^d(G,\mathbb{Z})$ contains an element of order $m := |G|$.

Sketch of the proof. It is obvious that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) and that (iv) $\Longrightarrow$ (v). To prove (iii) $\Longrightarrow$ (iv), first use dimension-shifting to show that there are isomorphisms as in (iii) for *all* integers n; now take $M = \mathbb{Z}$ and $n = 0$. Finally, the implication (v) $\Longrightarrow$ (i), which is perhaps the most surprising one *a priori*, follows from Proposition 6.5. $\square$

Combining Proposition 6.6 with the local calculation of cohomology (Section 6.2) and doing a little group theory, one can prove:

Theorem 6.7. *The following conditions are equivalent.*

(i) G has periodic cohomology.

(ii) Every abelian subgroup of G is cyclic.

(iii) For every prime p, every elementary abelian p-subgroup of G has rank at most 1.

(iv) The Sylow subgroups of G are cyclic or generalized quaternion groups.

The condition that G have periodic cohomology is very restrictive, and the groups with this property have been completely classified. A less restrictive (but still quite useful) condition is periodicity of the p-primary component $\widehat{H}^*(G)_{(p)}$ for a fixed prime p. More briefly, we say that G has *p-periodic cohomology*. There are various characterizations of this property analogous to the results stated above. For example, G has p-periodic cohomology (for a given prime p) if and only if every elementary abelian p-subgroup of G has rank at most 1. Roughly speaking, then, the p-primary part of the cohomology of G is very simple if and only if the structure of the p-subgroups of G is very simple.

We close by stating a beautiful and far-reaching generalization of the last assertion, due to Quillen [8]. In order to motivate it, suppose that G has p-periodic cohomology for some prime p dividing $|G|$. Then the ordinary mod p cohomology ring $H^*(G, \mathbb{F}_p)$ is finitely generated over a polynomial subring $\mathbb{F}_p[u]$ and hence has Krull dimension 1. Quillen's generalization asserts, among other things, the following:

Theorem 6.8. *For any finite group G and prime p, the Krull dimension of $H^*(G, \mathbb{F}_p)$ is the maximal rank of an elementary abelian p-subgroup of G.*

Thus the complexity of $H^*(G, \mathbb{F}_p)$ correlates precisely with the complexity of the p-subgroups of G.

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