ABSTRACT HOMOTOPY THEORY AND GENERALIZED SHEAF COHOMOLOGY

by

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Abstract

Cohomology groups $H^q(X,E)$ are defined, where X is a topological space and E is a sheaf on X with values in Kan's category of spectra. This includes, as special cases, the generalized cohomology of X with coefficients in a spectrum and the cohomology of X with coefficients in an abelian sheaf. The definition uses the abstract homotopy theory of Quillen's <u>Homotopical Algebra</u> applied to suitable categories of sheaves. In order to further study the homotopy category of sheaves of spectra, which is a non-additive analogue of the derived category of X, a homotopy theory more general than Quillen's is required. This is developed at the beginning of the paper and should be of independent interest. It yields, for example, an elementary conceptual proof of Verdier's hypercovering theorem, as well as an analogous theorem for the cohomology theory developed in this paper. A spectral sequence

$$H^p(X,\pi_{-q}E) \Longrightarrow H^{p+q}(X,E)$$

is constructed, a special case of which is a Leray spectral sequence in generalized sheaf cohomology for a map $Y \to X$. There is an appendix on stable homotopy theory in Kan's category of spectra. This again makes use of abstract homotopy theory.

Thesis Supervisor: Daniel G. Quillen
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INTRODUCTION

The main purpose of this paper is to define and study cohomology groups $H^q(X,E)$, where E is a sheaf on X with values in Kan's category of spectra, X being a topological space. If E is a sheaf of (generalized) Eilenberg-MacLane spectra, then $H^{\bullet}(X,E)$ reduces to the usual (hyper-) cohomology of X with coefficients in an abelian sheaf (or complex of abelian sheaves). On the other hand, if E is a constant sheaf and X is a CW complex, then $H^{\bullet}(X,E)$ reduces to the generalized cohomology of X in the usual sense.

The cohomology groups are defined by means of the theory of derived functors developed by Quillen [QHA]. More specifically, by doing homotopy theory in suitable categories of sheaves, it is possible to prove the existence of resolutions suitable for defining derived functors of the global section functor.

This procedure, however, while sufficient for the construction of the cohomology groups, does not lead to a completely satisfactory theory, since it gives no information at all about the homotopy category associated to the category of all sheaves of spectra, which is a natural non-additive analogue of the derived of X (i.e., the derived category of the category of abelian sheaves on X [H]). In order to remedy this defect, I have developed a new treatment of abstract homotopy theory, involving much weaker axioms than those of [QHA]. This theory is presented in Chapter I, and should be of independent interest. An immediate corollary of the first theorem of Chapter I, for example, is an elementary conceptual proof of Verdier's hypercovering theorem [SGA4, exp. V, appendix; AM], as well as an analogous theorem for the cohomology theory developed in this paper.

Chapter II applies the theory of Chapter I to the study of the homotopy theory of sheaves of spectra, and in Chapter III the cohomology groups are defined and their basic properties are derived. The most important of these properties is a spectral sequence $H^p(X,\pi_{-q}E) \Longrightarrow H^{p+q}(X,E)$, which generalizes the hypercohomology spectral sequence as well as the Atiyah-Hirzebruch spectral sequence. It also includes as a special case a Leray spectral sequence in generalized sheaf cohomology for a map $Y \rightarrow X$. Perhaps it should be mentioned that Chapter III also contains a sketch of an alternative approach to generalized sheaf cohomology. It involves canonical resolutions and avoids the homotopy theory of Chapters I and II, but it leads to technical difficulties, obscures the nature of the cohomology groups, and provides less information about the spectral sequence. It should also be mentioned that the theory of the present paper undoubtedly extends to more general sheaf cohomology [A; SGA4], but the details of this have not yet been worked out. This includes, for example, generalized equivariant cohomology and generalized etale cohomology of a scheme.

An appendix is included, developing stable homotopy theory in Kan's category of spectra by means of abstract homotopy theory. In an effort to make this paper more accessible, I have referred to the existing literature on Kan's category only for the easier results, and have made no use of the more technical results.

ACKNOWLEDGEMENT

The problem of studying the homotopy category of sheaves of spectra and of defining generalized sheaf cohomology groups was suggested to me by Daniel Quillen. I am grateful to him for helpful advice on how to attack the problem, without which I would never have gotten started. His motivation for suggesting this problem was the hope that the theory developed here would prove to be the right framework for defining (higher) algebraic K-groups of a ring (or scheme). Such applications, if they materialize, will be dealt with in future papers.

CHAPTER I

Abstract Homotopy Theory

The purpose of this chapter is to develop an abstract homotopy theory analogous to that of Quillen's [QHA], but with weaker axioms. Our theory will cover categories where there are good notions of weak equivalence and fibration, but not necessarily cofibration. A non-trivial application is given in Remark 1.13, and further applications will appear in Chapters II and III.

1. The axioms, the homotopy relation, and the homotopy category

It is convenient to split up abstract homotopy theory in a category $\underline{C}$ into two parts. The first is the study of the homotopy theoretical properties of a convenient full subcategory $\underline{C}_f$ of $\underline{C}$ ($\underline{C}_f$ would consist of the fibrant objects of $\underline{C}$ in the terminology of [QHA]). The second is the study of the relationship between $\underline{C}_f$ and $\underline{C}$. We will begin with the study of $\underline{C}_f$, leading up to Theorem 1.10, and then turn to the relationship between $\underline{C}_f$ and $\underline{C}$. Sections 2, 3, and 4 will then further develop the homotopy theory in $\underline{C}_f$. Even though $\underline{C}$ plays no role in the study of $\underline{C}_f$, we will continue to use the notation $\underline{C}_f$ to avoid later confusion.

Let $\underline{C}_f$ be a category with finite products and a final object e. Assume that $\underline{C}_f$ has two distinguished classes of maps, called <u>weak equivalences</u> and <u>fibrations</u>. A map will be called a <u>trivial fibration</u> if it is both a weak equivalence and a fibration. By a <u>path space</u> for an object B we mean an object B^I together with maps $B \xrightarrow{s} B^I \xrightarrow{(d_0,d_1)} B \times B$, where s is a weak equivalence, (d_0,d_1) is a fibration, and the composite is the diagonal map.

<u>Definition</u> 1.1. We call $\underline{C}_f$ a <u>category of fibrant objects for a homotopy theory</u>, or simply a category of fibrant objects, if the following axioms are satisfied.

(A) Let f and g be maps such that gf is defined. If two of f, g, gf are weak equivalences then so is the third. Any isomorphism is a weak equivalence.

(B) The composite of two fibrations is a fibration. Any isomorphism is a fibration.

(C) Given a diagram $A \xrightarrow{u} C \xleftarrow{v} B$ with v a fibration, the fibred product $A \times_C B$ exists and the projection $A \times_C B \xrightarrow{pr_1} A$ is a fibration. If v is a trivial fibration then so is pr_1.

(D) For any object B there exists at least one path space B^I (not necessarily functorial in B).

(E) For any object B the map $B \to e$ is a fibration.

We assume now that $\underline{C}_f$ is a category of fibrant objects.

Lemma 1.2. *Any map u can be factored u = pi where p is a fibration and i is a weak equivalence.*

Proof. Given $A \xrightarrow{u} B$, we choose a path space B^I and factor u as $A \xrightarrow{(id, su)} A \times_B B^I \xrightarrow{d_1 pr_2} B$. The fibred product exists because $B^I \to B$ is a (trivial) fibration (see [QHA]). The map $A \to A \times_B B^I$ is a weak equivalence because it is right inverse to pr_1, which is a trivial fibration because it is a base extension of $B^I \to B$. Finally, $A \times_B B^I \to B$ is a fibration because it factors as $A \times_B B^I \to A \times B \to B$, where the first is the base extension of $B^I \to B \times B$ by the map $A \times B \xrightarrow{u \times id} B \times B$, and the second map is the base extension of $A \to e$.

Remark 1.3. Lemma 1.2 applied to the diagonal map $B \to B \times B$ yields axiom (D), and in many applications it is no harder to prove Lemma 1.2 directly than to verify axiom (D). Note, however, that the proof actually shows the very useful fact that *the map* i *in Lemma* 1.2 *can always be taken to be right inverse to a trivial fibration.*

Lemma 1.4. *Any diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r] & B \end{tikzcd}$$

with i *a weak equivalence and* p *a fibration can be imbedded in a diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "i"'] & X' \arrow[r] \arrow[d, "t"] & E \arrow[d, "p"] \\ & X \arrow[r] & B \end{tikzcd}$$

with t *a trivial fibration.*

Proof. Apply Lemma 1.2 to the map $A \to X \times_B E$.

Definition 1.5. Two maps $f, g : A \to B$ will be called *strictly homotopic* if for some path space B^I there is a map $A \xrightarrow{h} B^I$ such that $d_0 h = f$ and $d_1 h = g$. We will write $f \overset{str}{\sim} g$.

Lemma 1.6. (i) *If* $f \overset{str}{\sim} g$ *and* u *is any map, then* $fu \overset{str}{\sim} gu$, *provided the compositions are defined.*

(ii) *Let* $B \xrightarrow{u} C$ *be a map and let* B^I, C^I *be path spaces. Then we can find a path space* $B^{I'}$, *a trivial fibration* $B^{I'} \to B^I$, *and a map* $B^{I'} \to C^I$, *such that the following diagram commutes.*

$$\begin{tikzcd} & B \arrow[dl] \arrow[dr] \arrow[rr, "u"] & & C \arrow[d] \\ B^I \arrow[dr] & & B^{I'} \arrow[ll] \arrow[dl] \arrow[r] & C^I \arrow[d] \\ & B \times B \arrow[rr, "u \times u"] & & C \times C. \end{tikzcd}$$

(iii) *Let* $B \xrightarrow{u} C$ *be a map and let* C^I *be a path space. If* $f \overset{str}{\sim} g : A \to B$ *then there is a trivial fibration* $t : A' \to A$ *such that* $uft \overset{str}{\sim} ugt$ *by a homotopy* $A' \to C^I$.

Proof. (i) is trivial. (ii) follows from Lemma 1.4 applied to the square

$$\begin{array}{ccc} B & \xrightarrow{su} & C^I \\ \downarrow & & \downarrow \\ B^I & \xrightarrow{(ud_0,ud_1)} & C \times C. \end{array}$$

For (iii), let $A \xrightarrow{h} B^I$ be a homotopy from f to g and form $B^{I'}$ as in (ii). If we let $A' = A \times_{B^I} B^{I'}$, the desired homotopy is the composite $A' \longrightarrow B^{I'} \longrightarrow C^I$.

Path spaces can be pieced together as in [QHA], which shows that strict homotopy is an equivalence relation. We now define homotopy for maps from A to B by saying that f and g are homotopic if there is a trivial fibration t such that $ft \overset{str}{\sim} gt$. This is still an equivalence relation because the fibred product of two trivial fibrations is again a trivial fibration, and it is easily seen (using Lemma 1.6 (iii)) that homotopy is compatible with composition on both sides, so that we can form a category $\pi\underline{C}_f$ with the same objects as $\underline{C}_f$ and with $\mathrm{Hom}_{\pi\underline{C}_f}(A,B) = \pi(A,B)$, where $\pi(A,B)$ is the set of homotopy classes of maps from A to B. Note that if $ft \overset{str}{\sim} gt$ for any weak equivalence t, not necessarily a fibration, we can deduce from Lemma 1.4 that f and g are in fact homotopic.

Lemma 1.8. *Given a diagram* $A \xrightarrow{u} C \xleftarrow{v} B$ *with* u *and* v *arbitrary, the map* $A \times_C C^I \times_C B \xrightarrow{pr_3} B$ *is a fibration and is a trivial fibration if* u *is a weak equivalence.*

Proof. The map pr_3 is a base extension of the map $A \times_C C^I \longrightarrow C$ constructed in the proof of Lemma 1.2. The result is now immediate.

Definition 1.9. Let $\underline{C}$ be any category with a distinguished class of maps called weak equivalences. Then by the homotopy category of $\underline{C}$ (denoted $\mathrm{Ho}\underline{C}$) we mean the category obtained from $\underline{C}$ by inverting the weak equivalences (see [GZ; H; QHA]).

Theorem 1.10. *Let $\underline{C}_f$ be a category of fibrant objects for a homotopy theory. Then the maps in $\mathrm{Ho}\underline{C}_f$ admit the explicit description,*

$$[A,B]_{\mathrm{Ho}\underline{C}_f} = \varinjlim \pi(A',B),$$

where the limit is taken over the category whose objects are weak equivalences $A' \to A$ and whose maps are commuting triangles in $\pi\underline{C}_f$. In technical terms, the class of weak equivalences admits a calculus of right fractions in $\pi\underline{C}_f$ (see [GZ]).

Proof. According to [GZ], we must verify (a) any diagram $A \xrightarrow{u} C \xleftarrow{v} B$ with v a weak equivalence can be imbedded in a square in $\pi\underline{C}_f$

$$\begin{array}{ccc} D & \to & B \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle v} \\ A & \xrightarrow{u} & C \end{array},$$

where t is a weak equivalence; and (b) given $A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{v} C$ with v a weak equivalence and $vf = vg$ in $\pi\underline{C}_f$, there is a weak equivalence t such that $ft = gt$ in $\pi\underline{C}_f$. The proof of (a) is contained in the statement of Lemma 1.8. For (b), we may assume there is a strict homotopy $A \xrightarrow{h} C^I$ from vf to vg, from which we obtain a map $A \xrightarrow{(f,h,g)} B \times_C C^I \times_C B$. Now Lemma 1.8 shows that the two projections from $B \times_C C^I \times_C B$ to B are weak equivalences, and it follows easily that by factoring the map $B \xrightarrow{(id, sv, id)} B \times_C C^I \times_C B$ as in Lemma 1.2 we will obtain a path space B^I with a trivial fibration $B^I \longrightarrow B \times_C C^I \times_C B$. We can then take $A' = A \times_{B \times_C C^I \times_C B} B^I$ and $t = pr_1$.

Remarks. 1.11. It follows from Remark 1.3 that to obtain $Ho\underline{C}_f$ we need only invert the trivial fibrations. Theorem 1.10 remains true if we replace "weak equivalence" by "trivial fibration". (It is worth observing, in this connection, that using Lemmas 1.2 and 1.4 one can prove that every map in $\pi\underline{C}_f$ is isomorphic to a fibration, and so every weak equivalence is isomorphic to a trivial fibration.)

1.12. In applications we are often given a functorial path space B^I. In this case it is more natural to define strict homotopy using this path space rather than allowing an arbitrary path space, and then (since this will already be compatible with composition on both sides) to let homotopy be the equivalence relation generated by strict homotopy. By applying Lemma 1.6 (iii) to id_B, we see that if $f \overset{str}{\sim} g$ in the old sense then there is a weak equivalence t such that $ft \overset{str}{\sim} gt$ in the present sense. Using this, we easily deduce from Theorem 1.10 the analogous theorem for the new definition of homotopy. Note, however, that our proof seems to make essential use of the possibility of using a flexible path space, since otherwise the map $B^I \longrightarrow B \times_C C^I \times_C B$ would probably not be a fibration. But this problem disappears if we use the reformulation of the theorem suggested in Remark 1.11.

1.13. Theorem 1.10 (together with Remark 1.12) has as an immediate corollary the hypercovering theorem of Verdier [SGA4, exp.V, appendix; AM]. (The attempt to obtain this corollary is, in fact, what led to the present generality of our theory.) To deduce the hypercovering theorem, we take for $\underline{C}_f$ the category $\underline{S}(X)_f$ of simplicial sheaves on X (i.e., simplicial objects in the category of sheaves of sets) which stalkwise satisfy Kan's extension condition. Weak equivalence and fibration are defined stalkwise, and it is trivial that all the axioms are satisfied. Now using Theorem 1.10 and Remark 1.12, and using standard and elementary properties of Eilenberg-MacLane complexes which extend to sheaves with no difficulty, a direct translation of Verdier's theorem is: For any abelian sheaf F, $H^q(X,F) \cong [e,K(F,q)]_{\mathrm{Ho}\underline{S}(X)_f}$.
On the other hand, it is well-known [H], and easy to prove using abstract homotopy theory, that the left-hand side of this equation is equal to $[\mathbb{Z}(e),K(F,q)]_{\mathrm{Ho}\underline{S}_{ab}(X)}$, where $\underline{S}_{ab}(X)$ is the category of simplicial abelian sheaves and $\mathbb{Z}(-)$ is the abelianization functor, and so Verdier's theorem follows from Lemma 1.20 (case (a)) at the end of this section. (We have used implicitly the fact that $\mathrm{Ho}\underline{S}_{ab}(X)$ can be identified with a full subcategory of the derived category of X [DP, § 3, and H], and that, under this identification $\mathbb{Z}(e)$ is the constant sheaf with stalk $\mathbb{Z}$, concentrated in dimension 0, and K(F,q) is F, concentrated in (co-) dimension -q.)

1.14. In many applications (in particular, whenever the axioms of [QHA] are satisfied) the direct system in Theorem 1.10 is essentially constant. It is not difficult to find sufficient conditions for this, but we will be content to consider only the following question: If $\underline{C}$ is a category with notions of fibration and weak equivalence, what axioms on the fibrations and weak equivalences will guarantee that $\underline{C}$ can be given the structure of closed model category [QHA], except possibly for the existence of limits? A useful answer to this is given by Proposition 1.16 below. First we recall some terminology from [QHA].

Definition 1.15. A map $A \xrightarrow{i} X$ will be said to have the *LLP(left lifting property)* with respect to a map $E \xrightarrow{p} B$, and p will be said to have the *RLP* (*right lifting property*) with respect to i, if for any solid arrow diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r] \arrow[ur, dotted] & B \end{tikzcd},$$

the dotted arrow exists. We call i a *cofibration* if it has the LLP with respect to all trivial fibrations.

Consider now the following axioms.

(F) Any map u can be factored u = sj where j is a cofibration and s is a weak equivalence.

(G) Any map u can be factored u = pi, where p is a fibration, i is a weak equivalence, and i has the LLP with respect to all fibrations.

Proposition 1.16. *Let $\underline{C}$ be a category with notions of weak equivalence and fibration. Assume that a retract of a weak equivalence (or fibration) is again a weak equivalence* (*or fibration*). *If $\underline{C}$ satisfies axioms* (A), (B), (C), (F), *and* (G), *then $\underline{C}$ also satisfies all of Quillen's axioms for a closed model category* [QHA, QRH], *except possibly for the existence of limits*.

Proof. By applying (G) to the map s in axiom (F), we see that s can be assumed to be a trivial fibration, so both factorization axioms are satisfied. It also follows from (G) that every trivial cofibration i is a retract of a map with the LLP with respect to all fibrations, so i also has this property. Thus the lifting axioms are satisfied, and the other axioms present no difficulty.

Remark 1.17. This proposition applies in particular to simplicial sets, with weak equivalence defined by means of the geometric realization functor, and yields a simple proof of [QHA, Chapter II, § 3, Theorem 3].

We now consider the second problem referred to at the beginning of this section. Thus we assume that $\underline{C}_f$ is a full subcategory of some category $\underline{C}$ and that weak equivalences are also defined in $\underline{C}$ and satisfy axiom (A). In practice, fibrations will also be defined in $\underline{C}$ and $\underline{C}_f$ will consist of the objects for which (E) holds, but all we need to assume for the following proposition is that $\underline{C}_f$ satisfies (A) through (E).

Proposition 1.18. *With the notation and assumptions of the preceding paragraph, assume either*

(1) *There is a functor* $\underline{C} \xrightarrow{fib} \underline{C}_f$ *and a natural weak*

equivalence $A \xrightarrow{i} \mathrm{fib}(A)$; *or*

(2) (weak form of axiom (G)) *For any* A *in* $\underline{C}$ *there is a weak equivalence* $A \xrightarrow{i} A'$, *where* A' *is in* $\underline{C}_f$ *and* i *has the LLP with respect to all fibrations* (*in* $\underline{C}_f$).

Then $\mathrm{Ho}\underline{C}_f \to \mathrm{Ho}\underline{C}$ *is an equivalence of categories.*

Proof. If (1) holds then there is an induced functor $\mathrm{Ho}\underline{C} \xrightarrow{\mathrm{fib}} \mathrm{Ho}\underline{C}_f$ quasi-inverse to $\mathrm{Ho}\underline{C}_f \to \mathrm{Ho}\underline{C}$. If (2) holds then the assignment $A \mapsto A'$ defines a functor $\underline{C} \to \pi\underline{C}_f$ which induces a quasi-inverse to $\mathrm{Ho}\underline{C}_f \to \mathrm{Ho}\underline{C}$ (cf. [QHA]).

Remark 1.19. If fibrations are defined in $\underline{C}$ and still satisfy axioms (B) and (C), and if $\underline{C}_f$ consists of those objects for which (E) holds (the *fibrant* objects), then we can deduce from 1.10, 1.11, and 1.18 the following description of $[X,B]_{\mathrm{Ho}\underline{C}}$ for arbitrary X and fibrant B, provided that either axiom (G) is satisfied in $\underline{C}$ or condition (1) of 1.18 is satisfied: *Any map from* X *to* B *in* $\mathrm{Ho}\underline{C}$ *can be written* ft^{-1}, *where* t *is a trivial fibration and* f *is a map in* $\underline{C}$; *any two maps can be written* ft^{-1} *and* gt^{-1} (*same* t), *and they are equal if and only if there is a trivial fibration* t' *such that* $ft' \overset{str}{\sim} gt'$. (I will prove here the hardest part of this assertion. Assume (1) holds and let f and g be equal in $\mathrm{Ho}\underline{C}$. I will show that there is a trivial fibration t such that $ft \overset{str}{\sim} gt$. Since $\mathrm{fib}(f) = \mathrm{fib}(g)$ in $\mathrm{Ho}\underline{C}$, we know from 1.10 that there is a trivial fibration $t'\colon Z \to \mathrm{fib}(X)$ such that $\mathrm{fib}(f)t' \overset{str}{\sim} \mathrm{fib}(g)t'$. Replacing X by $X \times_{\mathrm{fib}(X)} Z$, we may therefore assume that if $\overset{str}{\sim}$ ig, where $B \xrightarrow{i} \mathrm{fib}(B)$ is the natural map. The result now follows from the proof of 1.10.)

We will end this section with a very useful lemma on adjoint functors.

Lemma 1.20. *Let* $\underline{C}$ *and* $\underline{C}'$ *be categories in which there is a notion of weak equivalence, and let* $\underline{C}' \xrightarrow{L} \underline{C}$ *be left adjoint to* $\underline{C} \xrightarrow{R} \underline{C}'$. *Then each of the conditions* (a), (b), (c) *below is sufficient to imply that the left derived functor* $\underline{\underline{L}}$ *of* L *and the right derived functor* $\underline{\underline{R}}$ *of* R *exist* (*see* [QHA, I,§4] *for the definition of derived functor*), *and that* $\underline{\underline{L}}\colon \mathrm{Ho}\underline{C}' \to \mathrm{Ho}\underline{C}$ *is left adjoint to* $\underline{\underline{R}}\colon \mathrm{Ho}\underline{C} \to \mathrm{Ho}\underline{C}'$.

(a) L *and* R *preserve weak equivalences* (*in which case* $\underline{\underline{L}} = L$, $\underline{\underline{R}} = R$);

(b) $\underline{C}$ *has a full subcategory* $\underline{C}_f$ *satisfying the hypotheses of* 1.18, R *preserves weak equivalences in* $\underline{C}_f$, *and* L *preserves arbitrary weak equivalences* (*in which case* $\underline{\underline{L}} = L$);

(c) $\underline{C}$ *has a full subcategory* $\underline{C}_f$ *satisfying the hypotheses of* 1.18, $\underline{C}'$ *has a full subcategory* $\underline{C}'_c$ *satisfying the duals of the hypotheses of* 1.18, R *preserves weak equivalences in* $\underline{C}_f$, *and* L *preserves weak equivalences in* $\underline{C}'_c$.

Proof. I will give the proof in case (c), assuming that condition of (1) of 1.18 and its dual is satisfied. The other cases are similar or easier. We define $\underline{\underline{L}}(A) = L\,\mathrm{cof}(A)$ and $\underline{\underline{R}}(B) = R\,\mathrm{fib}(B)$, where $\mathrm{cof}\colon \underline{C}' \to \underline{C}'_c$ is the functor and $\mathrm{cof}(A) \xrightarrow{j} A$ is the natural map of condition (1) of the dual of 1.18. It is trivial to check that these are derived functors, and if $LR \xrightarrow{\alpha} \mathrm{Id}_{\underline{C}}$ and $\mathrm{Id}_{\underline{C}'} \xrightarrow{\beta} RL$

are the two given adjunction maps, we define a map $\underline{\underline{L}}\underline{\underline{R}}B \longrightarrow B$ in $\mathrm{Ho}\underline{C}$ as the composite

$$\underline{\underline{L}}\underline{\underline{R}}B = L\mathrm{cof}R\mathrm{fib}B \xrightarrow{Lj} LR\mathrm{fib}B \xrightarrow{\alpha} \mathrm{fib}B \xrightarrow{i^{-1}} B.$$

Dually, there is a map $A \longrightarrow \underline{\underline{R}}\underline{\underline{L}}A$ in $\mathrm{Ho}\underline{C}'$, and it is easy to see that these maps are adjunction maps. For example, to check that $\underline{\underline{L}}A \longrightarrow \underline{\underline{L}}\underline{\underline{R}}\underline{\underline{L}}A \longrightarrow \underline{\underline{L}}A$ is the identity, we may assume A is in $\underline{C}'_c$ and we must check that the map obtained by going across the top and down the right side of the following diagram is the identity in $\mathrm{Ho}\underline{C}$, which follows from the commutativity of the diagram:

$$\begin{tikzcd}
L\mathrm{cof}A \arrow[r, "L\mathrm{cof}j^{-1}"] \arrow[d, "Lj"'] \arrow[dr, equal] & L\mathrm{cof}^2A \arrow[r, "L\mathrm{cof}\beta"] \arrow[d, "Lj"'] & L\mathrm{cof}RL\mathrm{cof}A \arrow[r, "L\mathrm{cof}Ri"] \arrow[d, "Lj"'] & L\mathrm{cof}R\mathrm{fib}L\mathrm{cof}A \arrow[d, "Lj"'] \\
LA \arrow[r, "Lj^{-1}"] & L\mathrm{cof}A \arrow[r, "L\beta"] \arrow[dr, equal] & LRL\mathrm{cof}A \arrow[r, "LRi"] \arrow[d, "\alpha"'] & LR\mathrm{fib}L\mathrm{cof}A \arrow[d, "\alpha"'] \\
& & L\mathrm{cof}A \arrow[r, "i"] \arrow[dr, equal] & \mathrm{fib}L\mathrm{cof}A \arrow[d, "i^{-1}"'] \\
& & & L\mathrm{cof}A.
\end{tikzcd}$$

<u>Remarks</u>. 1.21. Except for the cases of the above lemma where condition (2) of 1.18 (or its dual) is assumed, no use is made of fibrations or cofibrations.

1.22. The power of adjoint functors is particularly striking under hypothesis (a) of the lemma, where the lemma enables us to say something about the maps in $\mathrm{Ho}\underline{C}$ (and $\mathrm{Ho}\underline{C}'$) without knowing anything about how to describe these maps in terms of the maps in $\underline{C}$ (or $\underline{C}'$). We have already seen a non-trivial application of this in 1.13.

2. The loop functor and fibration sequences

Before defining the loop space, it will be convenient to prove some lemmas about fibrations. Let $\underline{C}_f$ be a category of fibrant objects. If B is an object of $\underline{C}_f$, let $\underline{C}_{fib}/B$ denote the category of fibrations over B. It is trivial to check (using 1.2) that, if we define fibration and weak equivalence in $\underline{C}_{fib}/B$ by means of the forgetful functor to $\underline{C}_f$, then $\underline{C}_{fib}/B$ is also a category of fibrant objects.

Lemma 2.1. *If $B' \xrightarrow{u} B$ is a map in $\underline{C}_f$, then the base change functor $\underline{C}_{fib}/B \xrightarrow{u^{-1}} \underline{C}_{fib}/B'$ preserves fibrations and weak equivalences (and hence homotopies).*

Proof. Applying 1.3 to $\underline{C}_{fib}/B$, we see that it is sufficient to prove that u^{-1} preserves fibrations and trivial fibrations, which is clear. (One needs to observe here that if $E_1 \to E_2$ is a fibration of fibrations over B, then $B' \times_B E_1 = (B' \times_B E_2) \times_{E_2} E_1$.)

Lemma 2.2. *If u is a weak equivalence then u^{-1} induces an equivalence of categories*

$$\mathrm{Ho}(\underline{C}_{fib}/B) \to \mathrm{Ho}(\underline{C}_{fib}/B').$$

Proof. By 1.3, we may assume u is a trivial fibration. In this case u^{-1} is right adjoint to the forgetful functor $\underline{C}_{fib}/B' \to \underline{C}_{fib}/B$, and both adjunction maps are weak equivalences. The result now follows from 1.20.

Lemma 2.3. *The base extension of a weak equivalence by a fibration is a weak equivalence.*

Proof. This comes from a careful examination of the proof of the preceding lemma. Explicitly, let $E \xrightarrow{p} B$ be the fibration and let $B' \xrightarrow{u} B$ be the weak equivalence, which we may assume to be right inverse to a trivial fibration $B \xrightarrow{v} B'$. We may also assume that p is the base extension by v of a fibration $E' \xrightarrow{p'} B'$, since it is sufficient to prove the lemma with p replaced by the base extension of $p' = vp$. But then the base extension of u by p is right inverse to a trivial fibration $E \rightarrow E'$, so it is a weak equivalence.

Definition 2.4. Let $B' \xrightarrow{u} B$ be a map in $\underline{C}_f$, let $E' \xrightarrow{p'} B'$ and $E \xrightarrow{p} B$ be fibrations, and let $\bar{u}_0, \bar{u}_1 : E' \rightarrow E$ be two maps covering u. We say that $\bar{u}_0$ and $\bar{u}_1$ are fibre homotopic relative to u if the corresponding maps from E' to $u^{-1}E$ are homotopic as maps in $\underline{C}_{fib}/B'$.

This definition can be translated as follows. Let $E/B^{\,I}$ be a path space for p in $\underline{C}_{fib}/B$. Then $\bar{u}_0$ and $\bar{u}_1$ are fibre homotopic relative to u if and only if there is a diagram in $\underline{C}_f$

$$\begin{array}{ccc} E'' & \longrightarrow & E/B^{\,I} \\ {\scriptstyle t}\downarrow & & \downarrow \\ E' & \xrightarrow{(\bar{u}_0,\bar{u}_1)} & E \times_B E \end{array},$$

with t a trivial fibration. By 1.4 it sufficient for t to be a weak equivalence, and an important special case is:

Lemma 2.5. *With the notation of Definition 2.4, assume there is a weak equivalence t such that $\bar{u}_0 t = \bar{u}_1 t$. Then $\bar{u}_0$ and $\bar{u}_1$ are fibre homotopic relative to u.*

We can now define loop spaces.

Theorem 2.6. *Assume $\underline{C}_f$ is pointed (i.e., the final object e is also an initial object). Then there is a functor $\mathrm{Ho}\underline{C}_f \xrightarrow{\Omega} \mathrm{Ho}\underline{C}_f$ such that for any object B and any path space B^I, ΩB can be canonically with the fibre of $B^I \to B \times B$ (i.e., $e \times_{B\times B} B^I$). Furthermore, ΩB has a natural group structure.*

Proof. Let $\Omega^{(I)}B$ denote the fibre of $B^I \to B \times B$. If there exists a map of path spaces $B^I \xrightarrow{f} B^{I'}$, i.e., a map such that $fs = s'$ and $d'_i f = d_i$, $i = 0,1$, then Lemma 2.1 shows that $\Omega^{(I)}B \to \Omega^{(I')}B$ is a weak equivalence. Furthermore, Lemma 2.5 shows that any two such maps are fibre homotopic relative to $id_{B\times B}$ (take $t = s$), so we obtain, in this case, a well-defined map $\Omega^{(I)}B \to \Omega^{(I')}B$. But 1.6 (ii), applied to id_B, shows that for any two path spaces there is a third one which maps to both of them, so by what we have just done we can identify $\Omega^{(I)}B$ and $\Omega^{(I')}B$ for any two path spaces B^I and $B^{I'}$. We can now write ΩB instead of $\Omega^{(I)}B$, and by a similar use of Lemmas 2.1, 1.5, and 1.6 we can show that ΩB is functorial in B. Using Lemmas 2.3 and 2.1, we see that Ω preserves weak equivalences, and so, finally, we have a functor $\mathrm{Ho}\underline{C}_f \xrightarrow{\Omega} \mathrm{Ho}\underline{C}_f$. To give ΩB a group structure, let B^I and $B^{I'}$ be any two path spaces and let $B^{I+I'} = B^I \times_B B^{I'}$. Then there is an obvious map $\Omega^{(I)}B \times \Omega^{(I')}B \to \Omega^{(I+I')}B$ which gives us a product $\Omega B \times \Omega B \xrightarrow{m} \Omega B$ in $\mathrm{Ho}\underline{C}_f$. The product is easily seen to be well-defined and associative. The fact that e is an identity for this multiplication is immediate from the definitions and from the fact that $B^I \xrightarrow{(sd_0, id)} B^I \times_B B^I$ and $B^I \xrightarrow{(id, sd_1)} B^I \times_B B^I$ are maps of path spaces. Finally, the inverse $\Omega B \to \Omega B$ is induced by $B^I \xrightarrow{id} B^{I^{-1}}$, where $B^{I^{-1}}$ is the same object as B^I but with d_0 and d_1 reversed. To see that this actually is a (right) inverse, we need only show that the two maps

$$B^I \underset{(sd_0, sd_0)}{\overset{diag}{\rightrightarrows}} B^I \times_B B^{I^{-1}}$$

are fibre homotopic relative to $diag \cdot pr_1 : B \times B \longrightarrow B \times B$. This again follows from Lemma 2.5.

Proposition 2.7. *Let $\underline{C}_f$ be a pointed category of fibrant objects and let $E \xrightarrow{p} B$ be a fibration with fibre F. Then there is a natural map $F \times \Omega B \xrightarrow{a} F$ in $Ho\underline{C}_f$ which defines a right action of the group ΩB on* F.

Proof. As in [QHA, I, § 3] we construct path spaces E^I, B^I related by a fibration $E^I \to E \times_B B^I \times_B E$, and we then deduce a trivial fibration $E^I \to E \times_B B^I$. By base extension we get a trivial fibration $F \times_E E^I \times_E F \xrightarrow{t} F \times \Omega B$, and the desired map is then $pr_3 t^{-1}$. It is straightforward, using the techniques we have developed, to verify that this is well-defined and has the desired properties.

Observe, relative to a choice of B^I, the map a is obtained from a well-defined map $E \times_B B^I \xrightarrow{\alpha} pr_2{}^{-1} E$ in $Ho(\underline{C}_{fib} / B \times B)$ by applying the fibre functor $Ho(\underline{C}_{fib} / B \times B) \longrightarrow Ho\underline{C}_f$. (Intuitively, α lifts a path to E and then takes its endpoint.) In an important special case which we will consider in a moment, we will be given a weak equivalence $E \xrightarrow{w} E'$ in $\underline{C}_{fib}/B$ and a map $E \times_B B^I \xrightarrow{f} E'$ lying over $B \times B \xrightarrow{pr_2} B$. Now f can be regarded as a map in $\underline{C}_{fib}/B\times B$, and composing it with the inverse of $pr_2^{-1}(w)$ we obtain a map in the homotopy category, which we will want to prove coincides with α. For this it is sufficient that the diagram

$$\begin{tikzcd} E^I \arrow[r, "d_1"] \arrow[d] & E \arrow[d, "w"] \\ E \times_B B^I \arrow[r, "f"] & E' \end{tikzcd}$$

be fibre homotopy commutative relative to pr_2, and we see from 2.5 that it is sufficient that $fi = w$, where $i = (id, sp)$: $E \to E \times_B B^I$.

The special case referred to above is the following. Let $A \to B$ be a map in $\underline{C}_f$, which we convert to a fibration $E = A \times_B B^I \to B$ as in the proof of 1.2. Let $E' = E \times_B B^I \xrightarrow{d_1 pr_2} B$, let w be induced by the map of path spaces $B^I \xrightarrow{(id,\ sd_1)} B^I \times_B B^I$, and let $f = id$. Then $fi = w$ and we conclude that the action of ΩB on the fibre is induced by f. (Intuitively, ΩB acts by composition of paths.) We will use this in the proof of the next proposition.

Proposition 2.8. With the notation of 2.7, let the inclusion $F \hookrightarrow E$ be converted into a fibration as in the proof of 1.2. Then the fibre of this fibration can be identified (in $Ho\underline{C}_f$) with ΩB (with the "fibre inclusion" being the composite $\Omega B \xrightarrow{(e,\ ^{-1})} F \times \Omega B \xrightarrow{a} F$), and the action of ΩE on ΩB is given by the composite

$$\Omega B \times \Omega E \xrightarrow{id \times \Omega p} \Omega B \times \Omega B \xrightarrow{m} \Omega B.$$

Proof. The fibre, $F \times_E E^I \times_E e$, admits a trivial fibration to ΩB, namely, the base extension of $E^I \longrightarrow B^I \times_B E$ by $\Omega B \xrightarrow{(i,e)} B^I \times_B E$, where i is the inclusion of ΩB into B^I. The assertion in parentheses is immediate from the definitions, using the trivial fibration $F \times_E E^I \xrightarrow{pr_1} F$ to identify F with the total space of the fibration, and the assertion about the action follows from the remarks preceding this proposition and from the proof of 2.6 (in particular, the definition of m).

Corollary 2.9. *Under the hypotheses of 2.7, there is an exact sequence in* HoC_f

$$\cdots \longrightarrow \Omega E \longrightarrow \Omega B \longrightarrow F \longrightarrow E \longrightarrow B,$$

where exactness is interpreted as in [QHA, I, p. 3.8].

The proof is straightforward, using 1,10. Note that the homotopy lifting property which one usually uses still holds for our definition of homotopy, in view of the fact that $E^I \longrightarrow E \times_B B^I$ is a trivial fibration.

Corollary 2.10. *Let* C_f *be a pointed category of fibrant objects and let* $P \longrightarrow B$ *be a fibration with fibre* F, *where* P *is weakly equivalent to* e. *Then* F *is canonically isomorphic to* ΩB *in* HoC_f.

Proof. Exactness of $e \longrightarrow \Omega B \longrightarrow F \longrightarrow e$ means that the group ΩB acts "transitively and without fixed points" on F, so $\Omega B \longrightarrow F$ is an isomorphism.

Remark 2.11. If we apply the discussion preceding 2.8 to the diagonal map of B, then we see easily that the natural action of $\Omega B \times \Omega B$ on ΩB induced by any path fibration $B^I \longrightarrow B \times B$ is given by two-sided translation (i.e., $g\cdot(g',g") = g'^{-1}gg"$). This is useful in the following situation. Given two maps from A to B, we form the "equalizer" $K = A \underset{B\times B}{\times} B^I$ and we have a fibration sequence in $\mathrm{Ho}\underline{C}_f$

$$\cdots \longrightarrow \Omega K \longrightarrow \Omega A \xrightarrow{\beta} \Omega B \longrightarrow K \longrightarrow A,$$

where, because of the above description of the action of $\Omega B \times \Omega B$ on ΩB, we can identify β as the "difference" of the two maps ΩA to ΩB.

3. A theorem on inverse limits

The purpose of this section is to prove, in our abstract setting, a well-known theorem on the inverse limit of a tower of fibrations. It will be convenient to introduce, for any category of fibrant objects $\underline{C}_f$, the category $\mathrm{Tow}(\underline{C}_f)$ of diagrams

$$\cdots \longrightarrow A_i \xrightarrow{p_i} A_{i-1} \longrightarrow \cdots,$$

where each p_i is a fibration in $\underline{C}_f$ and where $A_i = e$ for sufficiently small i. A map of towers $(A_i) \longrightarrow (B_i)$ will be called a weak equivalence if each map $A_i \longrightarrow B_i$ is a weak equivalence in $\underline{C}_f$; the map is called a fibration if each map $A_i \longrightarrow A_{i-1} \underset{B_{i-1}}{\times} B_i$ is a fibration in $\underline{C}_f$.

It is easy to verify that $\mathrm{Tow}(\underline{C}_f)$ is a category of fibrant objects, with these definitions. To verify the first part of axiom (C), for example, given $(A_i) \xrightarrow{u} (C_i) \xleftarrow{v} (B_i)$ with v a fibration, we must show that $(A_i \underset{C_i}{\times} B_i) \longrightarrow (A_i)$ is a fibration (which also implies that the fibred product is in fact an object of $\mathrm{Tow}(\underline{C}_f)$). Now $(A_{i-1} \underset{C_{i-1}}{\times} B_{i-1}) \underset{A_{i-1}}{\times} A_i$ $= A_i \underset{C_{i-1}}{\times} B_{i-1} = A_i \underset{C_i}{\times} (C_i \underset{C_{i-1}}{\times} B_{i-1})$, and so the map $A_i \underset{C_i}{\times} B_i \longrightarrow (A_{i-1} \underset{C_{i-1}}{\times} B_{i-1}) \underset{A_{i-1}}{\times} A_i$ is obtained from the fibration $B_i \longrightarrow C_i \underset{C_{i-1}}{\times} B_{i-1}$ by applying u_i^{-1}, and is therefore a fibration by 2.1.

In order to obtain results on inverse limits we will need another axiom.

(I) Every tower of fibrations in $\underline{C}_f$ has an inverse limit, and the functor

$$\mathrm{Tow}(\underline{C}_f) \xrightarrow{\varprojlim} \underline{C}_f$$

preserves fibrations and trivial fibrations (and hence weak equivalences by 1.3).

<u>Remarks</u>. 3.1. If $\underline{C}_f$ consists of the fibrant objects of some category $\underline{C}$, i.e., those objects B such that $B \longrightarrow e$ is a fibration, and if fibrations in $\underline{C}$ are characterized by the RLP with respect to some class of maps (see 1.15), then (I) holds, provided, of course, that the inverse limits exist in $\underline{C}$.

3.2. Applying (I) to the fibration of any tower onto any of its truncations, we see that (I) implies that

$$\varprojlim_i A_i \longrightarrow A_j$$

is a fibration for each j.

3.3. If (I) holds then we easily deduce from 1.10 and 1.11 that the functor $\underline{C}_f \longrightarrow \mathrm{Ho}\underline{C}_f$ preserves countable products.

Assume now that (I) holds, let (A_i) be an object of $\mathrm{Tow}(\underline{C}_f)$, and consider the two maps

$$\prod A_i \underset{\prod p_i}{\overset{id}{\rightrightarrows}} \prod A_i .$$

We form a homotopy equalizer of these two maps as in 2.11, and we observe that there is an obvious map

$$\varprojlim A_i \xrightarrow{i} K.$$

<u>Lemma</u> 3.4. <u>The map i is a weak equivalence</u>.

<u>Proof</u>. Let $(A_i{}^I)$ be a path space for (A_i). Then we may take $\prod A_i{}^I$ as path space for $\prod A_i$, and then it is

immediate from the definitions that $K = \varprojlim K_i$, where $K_i = A_1{}^I \times_{A_1} \cdots \times_{A_{i-1}} A_i{}^I$ (assuming for simplicity of notation that $A_i = e$ for $i \leq 0$), and that i is induced by a map of towers

$$A_i \xrightarrow{(sp_2 \cdots p_i,\ \ldots,\ sp_i,\ s)} K_i .$$

By axiom (I) it is sufficient to prove that each of these maps is a weak equivalence, and this follows from the fact that the above map is right inverse to a composite of trivial fibrations,

$$A_1{}^I \times_{A_1} \cdots \times_{A_{i-1}} A_i{}^I \longrightarrow A_2{}^I \times_{A_2} \cdots \times_{A_{i-1}} A_i{}^I \longrightarrow \cdots$$
$$\cdots \longrightarrow A_{i-1}{}^I \times_{A_{i-1}} A_i{}^I \longrightarrow A_i{}^I \longrightarrow A_i .$$

In order to state our theorem, we recall that a map $K \xrightarrow{i} A$ in an arbitrary category is called a <u>weak equalizer</u> of two maps f, g with source A if (a) $fi = gi$ and (b) for any map $K' \xrightarrow{i'} A$ such that $fi' = gi'$ there is a map j (not necessarily unique) such that $ij = i'$. The construction in 2.11 (together with 1.10) shows that any pair of maps in $Ho\underline{C}_f$ has a weak equalizer.

<u>Theorem</u> 3.5. <u>Let $\underline{C}_f$ be a pointed category of fibrant objects for a homotopy theory, and assume $\underline{C}_f$ satisfies axiom</u> (I). <u>Then for any tower of fibrations</u> (A_i) <u>and any group valued functor</u> T <u>on $Ho\underline{C}_f$ which preserves countable products and weak equalizers, there is a natural exact sequence</u>

$$0 \longrightarrow R^1\varprojlim(T(\Omega A_i)) \longrightarrow T(\varprojlim A_i) \longrightarrow \varprojlim T(A_i) \longrightarrow 1.$$

Proof. This is obtained by (1) applying T to the fibration sequence constructed in 2.11 from the two maps of $\prod A_i$ to itself; (2) using 3.4 to identify $T(K)$ with $T(\varprojlim A_i)$; and (3) using the standard computation of $R^1\varprojlim$ (see, for example, [EM]). Independence of the choice of path space can be proved by the techniques of section 2.

Remark 3.6. The first theorem of this type seems to be due to Milnor [Mi]. We recover his theorem from 3.5 by taking for $\underline{C}_f$ the dual of the category of pointed topological spaces.

4. Higher-order structure

We end this chapter with a very brief indication of how a theory of higher homotopies can be carried out in a category of fibrant objects $\underline{C}_f$. Before defining higher path spaces, we recall the following definition [GZ]. If X is a simplicial set, the set of n+1 simplices of the nth coskeleton of X, $(\mathrm{cosk}_n X)_{n+1}$, is by definition the subset of $X_n \times \ldots \times X_n$ (n+2 factors) consisting of those $(x_0, \ldots, x_{n+1})$ such that $d_i x_j = d_{j-1} x_i$ for $0 \leq i < j \leq n+1$. Observe that this definition makes sense for a simplicial object in any category, provided that certain fibred products exist. Observe also that X need not have degeneracies for this definition to apply.

We can now define a complete path space for an object B in $\underline{C}_f$ to be a sequence of objects $B = B^{\Delta^0}, \ldots, B^{\Delta^n}, \ldots$ together with weak equivalences $s^{(n)}: B \longrightarrow B^{\Delta^n}$ for all n and face maps $d_i: B^{\Delta^n} \longrightarrow B^{\Delta^{n-1}}$ for $0 \leq i \leq n$, such that

(a) $d_i d_j = d_{j-1} d_i$ for $i < j$;

(b) $s^{(0)} = \mathrm{id}_B$,

$d_i s^{(n)} = s^{(n-1)}$ for $n > 0$ and all i ; and

(c) the natural maps $B^{\Delta^{n+1}} \longrightarrow B^{\dot{\Delta}^{n+1}}$ are fibrations,

where we have used the suggestive notation $B^{\dot{\Delta}^{n+1}} = (\mathrm{cosk}_n B^{\Delta})_{n+1}$.

Remarks. 4.1. If we are given B^{Δ^i} for $i \leq n$ satisfying the required conditions, then it is not difficult to see that the fibred products needed to define $B^{\dot{\Delta}^{n+1}}$ exist and that we have a map $B \longrightarrow B^{\dot{\Delta}^{n+1}}$. To construct $B^{\Delta^{n+1}}$, we need only factor this map as in 1.2.

4.2. If $\underline{C}_f$ is contained in a category $\underline{C}$ with finite inductive limits and in which axiom (G) of section 1 holds, then we can construct a complete path space which also has degeneracies, by applying axiom (G) to the map $(\mathrm{sk}_n \overline{B^{\Delta}})_{n+1} \to (\mathrm{cosk}_n \overline{B^{\Delta}})_{n+1}$ [GZ], where $\overline{B^{\Delta}}$ denotes a partial path space defined for $i \leq n$.

4.3. It is not difficult to apply the techniques developed in this chapter to prove that, in some reasonable sense, B^{Δ} is unique up to homotopy. It is convenient for this purpose to introduce an appropriate category of fibrant simplicial objects in $\underline{C}_f$, using a definition of "fibrant" analogous to the condition in (c) above.

4.4. As an example of what can be done with B^{Δ}, we remark that, exactly as in [BK, § 3], we can use B^{Δ} to construct a tower of fibrations, and hence "higher-order operations", associated to any cosimplicial object in $\underline{C}_f$.

CHAPTER II

A Sheafification of Stable Homotopy Theory

In this chapter we define a non-additive analogue StaHo(X) of the derived category of (the category of abelian sheaves on) X [H]. We show how the theory of Chapter I enables us to derive some basic properties of StaHo(X) (e.g., the existence of products, additivity, fibration sequences), and we end the chapter by developing the machinery to be used in Chapter III to define cohomology groups.

1. Local stable homotopy theory

Throughout this chapter X will denote a fixed topological space and Sp will denote the category of spectra (see Appendix). We refer to [G; BST] for standard terminology and results about sheaves. We can speak, in particular, about sheaves with values in the category of spectra. By definition, such a sheaf consists of a contravariant functor E on the category of open sets of X with values in Sp, such that for any open set U and any open cover $\{U_i\}$ of U, the sequence

$$E(U) \longrightarrow \prod E(U_i) \rightrightarrows \prod E(U_i \cap U_j)$$

is exact. We will often write $\Gamma(U,E) = E(U)$ and $\Gamma(E) = \Gamma(X,E)$. We will denote by Sp(X) the category of sheaves of spectra on X.

Equivalently, we can regard E as a sequence of sheaves of pointed sets E_n, indexed by all the integers, with face maps $d_i\colon E_n \to E_{n-1}$ and degeneracy maps $s_i\colon E_n \to E_{n+1}$ for $0 \leq i < \infty$, such that the usual simplicial identities hold (see Appendix) and such that for each n,

$$E_n = \bigcup_{N=0}^{\infty} \bigcap_{i>N} \ker d_i,$$

this union and intersection being taken in the category of sheaves. This condition can be restated as: Every section of E_n over any open set U locally has only finitely many non-trivial faces. Another reformulation: For every element of $E_{n,x}$ (the stalk of E_n at x) one can find a neighborhood U of x and a section s of E_n over U such that s(x) is the given element and s has only finitely many non-trivial faces.

The correspondence between our two definitions is as follows. If E is a sheaf of spectra as in our first definition, we let E_n be the sheaf associated to the presheaf $U \mapsto E(U)_n$. Conversely, given E_n as in the above paragraph, we let E(U) be the spectrum whose n-simplices are those sections of E_n over U with only finitely many non-trivial faces.

If $f: X \to Y$, we can define direct and inverse image functors f_* and f^*. If E is a sheaf on X then $f_*E(V) = E(f^{-1}V)$; and if F is sheaf on Y then f^*F is defined by $f^*F_n = f^*(F_n)$, where the f^* on the right is the usual inverse image of sheaves of sets. As usual, f^* is left adjoint to f_*. Finally, if E is a spectrum and U is any open set of X, E_U will denote the sheaf of spectra which is constant with stalk E over U and whose stalk at any point x not in U consists of the trivial spectrum e.

We are now ready to discuss homotopy theory in Sp(X). We call a map a <u>weak equivalence</u> if it induces stalk-wise weak equivalences in Sp, in the usual sense (see Appendix). Equivalently, if we define the <u>homotopy sheaf</u> $\pi_q E$ to be the sheaf (of abelian groups) associated to the presheaf $U \mapsto \pi_q E(U)$, then a map is a weak equivalence if and only if it induces an isomorphism on all homotopy sheaves. The <u>stable homotopy category over</u> X, StaHo(X), is defined to be the homotopy category associated to Sp(X) by Definiton 1.9 of Chapter I.

In order to study StaHo(X) we will need a notion of fibration, which we call local fibration to distinguish it from the notion of global fibration to be discussed in the next section. Thus a map will be called a <u>local fibration</u> if stalkwise it is a fibration in the sense of Kan (see Appendix). E is called <u>locally fibrant</u> if $E \longrightarrow e$ is a local fibration, i.e., if it stalkwise satisfies Kan's extension condition. We denote by $Sp_{loc\ f}(X)$ the category of locally fibrant sheaves of spectra.

<u>Proposition</u> 1.1. <u>With the above definitions,</u> $Sp_{loc\ f}(X)$ <u>is a category of fibrant objects for a homotopy theory</u> (Chapter I). <u>Furthermore condition</u> (1) <u>of</u> (I,1.18) <u>is satisfied by</u> $Sp_{loc\ f}(X)$ <u>and</u> Sp(X). <u>Finally, if we define the cofibrations in</u> Sp(X) <u>to be the injective maps, then</u> Sp(X) <u>satisfies the duals of axioms</u> (A) <u>through</u> (E) <u>of Chapter</u> I.

<u>Proof</u>. To verify condition (1) of (I,1.18), we can use the sheafification of the free group functor F of [K2], i.e., we can let FE be the sheaf associated to the presheaf $U \longmapsto F(E(U))$. For the first assertion, the only axiom which is not trivial to verify is (D). This will follow from the proof of Proposition 2.3 in the next section, but we will also give here a more direct (although much more difficult) construction. If E is a sheaf of group spctra, we can let $E^I(U) = \underline{Hom}(I,E(U))$ (see Appendix), where it is understood that we add a disjoint basepoint to I so that <u>Hom</u> is defined. Note that we need (Appendix,

A.13 and A.15) in order to define the maps $E \longrightarrow E^I \longrightarrow E \times E$. For general E, we can take $E^I = (E \times E) \underset{F(E)\times F(E)}{\times} (FE)^I$. Finally, all of the dual axioms except the dual of (D) are satisfied because they are satisfied stalkwise (Appendix, A.3). Instead of verifying the dual of (D), we will verify the dual of (I,1.2). Thus given $E \xrightarrow{u} E'$, let i be the inclusion of E into its cone (defined by sheafifying [KW1, 6.2], and we may then factor u as $E \xrightarrow{(u,i)} E' \times CE \xrightarrow{pr_1} E'$.

Corollary 1.2. *The following are equivalent for a map* f *in* StaHo(X):

(1) f *is an isomorphism*;

(2) f *induces stalkwise isomorphisms in* StaHo;

(3) f *induces isomorphisms on all homotopy sheaves*.

Proof. This is immediate from the fact (I,1.10) that any map in StaHo(X) is of the form gt^{-1}, where g and t are maps in Sp(X).

Corollary 1.3. *The functor* $Sp(X) \longrightarrow StaHo(X)$ *preserves arbitrary sums and finite products*. StaHo(X) *is an additive category*.

Proof. The first assertion can be proved without much difficulty, using either (I,1.19) or the dual of (I,1.10) (or both), together with the following facts, which need only be checked when X is a point: (a) an arbitrary sum or a finite product of weak equivalences is again a weak equivalence; and (b) the category of trivial cofibrations under a fixed object has arbitray sums (this follows from the fact that, when X is a point, trivial cofibrations can be characterized by lifting properties). I will indicate three proofs of the additivity of StaHo(X). In the first proof one begins by showing that $E \vee E' \to E \times E'$ is a weak equivalence; this need only be checked in case X is a point, where it can be proved by a stable range argument (see [K2, proof of 5.3 for an example of a stable range argument). This shows that every object of StaHo(X) has a natural commutative monoid structure, and the existence of inverses can now be proved exactly as in the proof [D, Satz 8.2] that every connected H-space has a homotopy inverse. In the second and third proofs shows more directly that every object in StaHo(X) has a natural group structure (which must then be abelian by the usual argument). The second proof uses the free group functor and the third proof uses Ω (I,2.6), once one has convinced oneself that Ω is a self-equivalence of StaHo(X) (cf. Appendix, A.13, A.14; or perhaps there is an easier way to see this).

Remark 1.4. Infinite products are more difficult to handle because they are not preserved by the stalk functors, so, in particular, it appears doubtful that $Sp_{loc\ f}(X)$ is closed under infinite products. The existence of certain infinite products in StaHo(X) follows from the results of the next section.

We end this section by giving two more or less explicit descriptions of the maps in StaHo(X). If E is a sheaf of spectra and G is a sheaf of group spectra, then it follows from 1.3 that the map

$$\mathrm{Hom}_{Sp(X)}(E,G) \to \mathrm{Hom}_{StaHo(X)}(E,G)$$

is a group homomorphism. To describe its kernel, we need a definition. A map f from E to G is called <u>null-homotopic</u> if there is a degree one map $E_n \xrightarrow{h} G_{n+1}$ such that $d_0h = f$, $d_ih = hd_{i-1}$ for $i > 0$, and $s_ih = hs_{i-1}$ for $i > 0$.

<u>Proposition</u> 1.5. <u>Let</u> E, G <u>be sheaves of spectra, with</u> G <u>a group. Then any map from</u> E <u>to</u> G <u>in</u> StaHo(X) <u>can be written in the form</u> ft^{-1} <u>with</u> t <u>a trivial fibration. Furthermore,</u> $ft^{-1} = 0$ <u>in</u> StaHo(X) <u>if and only if there is a trivial fibration</u> t' <u>such that</u> ft' <u>is null-homotopic</u>.

<u>Proof</u>. If we let $\Lambda G \to G$ be the standard contractible fibre space over G [K3,§2], our definition of null-homotopy is such that f is null-homotopic if and only if it lifts to ΛG. Now if f = 0 in StaHo(X), then by (I,1.19) we know that f lifts to $G^I \times_G e$, at least after replacing f by ft for some trivial fibration t. But then (I,1.4) applied to the square

$$\begin{array}{ccc} e & \longrightarrow & \Lambda G \\ \downarrow & & \downarrow \\ G^I \times_G e & \longrightarrow & G \end{array}$$

shows that ft' lifts to ΛG for some t'.

<u>Remark</u> 1.6. Without using Chapter I, I cannot even prove that the group of maps which is asserted in 1.5 to be the kernel of $\mathrm{Hom}_{Sp(X)}(E,G) \to [E,G]_{StaHo(X)}$ is a normal subgroup.

Our second description of StaHo(X) relates StaHo(X) to an unstable analogue. Thus we let S.(X) denote the category of sheaves of pointed simplicial sets on X. We define weak equivalence and fibration stalkwise, and it is trivial to check that the fibrant sheaves form a category of fibrant objects. This is even easier than for sheaves of spectra, because we can use the ordinary simplicial path space. Furthermore, we can verify condition (1) of (I, 1.18) by using Kan's Ex^{∞} functor [K1].

There is a functor $\mathrm{Sp}: S.(X) \longrightarrow \mathrm{Sp}(X)$ which "freely adds" to a simplicial sheaf K the extra degeneracies. In the terminology of [K2], Sp(K) is the spectrum associated to the prespectrum K, SK, S^2K, ... This functor is left adjoint to the functor $E \longmapsto E_{(0)}$, which assigns to a spectrum the 0-th term of the prespectrum associated to it [K2]. By (I,1.20) we obtain a pair of adjoint functors

$$\mathrm{HoS.}(X) \underset{\mathrm{Sp}}{\overset{\underline{\underline{(0)}}}{\rightleftarrows}} \mathrm{StaHo}(X),$$

where $E_{\underline{\underline{(0)}}} = E_{(0)}$ if E is locally fibrant. We can of course define the full prespectrum $E_{(q)}$ with maps $SE_{(q)} \longrightarrow E_{(q+1)}$, S being the sheafification of the ordinary simplicial suspension, and we have the following result.

Proposition 1.7. *For any* E, E' *in* StaHo(X) *there is a natural short exact sequence*, $0 \longrightarrow R^1\varprojlim_i [E_{(i)}, E'_{\underline{\underline{(i-1)}}}]_{\mathrm{HoS.}(X)} \longrightarrow [E,E']_{\mathrm{StaHo}(X)} \longrightarrow \varprojlim_i [E_{(i)}, E'_{\underline{\underline{(i)}}}]_{\mathrm{HoS.}(X)} \longrightarrow 0.$

Proof. It is immediate from the definitions that $E = \varinjlim S^{-i}\mathrm{Sp}(E_{(i)})$ and that $E_{(i)} = (S^iE)_{(0)}$, where S is the functor which raises dimensions by one. The dual of (I, 3.5) then gives us an exact sequence,

$0 \to R^1 \varprojlim\, [S^{-i}Sp(E_{(i)}),\ \Omega E']_{StaHo(X)} \to [E,E']_{StaHo(X)}$

$\to \varprojlim\, [S^{-i}Sp(E_{(i)}),E']_{StaHo(X)} \to 0$, and the result now follows from the observation that $[S^{-i}Sp(E_{(i)}),E']_{StaHo(X)}$ $= [Sp(E_{(i)}),S^{i}E']_{StaHo(X)} = [E_{(i)},(S^{i}E')_{\underline{(0)}}]_{HoS.(X)} =$ $[E_{(i)},E'_{\underline{(i)}}]_{HoS.(X)}$, and the observation that $(\Omega E')_{\underline{(i)}}$ $= E'_{\underline{(i-1)}}$ (which is simply a matter of checking definitions, using the ~~two~~ standard simplicial definitions of Ω.)

Remark 1.8. The above proof remains valid if $E_{(0)}$, $E_{(1)}, \ldots$ is replaced by any prespectrum whose associated spectrum is E.

2. Global stable homotopy theory

A map $E \to B$ of sheaves of spectra will be called a *global fibration* if for each inclusion $U \subset V$ of open sets of X, the map

$$\Gamma(V,E) \to \Gamma(U,E) \underset{\Gamma(U,B)}{\times} \Gamma(V,B)$$

is a fibration of spectra. We call E *globally fibrant* if the map $E \to e$ is a global fibration, and we denote by $Sp_{glob\ f}(X)$ the category of globally fibrant sheaves of spectra. Our interest in globally fibrant sheaves comes from the fact that they allow us to pass from local homotopy theoretical information to global information. The main example of this is Theorem 2.4 below.

Remarks. 2.1. The globally fibrant sheaves are a homotopy theoretical analogue of the flasque sheaves of [G].

2.2. The global fibrations are the maps with the RLP (I, 1.15) with respect to all maps of the form

$$\Lambda_V \underset{\Lambda_U}{\cup} \Delta_U \hookrightarrow \Delta_V .$$

(See Appendix and the beginning of section 1 for notation.)

Proposition 2.3. *The category of globally fibrant sheaves of spectra, with the notions of weak equivalence and global fibration, is a category of fibrant objects in the sense of Chapter* I. *Furthermore, axiom* (G) *of Chapter* I *holds in* Sp(X) *relative to the notion of global fibration.*

Proof. The only axiom that needs to be verified is axiom (G) (which, of course, implies axiom (D)). In view of 2.2, this can be done by a transfinite analogue of the "small object argument" [QHA, II, p. 3.4]. The point here is that if $\aleph$ is a cardinal such that any open cover of any open set of X has a subcover of cardinality $\leq \aleph$, and if α is the first infinite ordinal bigger than $\aleph$, then the functors $\Gamma(U,-)$ preserve well-ordered direct limits indexed by α, so that the objects $\Delta_U, \ldots$ are small relative to α.

In order to state our main result we need one more definition. A sheaf of spectra E is said to be <u>trivial in dimensions greater than</u> N if for each open set U and any two distinct n-simplices u,v of $\Gamma(U,E)$ for $n > N$, $d_i u \neq d_i v$ for at least one i. We say that E is <u>bounded below</u> if it is trivial in dimensions greater than N for some N.

<u>Theorem</u> 2.4. <u>The global section functor</u> Γ : Sp(X) $\to$ Sp <u>preserves weak equivalences of sheaves which are globally fibrant and bounded below.</u>

<u>Proof</u>. Let $E \to E'$ be such a weak equivalence, and let F be the "homotopy fibre", i.e., $F = E \underset{E'}{\times} \Lambda E'$, where $\Lambda E'$ is the standard contractible fibre space over E' [K3]. It is easy to check that F is also globally fibrant and bounded below, and so we are reduced to the case E' = e. Thus we assume that E is globally fibrant and bounded below and that $\pi_*(E) = 0$, and we will show by descending induction on q that $\pi_q \Gamma(U,E) = 0$ for all open U. This

is certainly true for large q because E is bounded below and $\Gamma(U,E)$ satisfies the extension condition (see the description of the homotopy groups in the Appendix). Assuming it now for q+1, we will prove it for q. Let s be a spherical section of E_q over U. By Zorn's lemma we can find a section t of E_{q+1} defined over some open $U_0 \subset U$, where (t, U_0) is maximal for the property $d_0 t = s$, $d_i t = e$ for $i > 0$. If $U_0 = U$ then $[s] = o$ in $\pi_q \Gamma(U,E)$ and we are done. If not, let x be any point of U not in U_0. Since $\pi_q(E) = 0$, there is a section t' over some neighborhood U_1 of x such that $d_0 t' = s$ and $d_i t' = e$ for $i > 0$. Since $\pi_{q+1}\ (U_0 \cap U_1, E) = 0$ by the induction hypothesis, the sections $t|U_0 \cap U_1$ and $t'|U_0 \cap U_1$ are homotopic as simplices of $\Gamma(U_0 \cap U_1, E)$, i.e., there is a section u of E_{q+2} over $U_0 \cap U_1$ such that $d_0 u = t$, $d_1 u = t'$, and $d_i u = e$ for $i > 1$. (To see this, let Δ be the spectrum generated by a simplex σ of dimension q+2 with relations $d_i \sigma = e$ for $i > 1$. Since $\dot{\Delta}$ is equivalent in StaHo to S^{q+1} (Appendix), the obvious map $\dot{\Delta} \to \Gamma(U_0 \cap U_1, E)$ is null-homotopic and therefore extends to Δ, $\dot{\Delta} \to \Delta$ being a cofibration.) Now let Λ be the subspectrum of Δ generated by $d_1 \sigma$. We have a square

$$\begin{array}{ccc} \Lambda & \xrightarrow{t'} & \Gamma(U_1, E) \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{u} & \Gamma(U_0 \cap U_1, E). \end{array}$$

From the definitión of fibration (see Appendix) we see

that there

is an extension of the homotopy u to a homotopy $\mathfrak{U}$ over U_1, with $d_1\bar{u} = t'$. Then $d_0\bar{u}$ agrees with t on $U_0 \cap U_1$, $d_0 d_0 \bar{u} = s$, and $d_i d_0 \bar{u} = e$ for $i > 0$, so t can be extended to $U_0 \cup U_1$, contradicting its maximality. This completes the proof.

Remarks. 2.5. The above proof is a homotopy theoretic analogue of the proof of [G,II,3.1.3]. It is more complicated than that in [G] because we must take a "homotopy difference" of t and t' instead of an actual difference. This complication disappears if we work with sheaves of group spectra, in which case the proof in [G] applies without change. (One should observe here that a sheaf of group spectra is globally fibrant if and only if its normalization (defined as for simplicial groups) is dimensionwise flasque.)

2.6. In the case of sheaves of abelian group spectra the boundedness assumption can be removed if we assume that X has finite cohomological dimension. I do not know of any way of removing the hypothesis in the non-abelian case.

2.7. We can also define a homotopy theoretic analogue of the soft (mou) sheaves of [G], and the analogue of 2.4 remains true, provided X is paracompact.

We will end this chapter with some further results on bounded below globally fibrant sheaves, which will be useful in Chapter III. We first define the Postnikov decomposition of a spectrum. If E is a spectrum and N is any integer, there is a spectrum $E_{[-\infty,N]}$ which is trivial in dimensions greater than N and there is a map $E \longrightarrow E_{[-\infty,N]}$

which is universal for maps of E into a spectrum trivial in dimensions greater than N. (If E satisfies Kan's extension condition, $E_{[-\infty,N]}$ is usually called the Nth Postnikov approximation to E.) We obtain $E_{[-\infty,N]}$ from E by identifying two simplices of E with the same N-dimensional faces. If E now denotes a sheaf of spectra, then $E_{[-\infty,N]}$ will denote the sheaf associated to the presheaf $U \mapsto E(U)_{[-\infty,N]}$.

Proposition 2.8. *Let $GrSp^N(X)$ be the category of sheaves of group spectra which are trivial in dimensions greater than N. Then $GrSp^N(X)$, together with the notions of global fibration and weak equivalence (and cofibration as defined in (I,1.15)) is a closed model category in the sense of* [QHA].

Proof. The existence of limits in $GrSp^N(X)$ presents no difficulties. Projective limits are computed as in Sp(X) and inductive limits are computed by first computing them in the category of sheaves of group spectra and then applying $_{[-\infty,N]}$. From 2.3 and the fact that the inclusion $GrSp^N(X) \hookrightarrow Sp(X)$ has a left adjoint, we see that fibrations in $GrSp^N(X)$ can be characterized by a lifting property and that axiom (G) of Chapter I can be verified by the transfinite small object argument as in the proof of 2.3. (One needs to use here the fact that the Postnikov functor has the right homotopy theoretic interpretation on the category of locally fibrant sheaves, and, in particular, on sheaves of group spectra.) If we use the criterion of (I,1.16), the proof will be complete once we construct the second factorization. This can again be done by the transfinite small object argument, since it is clear from the following lemma that trivial fibrations in $GrSp^N(X)$ are characterized by a lifting property analogous to that of 2.3.

Lemma 2.9. *Let* $E \to B$ *be a trivial global fibration in* $Sp(X)$ *whose fibre* F *is bounded below. Then for each inclusion of open sets* $U \subset V$, *the map*

$$\Gamma(V,E) \to \Gamma(U,E) \underset{\Gamma(U,B)}{\times} \Gamma(V,B)$$

is a trivial fibration of spectra.

Proof. The fibre of the above map is the same as the fibre of $\Gamma(V,F) \to \Gamma(U,F)$, which is aspherical since both of these spectra are aspherical by 2.4.

Remark 2.10. The analogue of 2.8 for spectra without group structure is false, even if X is a point. If it were true, then we could deduce (using lifting criteria and adjoint functors) that $_{[-\infty,N]}$ preserved trivial cofibrations and hence weak equivalences. But this is false, as we see by applying it to a weak equivalence $S \to S'$, where S is the sphere spectrum and S' satisfies Kan's extension condition.

CHAPTER III

Generalized Sheaf Cohomology

Throughout this chapter X will continue to be an arbitrary topological space, Sp(X) the category of sheaves of spectra on X, and StaHo(X) the associated homotopy category. We will define and study cohomology groups $H^q(X,E)$ for E an object of StaHo(X), provided that either $\pi_n E = 0$ for sufficiently large n or X has finite cohomological dimension. Section 1 contains several equivalent definitions of the cohomology groups. Thus we can define $H^q(X,E)$ as $\pi_{-q}\, R\Gamma(E)$, where $R\Gamma$ can be described either as a derived functor [QHA,I,§4] of the global section functor or as the right adjoint of f^*: StaHo $\longrightarrow$ StaHo(X), where f maps X to a point. [We should mention here that the existence of this right adjoint can be proved using Brown's representability theorem [B], but I have not been able to generalize this method to work for more general maps f.] The cohomology groups can also be defined by $H^q(X,E) = [S_X^{-q},E]_{\mathrm{StaHo}(X)}$, and the equivalence of the two definitions leads to an analogue of Verdier's hypercovering theorem; and, finally, they can be defined using a canonical resolution.

Section 2 develops the spectral sequence referred to in the Introduction to this paper, and section 3 introduces multiplicative structure into the spectral sequence.

1. Definitions of the cohomology groups

We define a full subcategory of $\mathrm{Sp}(X)$ (resp., $\mathrm{StaHo}(X)$), to be denoted $\mathrm{Sp}^+(X)$ (resp., $\mathrm{StaHo}^+(X)$), by the condition $\pi_q E = 0$ for q sufficiently large. It is clear from the description of the maps in $\mathrm{StaHo}(X)$ given by (I,1.10) that $\mathrm{StaHo}^+(X)$ is obtained from $\mathrm{Sp}^+(X)$ by inverting the weak equivalences.

Theorem 1.1. *Let* $\Gamma\colon \mathrm{Sp}^+(X) \longrightarrow \mathrm{Sp}$ *be the global section functor. Then the right derived functor* [QHA,I,§4] $R\Gamma\colon\ \mathrm{StaHo}^+(X) \longrightarrow \mathrm{StaHo}$ *exists; if* E *is globally fibrant and bounded below then* $R\Gamma E$ *can be identified with* ΓE. *Finally,* $R\Gamma$ *commutes with* Ω *and preserves fibration sequences coming from local fibrations* (I, section 2; II, section 1).

Proof. The existence of $R\Gamma$ on $\mathrm{HoGrSp}^N(X)$ is clear from (II, 2.8, 2.4) and from (I, 1.20) (see also QHA, I, §4). Now if $\mathrm{Sp}^N(X)$ is the full sub category of $\mathrm{Sp}(X)$ defined by $\pi_q E = 0$ for $q > N$, then we have a functor $_{[-\infty,N]}\circ F\colon$ $\mathrm{Sp}^N(X) \longrightarrow \mathrm{GrSp}^N(X)$, where F is the free group functor and $_{[-\infty,N]}$ is the Postnikov functor, and there is a natural weak equivalence $E \longrightarrow F(E)_{[-\infty,N]}$, from which it follows easily that $R\Gamma\colon \mathrm{HoSp}^N(X) \longrightarrow \mathrm{StaHo}$ exists and can be computed by choosing a weak equivalence $E \longrightarrow E'$, where E' is in $\mathrm{GrSp}^N_{glof}(X)$, and applying Γ to E'. In view of (II, 2.4), the same description holds if E' is only required to be globally fibrant and bounded below, but not necessarily a group. It is now clear that we can let

N approach ∞ and deduce the existence of $R\Gamma$: $\mathrm{StaHo}^{+}(X) \longrightarrow \mathrm{StaHo}$. The last assertion of the theorem is clear from [QHA], together with the fact that we can convert a fibration to a global fibration without changing the weak homotopy type of the fibre.

Definition 1.2. If E is in $\mathrm{StaHo}^{+}(X)$, we define the generalized sheaf cohomology groups by $H^q(X,E) = \pi_{-q}R\Gamma E$. If X has finite cohomological dimension and E is arbitrary in StaHo(X), we define $H^q(X,E) = H^q(X, \underline{\underline{E_{[-\infty,N]}}})$ for large N (see 1.4 below), where the double underlining indicates as usual that E must first be replaced by a locally fibrant E' if necessary.

Remarks. 1.3. If we assume that there is an integer p such that X locally has cohomological dimension $\leq$ p, then it appears that we can prove the existence of $R\Gamma$: StaHo(X) $\longrightarrow$ StaHo by taking a resolution $E \longrightarrow E'$ constructed either by resolving the postnikov tower of E and taking the inverse limit or by applying the total spectrum construction (Appendix, definition preceding A.11.) to Godement's standard resolution of E (i.e., the resolution obtained from the triple $i_* i^*$, where i is the natural map from X_{dis} to X, X_{dis} being the underlying set of X with the discrete topology). The assumption on X seems to be necessary in order to prove that the map $E \longrightarrow E'$ is a weak equivalence. Unfortunately, I have not been able to find a reasonable description of a class of sheaves for which $R\Gamma E = \Gamma E$. In particular, I do not know if the globally fibrant sheaves have this property. Note that the standard resolution can be used to define the cohomology groups in a way that avoids the homotopy theory of Chapters I and II, but it is clearly more desirable to be able to choose resolutions more flexibly, as in the abelian case (cf. [G, II, § 4.7]).

1.4. If $K^\bullet$ is a (co-) chain complex of abelian sheaves, then $K^\bullet$ can be regarded as the normalization of a sheaf of abelian spectra E (cf. [DP, § 3] for the unstable analogue). It is clear from the proof of Theorem 1.1 that $H^q(X,E)$ as defined above agrees with the usual (hyper-) cohomology group $H^q(X,K^\bullet)$. In particular, if $K^\bullet$ consists of an abelian sheaf F concentrated in (co-) dimension $-n$, then E is precisely the stable Eilenberg-MacLane sheaf $K(F,n)$ (i.e., the sheaf $U \mapsto K(F(U),n)$), so we see that $H^q(X,K(F,n)) = H^{q+n}(X,F)$. This, together with Lemma 2.4 of the next section, shows that if X has finite cohomological dimension p and E is arbitrary, then $H^q(X,E) = H^q(X,E_{[-\infty,N]})$ for $N \geq p-q$.

1.5. Theorem 1.1 remains valid if Γ is replaced by Γ_Φ (sections with support in Φ) where Φ is a family of closed subsets of X closed under finite union. In fact, we need only check that (II,2.7) remains valid if Γ is replaced by Γ_Φ, and this follows easily from the proof of (II,2.7). (In the notation of that proof, if s has support in Z then we may assume that $U_0 \supset U - Z$ and that $t = e$ on $U - Z$.)

1.6. Theorem 1.1 also remains valid if we replace Γ by f_*, where $f: X \longrightarrow Y$. We write $R^q f_*(E) = \pi_{-q}\, Rf_*(E)$.

This is an abelian sheaf on Y and is the sheaf associated to the presheaf $V \mapsto H^q(f^{-1}V,E)$. It follows from (I,1.20) and from the proof of 1.1 that Rf_* is right adjoint to f^*: $\mathrm{StaHo}^+(Y) \to \mathrm{StaHo}^+(X)$.

Proposition 1.7. *If E is in* $\mathrm{StaHo}^+(X)$ *or if X satisfies the condition of Remark 1.3 and E is arbitrary, then* $H^q(X,E) \cong [S_X^{-q}, E]_{\mathrm{StaHo}(X)}$, *where* S_X^{-q} *is the constant sheaf whose stalk is the sphere spectrum of dimension* $-q$ (see Appendix).

Proof. We will assume first that E is in $\mathrm{StaHo}^+(X)$, and we may in fact assume that E is $\mathrm{GrSp}^N(X)$. Then $H^q(X,E)$ $= [S^{-q}, R\Gamma E]_{\mathrm{StaHo}} \cong [F(S^{-q})_{[-\infty,N]}, R\Gamma E]_{\mathrm{HoGrSp}^N} \cong$ $[F(S_X^{-q})_{[-\infty,N]}, E]_{\mathrm{HoGrSp}^N(X)} \cong [S_X^{-q}, E]_{\mathrm{StaHo}(X)}$, where the first isomorphism is a definition and all the others follow from (I,1.20). The other case follows from this, once we verify that $[S_X^{-q},E] = [S_X^{-q},E_{[-\infty,N]}]$ for large N under the appropriate finiteness hypotheses. For this we replace the Postnikov tower of E by a tower of global fibrations whose Nth term is in $\mathrm{GrSp}^N(X)$ and we then prove the desired result by doing homotopy theory (using II, 2.8, for example). Details will be omitted.

Corollary 1.8. *Under the same hypotheses,* $H^q(X,E) \cong [e,E_{(q)}]_{\mathrm{HoS}(X)}$, *where* S(X) *is the category of simplicial sheaves on* X, e *is the final object, and* $E_{(q)}$ *is as in the discussion preceding* (II,1.7).

Proof. Since the stable sphere is obtained from the unstable sphere by applying the functor Sp introduced in Chapter II to the unstable sphere, it is clear that

$[S_X^{-q},E]_{\mathrm{StaHo}(X)} \cong [S_X^0,\underline{\underline{E}}_{(q)}]_{\mathrm{HoS.}(X)} \cong [e,\underline{\underline{E}}_{(q)}]_{\mathrm{HoS}(X)}$, so the corollary is simply a restatement of the proposition.

Remark 1.9. This corollary implies the analogue for generalized cohomology of Verdier's hypercovering theorem (cf. I,1.13), i.e., $H^q(X,E) \cong \varinjlim_K \pi(K,\underline{\underline{E}}_{(q)})$, where K ranges over the hypercoverings of X and π denotes simplicial homotopy classes of maps, if the hypotheses of 1.8 hold.

Remark 1.10. If E is a spectrum we can define singular cohomology groups $H^q_{\mathrm{sing}}(X,E) = [S(X),\underline{\underline{E}}_{(q)}]_{\mathrm{Ho}}$, where S(X) is the Eilenberg complex and Ho is the homotopy category of simplicial sets. If X is paracompact and homologically locally connected then it can be shown that $H^q_{\mathrm{sing}}(X,E) \cong H^q(X,E_X)$, E_X being the constant sheaf with stalk E. We will only sketch the proof since it is in any case clear from the results of the next two sections that $H^\bullet(X,E_X)$ shares with $H^\bullet_{\mathrm{sing}}(X,E)$ all the usual cohomological properties. The proof is based on consideration of the sheaf associated to the presheaf $U \longmapsto \underline{\mathrm{Hom}}(S(U)^+,E)$ (see Appendix; the + indicates that a disjoint basepoint has been added so that Hom is defined). Using this sheaf one obtains a map $H^q_{\mathrm{sing}}(X,E) \longrightarrow H^q(X,E_X)$, which will be an isomorphism for all E if it is an isomorphism for $E = K(\pi,n)$ by 2.4 below, and this is a known result [BST; cf. also G, Ex. 3.9.1].

2. Long exact sequences and the fundamental spectral sequence

Proposition 2.1. (i) *If* $\Omega E'' \to E' \to E \to E''$ *is a fibration sequence in* StaHo(X), *then the sequence*

$$\cdots \to H^{q-1}(X,E'') \to H^q(X,E') \to H^q(X,E) \to H^q(X,E'') \to \cdots$$

is exact, provided that all of the cohomology groups are defined.

(ii) *If* U *is an open set of* X *and* E *is an object of* StaHo(X) *then there is a long exact sequence*

$$\cdots \to H^{q-1}(U,E) \to H^q(X,U;E) \to H^q(X,E) \to H^q(U,E) \to \cdots$$

provided all the groups are defined, where the relative group is by definition the cohomology of X *with supports in* X-U (see 1.5).

(iii) *If* U,V *are open sets of* X, $X = U \cup V$, *and* E *is an object of* StaHo(X), *then there is a long exact sequence*

$$\cdots \to H^{q-1}(U \cap V,E) \to H^q(X,E) \to H^q(U,E) \oplus H^q(V,E) \to H^q(U \cap V,E) \to \cdots$$

provided that all the groups are defined.

Proof. (i) is clear if all the sheaves are bounded below. In the general case, we imbed $E_{[-\infty,n]} \to E''_{[-\infty,n]}$ in a fibration sequence with fibre F_n. We may assume that $E'_{[-\infty,n]} \to E_{[-\infty,n]}$ factors through F_n, and it is not difficult to see (using 1.4) that if n is sufficiently large (for fixed q), $H^q(X,E'_{[-\infty,n]}) = H^q(X,F_n)$, so the exactness follows from the result for the bounded case. (ii) is the homotopy sequence of the fibration $\Gamma(X,E) \to \Gamma(U,E)$, where we have assumed (as we may) that E is globally fibrant and bounded below. (iii) is the homotopy sequence of a fibration $\Gamma(U,E)\times\Gamma(V,E)\to\Gamma(U\cap V,E)$, where we assume E to be a globally fibrant and bounded below sheaf of group spectra, and where the above fibration is the composite

$$\Gamma(U,E)\times\Gamma(V,E)\longrightarrow\Gamma(U\cap V,E)\times\Gamma(U\cap V,E)\longrightarrow$$
$$\Gamma(U\cap V,E)\times\Gamma(U\cap V,E)\xrightarrow{pr_2}\Gamma(U\cap V,E),$$

the second map being the automorphism (of the underlying set spectrum) given by $(x,y)\longmapsto(x,\ xy^{-1})$.

Remarks. 2.2. If X is paracompact then using (II,2.7) we can obtain long exact sequences analogous to (ii) and (iii) involving closed subsets of X.

2.3. Using the stable Bousfield-Kan spectral sequence (Appendix, A.12) we can generalize (iii) to a spectral sequence for an open cover (or hypercovering) of X.

We turn now to the construction of a spectral sequence which is a non-additive generalization of the hyperhomology spectral sequence and which is a sheaf theoretic generalization of the Atiyah-Hirzebruch spectral sequence.

Lemma 2.4. *Let* E *be a sheaf of spectra and suppose that for some* $n, \pi_q E = 0$ *for* $q \neq n$. *Then* E *is canonically isomorphic in* StaHo(X) *to the Eilenberg-MacLane sheaf* $K(\pi_n E, n)$.

Proof. Since the functor $K^{\bullet}\longmapsto E$ defined in Remark 1.4 is right adjoint to the normalized chain complex functor $E\longmapsto C_{\bullet}(E)$, a map from E to $K(\pi_n E,n)$ in StaHo(X) is the same as a map from $C_{\bullet}(E)$ to $\pi_n E$ in the homotopy category of chain complexes of abelian sheaves, where $\pi_n E$ is regarded as a complex concentrated in dimension n. (Note that we have used (I, 1.20) again.) Now the group of such maps can be computed by replacing $C_\bullet(E)$ by a complex which is zero in dimensions less than n and replacing $\pi_n E$ by a complex of injectives which is zero in dimensions bigger than n and then computing homotopy classes of chain maps. It is trivial to check that this group is $\mathrm{Hom}(H_n C_\bullet(E), \pi_n E)$, and to complete the proof we need only find an isomorphism $H_n C_\bullet(E) \cong \pi_n E$. For this, we consider the "Hurewicz map" $E \longrightarrow \mathbb{Z}E$, $\mathbb{Z}E$ being the (reduced) free abelian sheaf generated by E (so that $C_\bullet(E)$ is the normalization of $\mathbb{Z}E$), and observe that by the ordinary Hurewicz theorem applied stalkwise, this map induces an isomorphism on π_n. The result now follows from the fact that $H_n C_\bullet(E) = \pi_n \mathbb{Z}E$.

Theorem 2.5. *For any sheaf of spectra E there is a first and fourth quadrant spectral sequence of cohomological type with $E_2^{pq} = H^p(X, \pi_{-q}E)$. If X has finite cohomological dimension or if $\pi_q E = 0$ for q sufficiently large, then the spectral sequence converges to $H^{p+q}(X,E)$.*

Proof. We will assume E is locally fibrant. It follows from Lemma 2.4 that we have fibrations $K(\pi_n E, n) \hookrightarrow E_{[-\infty, n]} \longrightarrow E_{[-\infty, n-1]}$. These give rise to long exact sequences in cohomology which fit together to form an exact couple and hence a spectral sequence. The identification of the E_2 term follows from 1.4 and the convergence assertion is trivial.

Corollary 2.6 (*Leray spectral sequence*). *Let* $f: Y \to X$ *and let* E *be an object of* $\mathrm{StaHo}^{+}(Y)$. *Then there is a spectral sequence*

$$E_2^{pq} = H^p(X, R^q f_* E) \implies H^{p+q}(Y, E).$$

The same is true for arbitrary E *in* $\mathrm{StaHo}(Y)$ *provided that* X, Y, *and* f *have finite cohomological dimension* (i.e., *for large* q $R^q f_* F = 0$ *for all abelian sheaves on* Y), *where we define* $R^q f_* E = R^q f_* E_{[-\infty, n]}$ *for large* n (see 1.6).

Proof. If E is in $\mathrm{StaHo}^{+}(Y)$, this is the spectral sequence of Theorem 2.5 applied to $Rf_* E$. The generalization under finiteness hypotheses is obtained by an obvious passage to the limit.

Remark 2.7. The spectral sequence of Theorem 2.5 can also be obtained as the Bousfield-Kan spectral sequence (Appendix, A.12) of the cosimplicial spectrum obtained by applying Γ to the Godement resolution of E (see 1.3). This method, while more direct in the sense that no sheafified homotopy theory is used (if we use the total spectrum construction to define the cohomology groups), yields no information as to how to compute the differentials, whereas from the method we have used, the differentials can be "read off" from the k-invariants of E.

3. Multiplicative structure

In order to define a smash product in StaHo(X), we will introduce the category Bisp(X) of sheaves of bispectra (see Appendix). It is obvious that everything we have done for sheaves of spectra could have been done equally well for sheaves of bispectra, and using the adjoint functors between spectra and bispectra [KW1] we easily see that we can identify HoBisp(X) with StaHo(X). In particular, if we sheafify the external smash product defined in the Appendix, we obtain a functor $Sp(X) \times Sp(X) \longrightarrow Bisp(X)$ which induces $StaHo(X) \times StaHo(X) \longrightarrow StaHo(X)$. (The external smash product is given explicitly by letting $E \wedge E'$ be the sheaf of bispectra associated to the presheaf $U \longmapsto E(U) \wedge E'(U)$.) The standard pairing $\pi_p E(U) \otimes \pi_q E'(U) \longrightarrow \pi_{p+q} E(U) \wedge E'(U)$ (Appendix, A.9) yields, upon passage to associated sheaves, a pairing

$$(3.1) \qquad \pi_p E \otimes \pi_q E' \longrightarrow \pi_{p+q} E \wedge E'.$$

If E, E', E'' are in $StaHo^+(X)$ and we are given a map $E \wedge E' \longrightarrow E''$ in StaHo(X), then I claim there is a natural map

$$(3.2) \quad (R\Gamma E) \wedge (R\Gamma E') \longrightarrow R\Gamma E''.$$

In fact, we may assume that E and E' are globally fibrant and bounded below, in which case the map is given by smash product of sections in the obvious sense. Using the pairing of (Appendix, A.9) again, we obtain from 3.2 <u>cup products</u>

$$(3.3) \quad H^p(X,E) \otimes H^q(X,E') \xrightarrow{\smile} H^{p+q}(X,E'').$$

(This definition extends to arbitrary E, E', and E'' if X has finite cohomological dimension, using Lemma 3.5 (ii) below.)

Example 3.4. Let F and G be abelian sheaves and let E, E', and E'' be $K(F,n)$, $K(G,m)$, $K(F\otimes G,n+m)$. It follows from the proof of Lemma 2.4, together with (Appendix, A.7, A.9), that there is a unique pairing $E\wedge E' \longrightarrow E''$ such that (3.1) induces the identity of $F\otimes G$; the resulting cup product $H^p(X,E)\otimes H^q(X,E') \longrightarrow H^{p+q}(X,E'')$ agrees with the usual cup product $H^{p+n}(X,F)\otimes H^{q+m}(X,G) \longrightarrow H^{p+q+n+m}(X,F\otimes G)$ under the identifications of 1.4.

We now study products in the spectral sequence. We first need to observe that the exact couple defined in the proof of Theorem 2.5 is actually part of a spectral system (i.e., an $H(p,q)$ system as in [CE, Chapter XV]). Thus if we define, for E a locally fibrant sheaf of spectra or bispectra $E_{[p,\infty]}$ = the fibre of $E \longrightarrow E_{[-\infty,p-1]}$, and if we define, for $-\infty \leq p \leq q \leq \infty$, $E_{[p,q]} = (E_{[p,\infty]})_{[-\infty,q]}$, then the spectral system is given by $H(p,q) = H^\bullet(X,E_{[p,q-1]})$, where, of course, $E_{[p,p-1]} = e$. The maps $H(p,q) \longrightarrow H(p',q')$ for $(p',q') \leq (p,q)$, which we will denote by λ, come from the natural maps $E_{[p,q-1]} \longrightarrow E_{[p',q'-1]}$; the maps δ: $H(p,q) \longrightarrow H(q,r)$ are the connecting homomorphisms in the long exact cohomology sequences associated to the fibrations $E_{[q,r-1]} \hookrightarrow E_{[p,r-1]} \longrightarrow E_{[p,q-1]}$. Note that the exact couple of the proof of Theorem 2.5 consists of the exact sequences $\cdots \longrightarrow H(q-1,q) \longrightarrow H(-\infty,q) \longrightarrow H(-\infty,q-1)$

$\longrightarrow H(q-1,q) \longrightarrow \cdots$, but the groups $H(-\infty,q)$ will play no role in our construction of pairings.

Lemma 3.5. *Let E, E' be sheaves of spectra, let p and q be (finite) integers, and assume $0 \leq r \leq \infty$. Then*

(i) $E_{[p,\infty]} \wedge E'_{[q,\infty]} \subset (E \wedge E')_{[p+q,\infty]}$.

(ii) The inclusions of (i) induce, by passage to the quotients, maps $E_{[p,p+r]} \wedge E'_{[q,q+r]} \longrightarrow (E \wedge E')_{[p+q,p+q+r]}$.

Proof. This is simply a matter of checking the definitions.

If we are given a pairing $E \wedge E' \longrightarrow E''$, the pairing (3.1) induces (via cup product) a pairing

$$E_2^{pq} \otimes E'^{p'q'}_2 \longrightarrow E''^{p+p'\ q+q'}_2 \qquad (3.6)$$

of the spectral sequences of E, E', and E''.

Theorem 3.7. *Let E, E', E'' be sheaves of spectra, and let E_r, E'_r, E''_r be the spectral sequences of Theorem 2.5. Then given any map $E \wedge E' \longrightarrow E''$ in StaHo(X), the pairing (3.6) extends to a pairing of spectral sequences $E_r \otimes E'_r \longrightarrow E''_r$ which on E_∞ is compatible with the cup product $H^\bullet(X,E) \otimes H^\bullet(X,E') \longrightarrow H^\bullet(X,E'')$ (provided that either X has finite cohomological dimension or all the sheaves are in StaHo$^+$(X)).*

Proof. Let H, H', H'' be the three spectral systems. Lemma 3.5 gives us cup products $H(p,p+r) \otimes H'(q,q+r) \longrightarrow H''(p+q,p+q+r)$, which will induce a pairing of spectral sequences provided we verify commutativity of the following (cf. [Do,IIA]):

$$\begin{tikzcd} H(p,p+r)\otimes H'(q,q+r) \arrow[r] \arrow[d, "\delta\otimes\lambda\ +\ \lambda\otimes\delta"'] & H''(p+q,p+q+r) \arrow[d, "\delta"] \\ \begin{array}{c} H(p+r,p+r+1)\otimes H'(q,q+1) \\ + \\ H(p,p+1)\otimes H'(q+r,q+r+1) \end{array} \arrow[r] & H''(p+q+r,p+q+r+1) \end{tikzcd}.$$

(Note that according to the sign convention $\lambda\otimes\delta\ (u\otimes u') = (-1)^p\lambda(u)\otimes\delta(u')$ if u is a cohomology class of dimension p.) This follows from Lemma 3.8 below applied to the fibrations

$$\begin{array}{c} E_{[p+r,p+r]} \hookrightarrow E_{[p,p+r]} \longrightarrow E_{[p,p+r-1]} \\ E'_{[q+r,q+r]} \hookrightarrow E'_{[q,q+r]} \longrightarrow E'_{[q,q+r-1]} \\ E''_{[p+q+r,p+q+r]} \hookrightarrow E''_{[p+q,p+q+r]} \longrightarrow E''_{[p+q,p+q+r-1]}. \end{array}$$

Note that we have used the dual of (I,1.10) applied to sheaves of bispectra in order to assume that we have a map $E \wedge E' \longrightarrow E''$ in $\mathrm{Bisp}(X)$.

Lemma 3.8. *Let* $F \xhookrightarrow{i} E \xrightarrow{p} B$ *and* $F' \xhookrightarrow{i'} E' \xrightarrow{p'} B'$ *be local fibrations of locally fibrant, bounded below sheaves of spectra, let* $F'' \xhookrightarrow{i''} E'' \xrightarrow{p''} B''$ *be a local fibration of locally fibrant, bounded below sheaves of bispectra, and assume given maps* $E \wedge E' \xrightarrow{\alpha} E''$, $B \wedge B' \xrightarrow{\beta} B''$, $F \wedge B' \xrightarrow{\gamma} F''$, *and* $B \wedge F' \xrightarrow{\mu} F''$ *such that* $p''\alpha = \beta\circ p\wedge p'$, $i''\gamma\circ(F\wedge p') = \alpha\circ(i\wedge E')$, *and* $i''\mu\circ(p\wedge F') = \alpha\circ(E\wedge i')$. *Letting* δ *denote the connecting homomorphism in all three cohomology exact sequences, we have (using the cup products obtained from the four given maps)*

$$\delta(u\smile u') = \delta(u)\smile u' + (-1)^p u\smile\delta(u')$$

in $H^\bullet(X,F'')$, *where* $u\in H^p(X,B)$ *and* $u'\in H^\bullet(X,B')$.

Proof. We will assume first that all sheaves are globally fibrant and all fibrations are global fibrations. An element of $H^p(X,B)$ is represented by a section s of the spherical $-p$ simplices of B, and an element of $H^q(X,B')$ is represented by a section s' of the spherical $-q$ simplices of B'. Let t be a lifting of s to E such that $d_i t = e$ for $i > 0$ and let t' be a similar lifting of s'. Then $\delta[s] = [d_0 t]$, where $d_0 t$ is regarded as a section of F, and $\delta[s'] = [d_0 t']$ (see Appendix, A.5). On the other hand, by definition, $[s] \smile [s'] = \varepsilon_{-p}[s'']$, where s" is the image in B" of $s \wedge s'$ and ε_{-p} is as in (Appendix, A.7). The image t" in E" of $t \wedge t'$ is a lifting of s" to E", and its non-trivial faces are $d_{-(p+1)}t'' = \gamma(d_0 t \wedge s')$ and $d_{-p}t'' = \mu(s \wedge d_0 t')$. Therefore (Appendix, A.5), $\delta([s] \smile [s'])$
$= \varepsilon_{-p}\delta[s''] = \varepsilon_{-p}((-1)^{p+1}[d_{-(p+1)}t''] + (-1)^p[d_{-p}t''])$
$= \varepsilon_{-p}(-1)^{p+1}\varepsilon_{-p-1}\delta[s] \smile [s'] + \varepsilon_{-p}(-1)^p\varepsilon_{-p}[s] \smile \delta[s']$
$= \delta[s] \smile [s'] + (-1)^p[s] \smile \delta[s']$.

We will now reduce the general case to the special case just considered by showing that we can map the given sheaves to sheaves as in the above paragraph in a way compatible with all the given data and such that the maps are weak equivalences. By applying the free group functor and then an appropriate Postnikov functor to p we can imbed it by a weak equivalence into a map in $GrSp^N(X)$ for some N, which we can then imbed in a global fibration $\overline{p}$ of globally fibrant sheaves by (II, 2.8) and [QHA]. Similarly, we imbed p' by a weak equivalence into $\overline{p'}$.

Next we can replace B'' by $B'' \underset{B\wedge B'}{\cup} \overline{B}\wedge\overline{B'}$ and we can therefore assume that β factors as $B\wedge B' \longrightarrow \overline{B}\wedge\overline{B'} \xrightarrow{\overline{\beta}} B''$. The rest of the given maps can be summarized by a diagram

$$\begin{tikzcd} & Z \arrow[r] \arrow[dl] & E'' \arrow[d, "p''"] \\ B \wedge B' \arrow[rr, "\beta"] & & B'', \end{tikzcd}$$

where $Z = (B\wedge F') \underset{E\wedge F'}{\cup} (E\wedge E') \underset{F\wedge E'}{\cup} (F\wedge B')$ and the unlabelled horizontal and vertical maps are, respectively, $(i''\mu, \alpha, i''\gamma)$ and $(e, p\wedge p', e)$. If we verify that $Z \longrightarrow \overline{Z}$ is a stalkwise trivial cofibration, then we can replace E'' by $\overline{E''} = \overline{Z} \underset{Z}{\cup} E''$ and then convert the resulting map $\overline{E''} \longrightarrow B''$ into a fibration, and this will complete the reduction. Now one sees by inspection that the map $Z \longrightarrow \overline{Z}$ is injective, and to prove it is a weak equivalence we observe that we have a cocartesian square

$$\begin{tikzcd} (E\wedge F') \underset{F\wedge F'}{\cup} (F\wedge E') \arrow[r] \arrow[d] & (B \quad F') \wedge (F \quad B') \arrow[d] \\ E\wedge E' \arrow[r] & Z \end{tikzcd}$$

and similarly for $\overline{Z}$, where the vertical arrows are injective. But it follows from [KW1,5.5] that these squares give rise to Mayer-Vietoris sequences in homotopy, which reduces us to proving a weak equivalence of the spectra in the upper left-hand corner, and this again follows from the Mayer-Vietoris sequence.

APPENDIX

Kan's Category of Spectra

In this appendix we will show how one can do stable homotopy theory in Kan's category of spectra using the abstract homotopy theory of [QHA]. This is vital for the present paper, because the treatment found in the literature [K2; KW2,appendix; BD] does not generalize to sheaves easily (if it generalizes at all), and furthermore, the present treatment is much simpler even when we are not concerned with sheaves. We will use freely standard results and definitions from simplicial homotopy theory [GZ;L;M; see also QHA and (I,1.17) of this paper], but we will refer to the literature on spectra only for relatively easy results. We begin by recalling some definitions from [K2], in a slightly different (but equivalent) form.

By a <u>spectrum</u> we mean a sequence of sets E_n with basepoint e, indexed by all the integers n, together with face operators $d_i: E_n \to E_{n-1}$ and degeneracy operators $s_i: E_n \to E_{n+1}$, for $0 \leqslant i < \infty$, such that (a) the usual simplicial identities hold ($d_i d_j = d_{j-1} d_i$ for $i < j$, etc.) and (b) each simplex of E has only finitely many faces different from e. The same definition defines <u>bispectrum</u>, except that the face and degeneracy operators are now defined for all integers i. Until we deal with smash products below, we will not mention bispectra, but it is to be understood that all definitions and results given for spectra apply equally well to bispectra.

Examples. A.1. For each pair of integers n,k with $k \geq 1$, there is a spectrum generated by an n simplex x, subject to the relations $d_i x = e$ for $i \geq k$. We will generically denote such a spectrum by Δ, we will let $\dot{\Delta}$ be the subspectrum generated by the faces $d_i x$, and we will denote by Λ any subspectrum of $\dot{\Delta}$ generated by all faces $d_i x$ except one.

A.2. The same definition as in A.1 but with $k = 0$ defines the n-sphere S^n.

A map of spectra will be called a fibration if it has the RLP (I,1.15) with respect to all inclusions $\Lambda \hookrightarrow \Delta$. We will say that E satisfies Kan's extension condition if $E \rightarrow e$ is a fibration.

Homotopy groups of spectra are defined in [K2], and a map is called a weak equivalence if it induces an isomorphism on all homotopy groups. We remark that by applying (I,1.20) to the adjoint functors between simplicial sets and spectra (see discussion preceding II,1.7) we easily see that $\pi_q E = [S^q, E]_{StaHo}$, where StaHo is the homotopy category (I,1.9) of spectra. If E satisfies the extension condition then it is easy to use A.3 below to make this explicit. Thus an element of $\pi_q E$ is an equivalence class $[x]$, where x is a spherical q simplex of E (i.e., $d_i x = e$ for all i), and where $[x] = [x']$ if and only if there is a $q+1$ simplex h such that $d_0 h = x$, $d_1 h = x'$, and $d_i h = e$ for $i > 1$. (This description can also be deduced from the corresponding description in the case of simplicial sets.)

Proposition A.3. *With the above definitions of fibration and weak equivalence, and with the cofibrations defined to be the injective maps, the category of spectra forms a closed model category in the sense of* [QHA].

Proof. We will use the criterion of (I,1.16). By definition, the fibrations are characterized by the RLP with respect to all $\Lambda \hookrightarrow \Delta$, and it is easy to see (using the corresponding result for simplicial sets) that the trivial fibrations are characterized by the RLP with respect to all $\dot{\Delta} \hookrightarrow \Delta$. The factorizations required by (I,1.16) can thus be constructed by the small object argument as in [QHA,II, p.3.4]. We need only verify that a map obtained by cobase extension from a sum of maps of the form $\Lambda \hookrightarrow \Delta$ is a weak equivalence. For this we observe that the spectra Δ can be obtained, up to dimension shift, by applying the functor Sp (II,discussion preceding 1.7) to the pointed simplicial set obtained from an ordinary simplex by collapsing its last face to a point. We obtain Λ by a similar process, and the desired result can now be deduced from the fact that the category of pointed simplicial sets satisfies the axioms[QHA,II,§3; see also (I,1.17)of this paper] together with the fact [K2] that every spectrum comes from a prespectrum and every map from a map of prespectra. Finally, to see that every inclusion is a cofibration in the sense of (I,1.15), we factor an inclusion $E' \hookrightarrow E$ as $E' \hookrightarrow E' \cup Sp(E_{(0)}) \hookrightarrow E' \cup S^{-1}Sp(E_{(1)}) \hookrightarrow \cdots \hookrightarrow E$, where the notation is as in (II, discussion preceding 1.7), and we observe that each of these inclusions (up to dimension shift) is a cobase extension of a map obtained by applying Sp to a cofibration of simplicial sets.

Remark A.4. It is convenient to observe at this time that $\dot{\Delta}$ is canonically isomorphic in StaHo to a sphere. We can deduce this from the corresponding fact about simplicial sets, as in the above proof, or we can simply compute that $\dot{\Delta}$ has the same homology as a sphere. Note that we have a canonical cycle $\sum(-1)^i d_i x$ generating the non-zero homology group.

We recall now from the homotopy theory of simplicial sets (or from [QHA, I, §3]) the explicit description of the boundary homomorphism $\pi_n B \xrightarrow{\partial} \pi_{n-1} F$, where $F \hookrightarrow E \to B$ is a fibration. We observe that $\Delta/\dot{\Delta}$ is a sphere and we represent an element $[s]$ of $\pi_n B$ by a map $\Delta/\dot{\Delta} \xrightarrow{s} B$, which we lift to a map $\Delta \xrightarrow{t} E$. Then $t \,|\, \dot{\Delta} : \dot{\Delta} \to F$ represents $\partial[s]$. This description, together with a suitable version of the homotopy addition theorem (which is proved by applying the Hurewicz theorem to the wedge of spheres obtained from $\dot{\Delta}$ by collapsing all $d_i d_j x$ to e), yields the following.

A.5. Let $F \hookrightarrow E \to B$ be a fibration, let s be a spherical simplex in B, and let t be a lifting of s to E such that all faces $d_i t$ are spherical. Then

$$[s] = \sum(-1)^i [d_i t]$$

in $\pi_* F$.

We consider now the smash product of spectra. Our definition is an adaptation of the definition in [KW1]. If E and E' are spectra, we define their (external) <u>smash product</u> to be the bispectrum $E \wedge E'$ with a $p+q$ simplex $x \wedge x'$ for each p simplex x of E and q simplex x' of E', subject to the identification $e \wedge x' = e = x \wedge e$, with faces defined by

$$d_i(x \wedge x') = \begin{cases} d_{p-i-1} x \wedge x' & \text{for } -\infty < i < p \\ x \wedge d_{i-p} x' & \text{for } p \leqslant i < \infty, \end{cases}$$

and degeneracies defined similarly.

<u>Remarks</u>. A.6. The smash product functor preserves weak equivalences in both variables and induces a functor StaHo $\times$ StaHo $\longrightarrow$ HoBisp which can be converted to an internal smash product in StaHo using the equivalence of StaHo and HoBisp established in [KW1].

A.7. Letting $C(E)$ be the unnormalized chain complex of E, i.e., the (reduced) free abelian group generated by E, with differential $d = \sum (-1)^i d_i$, there is an isomorphism

$$\text{(A.8)} \qquad C(E) \otimes C(E') \longrightarrow C(E \wedge E')$$

defined by $x \otimes x' \longmapsto \varepsilon_p \, x \wedge x'$ (p= degree of x), where ε_p is defined for all integers p by $\varepsilon_0 = 1$, $\varepsilon_p = (-1)^{p-1} \varepsilon_{p-1}$.

A.9. There is a pairing $\pi_p E \otimes \pi_q E' \longrightarrow \pi_{p+q} \; E \wedge E'$ defined by $[s] \otimes [s'] \longmapsto \varepsilon_p \, [s \wedge s']$, s and s' being spherical and ε_p being as in A.7. The sign is used so that the pairing will be compatible, under the Hurewicz map, with the pairing of homology classes induced by (A.8).

We end this appendix with some technical results which are not necessary for this paper, but which are necessary for alternative treatments which have been indicated for various parts of this paper. If K is a pointed simplicial set and E is a spectrum, we can define a spectrum $K \wedge E$ by formulas similar to those defining the smash product of two spectra. The functor $K \wedge -$ has a right adjoint $\underline{\mathrm{Hom}}(K,-)$, which is what we are interested in here, and which can be defined explicitly as follows. For each integer $p \geq 0$, $\mathrm{Hom}_{(\text{pointed sets})}(K_p, E)$ is a graded set with operators d_i and s_i (coming from those in E) satisfying the simplicial identities. If we let X^p be the subobject consisting of those elements with only finitely many non-trivial faces, then X^p is a spectrum, and the faces and degeneracies in K induce coface and codegeneracy operators δ_i and σ_i in $X = \{X^p\}$, making X a cosimplicial spectrum, i.e., a cosimplicial object in the category of spectra. We now define the __function spectrum__ $\underline{\mathrm{Hom}}(K,E)$ as the __total spectrum__ $T(X)$ where, by definition, an n simplex of $T(X)$ is a sequence $(x^p_{p+n} \in X^p_{p+n})$ such that (a) $d_i x^p_{p+n} = \delta_i x^{p-1}_{p+n-1}$ and $s_i x^p_{p+n} = \sigma_i x^{p+1}_{p+n+1}$, for $0 \leq i \leq p$, where for $p = 0$ we set $x^{-1}_{n-1} = e$; and (b) (x^p_{p+n}) has only finitely many non-trivial faces, these being defined by

$$(A.10) \qquad d_i \, (x^p_{p+n}) = (d_{i+p+1} x^p_{p+n}).$$

Replacing "d" by "s" in (A.10), we obtain a definition of degeneracies in $T(X)$, which then becomes a spectrum.

(Note that an n simplex of Hom(K,E) is just a map of degree n from K to E satisfying a certain natural condition.)

Remarks. A.11. There is an unstable analogue of the above construction of the total spectrum, which was also discovered (independently) by Bousfield and Kan [BK,§3]. It assigns a total simplicial set to a cosimplicial object in the category of simplicial sets. The situation is much better in the stable case, since in the unstable case it is virtually impossible to explicitly describe the n simplices of T(X) for $n > 0$.

A.12. There is an obvious way to write T(X) as the inverse limit of a tower, the maps of which are fibrations if X is "fibrant" in a suitable sense (involving a condition on the codegeneracies analogous to the condition on the faces of the simplicial objects considered in Chapter I, section 4 of this paper). The homotopy exact couple of this tower yields a spectral sequence whose E_1 term is the normalization of the cosimplicial graded abelian group $p \longmapsto \pi_* X^p$. (This spectral sequence, or rather the unstable version of it, is due to Bousfield and Kan, who study it in a series of papers to appear.) Thus the total spectrum should be thought of as a non-additive generalization of the total complex associated to a double complex of abelian groups.

A. 13. There is , unfortunately, one technical complication that makes the function spectrum less pleasant to work with than it would otherwise be. This arises from the fact that $\underline{\mathrm{Hom}}(S^0,E)$ is not equal to E. It does, however, have (in a natural way) the same weak homotopy type as E, provided that E satisfies Kan's extension condition. This follows from Lemma A.15 below, which also has as an immediate corollary the fact that suspension and loop in StaHo can be identified with dimension shifting functors. (One needs to observe here that $\underline{\mathrm{Hom}}(S^0,E)$ differs by a dimension shift from ΩE as defined in [K3], and coincides with $\widetilde{E}$ of A.15.)

A.14. For the sake of completeness, here is an outline of a better proof that loop is a dimension shift. It involves techniques which are also useful for other purposes. Consider the prespectrum associated to ΩE, Ω being defined as in [K3]. It is sufficient to prove that this differs from the prespectrum of E by a shift in indexing. Now an inspection of the definitions shows that this would be true if the two simplicial definitions of Ω (i.e., the adjoints of left and right join with S^0) were the same. They are not the same, but we can make the above argument work if we use a model for homotopy theory which has a better join functor. An example of such a model is the category of "simplicial sets with permutations", i.e., the category of contravariant functors on the category of unordered simplices (or non-empty finite sets) with values in the category of sets. This category has a commutative, associative join modelled on the join of unordered (geometric) simplicial complexes, and defined explicitly using shuffles. We define fibration and weak equivalence by means of the forgetful functor to simplicial sets, and it is easy to show, using (I,1.16), that we can do homotopy theory in this category and that the homotopy theory coincides with the ordinary homotopy theory of [QHA, II, §3].

Lemma A.15. *Let* E *be a group spectrum. There is a natural map of pointed sets* $E \xrightarrow{f} E$ *such that* $fd_i = d_{i+1}f$ *for* $i \geq 0$, $d_0 f = e$, *and* $fs_i = s_{i+1}f$ *for* $i \geq 0$. *Furthermore* f *induces an isomorphism* $E \to \widetilde{E}$, *where* $\widetilde{E}$ *is the spectrum with* $\widetilde{E}_n = \ker d_0 \subset E_n$ *and* $\widetilde{d}_i = d_{i+1}|E$, $\widetilde{s}_i = s_{i+1}|E$. *If* E *is abelian then* f *is a group homomorphism.*

Proof. To motivate the proof we will first assume E is abelian. Then there is an isomorphism of the normalized chain complexes of E and $\widetilde{E}$ which takes x to $(-1)^p(x-s_0d_0x)$ if p = degree x. This extends uniquely [DP, § 3] to a map f having the required properties. We can describe f recursively as follows. Assume we know f on all simplices y such that $d_iy = 0$ for $i > n-1$, and let x satisfy $d_ix = 0$ for $i > n$. Then d_nx and $x-s_{n-1}d_nx$ are killed by d_i for $i > n-1$, so $f(d_nx)$ and $f(x - s_{n-1}d_nx)$ are known and we have $f(x) = f(x - s_{n-1}d_nx) + \widetilde{s}_{n-1}f(d_nx) = f(x - s_{n-1}d_nx) + s_nf(d_nx)$. The attempt to make this work in the non-abelian case leads to the following. Let $g_0(x) = x$ and let g_i be defined recursively by

$$g_{i+1}(x) = \begin{cases} g_i(x)\cdot s_ig_i(d_ix)^{-1} & \text{if } i \text{ is even} \\ s_ig_i(d_ix)^{-1}\cdot g_i(x) & \text{if } i \text{ is odd.} \end{cases}$$

We then define $f(x) = \begin{cases} g_i(x) \text{ for large } i & \text{if degree } x \text{ is eve[n]} \\ g_i(x)^{-1} \text{ for large } i & \text{if degree } x \text{ is o[dd]} \end{cases}$

One can check that f has the right properties with respect to face and degeneracy operators, and it is quite trivial to look at the homotopy groups and see that f is a weak equivalence. To see that it is actually an isomorphism, one can solve explicitly for the inverse. Unfortunately there does not seem to be a simple recursive definition of the inverse analogous to the definition of f. Details will be omitted.

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