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ALGEBRAIC K-THEORY AS GENERALIZED SHEAF COHOMOLOGY

Kenneth S. Brown[1] and Stephen M. Gersten

Let X be a topological space. In [2] it was shown how one could define "generalized sheaf cohomology" of X; this generalizes ordinary sheaf cohomology, as well as generalized cohomology in the sense of [11]. In case $X = \operatorname{Spec} A$, where A is a regular commutative ring, it was announced in [5] that the (Karoubi-Villamayor) K-groups of A could be obtained as generalized sheaf cohomology groups of X. As a consequence, one obtains a "local to global" (or "Atiyah-Hirzebruch") spectral sequence

$$E_2^{pq} = H^p(X, \mathcal{K}_{-q}) \Rightarrow K_{-(p+q)}(A),$$

where $\mathcal{K}_{-q}$ is the (abelian) sheaf of local K-groups of A. [Note: In the Karoubi-Villamayor notation, one would write $\mathcal{K}^q$ and K^{p+q} instead of $\mathcal{K}_{-q}$ and $K_{-(p+q)}$.]

This application of the results of [2] to K-theory requires an improvement of those results. In particular, one needs to eliminate a boundedness assumption on the coefficient sheaves for generalized sheaf cohomology; this can be done provided X is a Noetherian space of finite Krull dimension. It is the purpose of the present paper to present this improvement and to give the application to K-theory. Since the improvement vastly simplifies the generalized sheaf cohomology theory, we give in Sections 1 and 2 an account of that theory independent of [2].

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Specifically, Section 1 develops the homotopy theory of simplicial sheaves on a Noetherian space X in which the irreducible closed subsets satisfy the ascending chain condition. Section 2 uses this homotopy theory to define generalized sheaf cohomology and to derive the local to global spectral sequence, in case X has finite Krull dimension. Finally, in Section 3 we show that the groups $G_i(X)$ defined in [8, §5] for a Noetherian scheme X are generalized sheaf cohomology groups, and we obtain as a corollary a spectral sequence in G-theory which generalizes the K-theory spectral sequence referred to above.

1. The homotopy theory of simplicial sheaves.

Let X be a topological space. By a simplicial sheaf on X we mean a sheaf on X with values in the category of simplicial sets, or, equivalently, a simplicial object in the category of sheaves of sets on X. If K is a simplicial sheaf we will denote by $\Gamma(U,K)$ the simplicial set of sections over the open set U.

A map of simplicial sheaves will be called a weak equivalence if it induces a stalkwise weak equivalence of simplicial sets at each point of X, where a map of simplicial sets is called a weak equivalence if its geometric realization is a homotopy equivalence. A map $p:E \to B$ of simplicial sheaves will be called a global fibration if, for any inclusion $U \subset V$ of open sets, the map

$$\Gamma(V,E) \xrightarrow{(\Gamma(V,p),\mathrm{res})} \Gamma(V,B) \times_{\Gamma(U,B)} \Gamma(U,E)$$

is a fibration in the sense of Kan, where "res" denotes restriction of sections. Taking $U = \emptyset$ and noting that $\Gamma(\emptyset,E) = \Gamma(\emptyset,B) = *$ (the simplicial set with exactly one simplex in each dimension), we see in particular that $\Gamma(V,p):\Gamma(V,E) \to \Gamma(V,B)$ is a fibration.

Finally, a simplicial sheaf K is called flasque if the unique map $K \to *$ is a global fibration, where $*$ here denotes the simplicial sheaf defined by $\Gamma(U,*) = *$. In other words, K is flasque if each restriction map $\Gamma(V,K) \to \Gamma(U,K)$ is a fibration. Taking $U = \emptyset$ we see in particular that $\Gamma(V,K)$ is a Kan complex.

Recall that X is called Noetherian if the open sets satisfy the ascending chain condition. We can now state the main result on which the homotopy theory and the cohomology theory of sheaves will be based.

THEOREM 1. *Let X be a Noetherian space in which the irreducible closed sets satisfy the ascending chain condition. If K is a flasque simplicial sheaf on X such that the map $K \to *$ is a weak equi-*

valence, then for any U, $\Gamma(U,K)$ is non-empty and $\pi_*\Gamma(U,K) = 0$. [Hence $\Gamma(U,K)$, being a Kan complex, is contractible.]

This theorem applies in particular to $X = \operatorname{Spec} A$ for A an arbitrary commutative Noetherian ring, since Noetherian local rings have finite Krull dimension.

Proof: Let us first observe that it suffices to prove that if $\Gamma(U,K) \neq \emptyset$ then $\pi_*\Gamma(U,K) = 0$. In fact, if this is proved then we can prove the theorem as follows. Since X is Noetherian, there is a maximal open set U such that $\Gamma(U,K) \neq \emptyset$. If $U = X$ then we are done. Otherwise, note that the hypothesis on K implies that the stalks of K are non-empty, so we can find an open set $V \not\subset U$ with $\Gamma(V,K) \neq \emptyset$. Consider now the cartesian square of fibrations

$$(1) \qquad \begin{array}{ccc} \Gamma(U \cup V,K) & \longrightarrow & \Gamma(U,K) \\ \downarrow & & \downarrow \\ \Gamma(V,K) & \longrightarrow & \Gamma(U \cap V,K) \end{array} .$$

Now the complex in the lower right-hand corner is known to be contractible (hence, in particular, connected), since we are assuming that the theorem has been proved under the assumption of non-emptiness. But then all the maps in the square are surjective; thus $\Gamma(U \cup V,K) \neq \emptyset$, which contradicts the maximality of U and establishes our assertion.

We may assume, then, after replacing X by U if necessary, that $\Gamma(X,K) \neq \emptyset$. We choose a basepoint in $\Gamma(X,K)$, which gives us by restriction a basepoint in all $\Gamma(U,K)$. For any open sets U, V of X, the square (1) gives rise to a Mayer-Vietoris sequence in homotopy, from which we isolate exact sequences

$$(2) \qquad T_{q+1}(U \cap V) \xrightarrow{\partial} T_q(U \cup V) \to T_q(U) \times T_q(V),$$

where we have set $T_q(-) = \pi_q\Gamma(-,K)$. The theorem follows now from

THEOREM 1'. *With the same hypotheses on* X, *let* $\{T_q\}$ *be a family of presheaves of pointed sets together with exact sequences* (2), *where* ∂ *is assumed to be natural in the sense that if* $V' \subset V$ *then*

$$\begin{CD} T_{q+1}(U \cap V) @>{\partial}>> T_q(U \cup V) \\ @V{\text{res}}VV @VV{\text{res}}V \\ T_{q+1}(U \cap V') @>{\partial}>> T_q(U \cup V') \end{CD}$$

commutes. Assume further that $T_q(\emptyset) = *$ *for all* q. *If all stalks of all* T_q *are trivial then* T_q *is trivial for all* q, i.e., $T_q(U) = *$ *for all* U.

Proof: Let Y be an arbitrary open subset of X, let $y \in T_q(Y)$ for some q, and let U be a maximal open subset of Y such that $y|U = *$. A closed irreducible set C in X will be called *bad* if for some such Y, y, q, and U, C meets Y but misses U. If there are no bad sets then $T_q(Y) = *$ and we are done, since if $U \neq Y$ we can take C to be the closure in X of an irreducible component of $Y-U$. If there are bad sets then by the hypothesis on X we can let C be a maximal one, and we let Y, y, q, U be as in the definition of "bad". Since the stalk of T_q is trivial at any point of $C \cap Y$, we can find an open set $V \subset Y$ such that V meets C and $y|V = *$. Then the exact sequence (2) implies that $y|U \cup V = \partial z$ for some $z \in T_{q+1}(U \cap V)$.

Now let W be a maximal open subset of $U \cap V$ such that $z|W = *$. [Note that we need here the hypothesis that $T_{q+1}(\emptyset) = *$, since even if we only consider non-empty Y above, it may happen that $U \cap V = \emptyset$.] We claim that C is an irreducible component of $X-W$. In fact, if C is properly contained in a closed irreducible

subset D of X-W, then the maximality of C implies that D is not bad, so D meets U. On the other hand D meets V, since C does; being irreducible, D must meet $U \cap V$. Applying the definition of "bad" with Y, y, q, and U replaced by $U \cap V$, z, q+1, and W, we see that either D meets W or D is bad. But we have already noted that D is not bad; since $D \subset X-W$, we have obtained a contradiction, thus proving the claim.

We now let F be the union of the irreducible components of X-W other than C, and we let $V' = V-F$. Then V' still meets C, and $U \cap V' \subset W$ because $U \cap V' \cap (X-W) \subset U \cap C = \emptyset$. Therefore $z|U \cap V' = *$, so $y|U \cup V' = \partial(z|U \cap V') = *$, which contradicts the maximality of U and completes the proof.

We now develop homotopy theory in the category $\mathcal{S}(X)$ of simplicial sheaves on X.

<u>THEOREM 2</u>. <u>With the same hypotheses on</u> X <u>as in Theorem 1, the category</u> $\mathcal{S}(X)$, <u>together with the notions of weak equivalence and (global) fibration, is a closed model category in the sense of</u> [9].

Before beginning the proof we explain what the statement of Theorem 2 means. Consider a commutative square

(3)
$$\begin{array}{ccc} K & \longrightarrow & E \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ L & \longrightarrow & B \end{array}$$

in $\mathcal{S}(X)$. By a <u>lifting</u> in the square we man a map $L \to E$ such that the resulting diagram still commutes. We call a map $i: K \to L$ a <u>cofibration</u> if a lifting exists in every square (3) in which p is a global fibration and a weak equivalence. We can now state the properties of $\mathcal{S}(X)$ which we will prove.[1]

[1] We are listing the axioms for closed model categories labelled CM 4 (ii) and CM 5 in [10, p.233]. The other axioms are trivially true in the present case.

Lifting property. A lifting exists in every square (3) in which i is a cofibration and a weak equivalence and p is a global fibration.

Factorizations. Any map f in $\mathcal{S}(X)$ can be factored $f = pi$ where i is a cofibration, p is a global fibration, and either i or p can be taken to be a weak equivalence.

We now begin the proof by reformulating the definition of "global fibration" in terms of lifting properties. Let Δ^n be the (semi-simplicial) n-simplex and let $\Lambda^{n,k}$ be the subcomplex generated by all faces of the boundary except the k-th. Then from the definition of "global fibration" and the definition of "fibration" for simplicial sets, we see that a map $p:E \to B$ in $\mathcal{S}(X)$ is a global fibration if and only if a lifting exists in every square of the form

(4)
$$\begin{tikzcd} \Lambda^{n,k} \arrow[r] \arrow[d, hook] & \Gamma(V,E) \arrow[d] \\ \Delta^n \arrow[r] & \Gamma(V,B) \times_{\Gamma(U,B)} \Gamma(U,E) \end{tikzcd} .$$

If A is any simplicial set and W is any open subset of X, let us denote by $\underline{A}_W$ the simplicial sheaf which is constant with stalk A on W and which has empty stalks outside of W. With this notation, (4) can be rewritten as a square in $\mathcal{S}(X)$

(4')
$$\begin{tikzcd} \underline{\Lambda}^{n,k}_V \cup_{\underline{\Lambda}^{n,k}_U} \underline{\Delta}^n_U \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ \underline{\Delta}^n_V \arrow[r] & B \end{tikzcd} ,$$

and we see then that p is a global fibration if and only if every square (4') has a lifting.

In this type of situation, there is a standard way of factoring a map f as pi, where p is a global fibration and where i is the composite of a countable sequence of inclusions, each of which is a pushout of a disjoint union of inclusions

$$(5) \qquad \underline{\Lambda}_V^{n,k} \underset{\underline{\Lambda}_U^{n,k}}{\cup} \underline{\Delta}_U^n \hookrightarrow \underline{\Delta}_V^n \ .$$

(See, for example, [4, Ch. VI, 5.5] or [9, Ch. II, p.3.4].[1])
Such a map i is a weak equivalence because each inclusion (5) is stalkwise either an identity map or a map whose geometric realization is an elementary expansion. Furthermore i is a cofibration, because it even satisfies a stronger lifting property than required in the definition of "cofibration", namely, a lifting exists in every square (3) such that p is a global fibration. This completes the proof of the existence of one of the factorizations.

Turning now to the lifting property, let $i:K \to L$ be a cofibration and a weak equivalence, and write $i = pj$, where p is a global fibration and j is a cofibration and weak equivalence as just constructed. Then p must also be a weak equivalence; thus (since i is a cofibration) there is a lifting in

[1] The construction just referred to seems to require that the sheaves $\underline{\Lambda}_V^{n,k} \underset{\underline{\Lambda}_U^{n,k}}{\cup} \underline{\Delta}_U^n$ be "small", and, in fact, they are, as one can see using the quasi-compactness of X. The smallness is not really essential, however, since a transfinite analogue of the construction works for any X.

$$\begin{tikzcd} K \arrow[r, "j"] \arrow[d, "i"'] & K' \arrow[d, "p"] \\ L \arrow[r, "id"] & L \end{tikzcd} .$$

In other words, i is a retract of j, and it is therefore sufficient to check the lifting property with i replaced by j. But we already remarked at the end of the previous paragraph that j has the required lifting property.

Finally, the construction of the second factorization is similar to that of the first, in view of the following.

LEMMA. *A map $p:E \to B$ in $\mathcal{S}(X)$ is a global fibration and a weak equivalence if and only if a lifting exists in every square of the form*

$$(6') \qquad \begin{tikzcd} \dot{\underline{\Delta}}^n_V \cup_{\dot{\underline{\Delta}}^n_U} \underline{\Delta}^n_U \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ \underline{\Delta}^n_V \arrow[r] & B \end{tikzcd} .$$

Proof: We first reformulate the statement of the lemma. The square (6') can be rewritten as a square in the category of simplicial sets

$$(6) \qquad \begin{tikzcd} \dot{\Delta}^n \arrow[r] \arrow[d, hook] & \Gamma(V,E) \arrow[d] \\ \Delta^n \arrow[r] & \Gamma(V,B) \times_{\Gamma(U,B)} \Gamma(U,E) \end{tikzcd} ,$$

and it is well known (see, for example, [9, Ch. II, §2,3]) that all squares (6) will have a lifting if and only if

$$(7) \qquad \Gamma(V,E) \to \Gamma(V,B) \underset{\Gamma(U,B)}{\times} \Gamma(U,E)$$

is a fibration and a weak equivalence. Thus we must prove that p is a global fibration and a weak equivalence if and only if, for every inclusion of open sets $U \subset V$, the map (7) is a fibration and a weak equivalence.

Suppose, then, that p is a global fibration and a weak equivalence. Then (7) is a fibration, and to check that it is a weak equivalence, we must show that every fibre is non-empty and contractible. Thus let $*$ be an arbitrary basepoint in the target of (7). The first component of $*$ is a basepoint in $\Gamma(V,B)$, which gives us (by restriction) a basepoint in $\Gamma(W,B)$ for $W \subset V$. Relative to this basepoint we can define a simplicial sheaf F on V by letting $\Gamma(W,F)$ be the fibre of $\Gamma(W,E) \to \Gamma(W,B)$. The second component of $*$ can now be regarded as a basepoint in $\Gamma(U,F)$. Consider now the Cartesian square

$$\begin{tikzcd}
\Gamma(V,F) \arrow[r, "\Gamma(V,i)"] \arrow[d, "\mathrm{res}"'] & \Gamma(V,E) \arrow[d] \\
\Gamma(U,F) \arrow[r, "(*,\Gamma(U,i))"] & \Gamma(V,B) \underset{\Gamma(U,B)}{\times} \Gamma(U,E)
\end{tikzcd}$$

where $i:F \to E$ is the inclusion. Since the bottom horizontal map is basepoint preserving, the fibre of (7) is the same as the fibre of $\Gamma(V,F) \to \Gamma(U,F)$; the contractibility of this fibre will follow from the fact that $\Gamma(V,F)$ and $\Gamma(U,F)$ are both contractible by Theorem 1. In fact, in order to apply Theorem 1, we need only verify that the stalks of F are contractible. But this follows from the fact that the stalk F_x is the fibre of $p_x:E_x \to B_x$, which is a fibration since

it is a filtered direct limit of fibrations $\Gamma(W,E) \to \Gamma(W,B)$, and which is a weak equivalence by hypothesis.

Conversely, if each map 7) is a fibration and a weak equivalence, then trivially p is a global fibration. Furthermore, taking $U = \emptyset$, we see that $\Gamma(V,E) \to \Gamma(V,B)$ is a weak equivalence; by passage to the direct limit over the open neighborhoods of a point, we deduce that p is stalkwise a weak equivalence, hence p is a weak equivalence. This completes the proof of the lemma and of Theorem 2.

It will be convenient to have at our disposal the <u>homotopy category</u> $\mathcal{H}o(X)$ associated to $\mathcal{S}(X)$ [9, Ch. I, p.1.12]. Recall that $\mathcal{H}o(X)$ is characterized by the following properties:

(1) $\mathcal{H}o(X)$ has the same objects as $\mathcal{S}(X)$.

(2) There is a functor $\gamma: \mathcal{S}(X) \to \mathcal{H}o(X)$ which is the identity on objects and which satisfies

(3) $\gamma(s)$ is an isomorphism if s is a weak equivalence, and $\mathcal{H}o(X)$ is the universal target of a functor on $\mathcal{S}(X)$ with this property.

<u>REMARK</u>. Using Theorem 2 it is easy to give a simple concrete description of $\mathcal{H}o(X)$. One first shows that every object in $\mathcal{H}o(X)$ is isomorphic to one which is both flasque and cofibrant.[1] Next one shows that, for such objects, maps in $\mathcal{H}o(X)$ are the same as simplicial homotopy classes of maps in $\mathcal{S}(X)$. More precisely, if K is cofibrant and L is flasque, then γ induces

$$\pi(K,L) \xrightarrow{\approx} [K,L],$$

where $\pi(\cdot,\cdot)$ denotes simplicial homotopy classes of maps and $[\cdot,\cdot]$ denotes maps in $\mathcal{H}o(X)$. (Part of the assertion is that simplicial

[1] An object K is <u>cofibrant</u> if the map $\emptyset \to K$ is a cofibration. Here $\emptyset$ is the empty sheaf, but in general the same definition applies with $\emptyset$ being the initial object of whatever category is being discussed.

homotopy is an equivalence relation for such K, L.) These assertions are immediate consequences of the results of [9, Ch.I, §1]. One only needs to verify that, if L is flasque, then the simplicial path space L^I (defined by $\Gamma(U,L^I) = \Gamma(\dot{U},L)^I$) is a path space for L in the sense of [9,Ch.I, p.1.5].

We will denote by $\mathcal{S}$ and $\mathcal{H}o$ the category of simplicial sets and its homotopy category, respectively.

PROPOSITION 1. *For any open set* U *there is a functor* $R\Gamma(U,-)\colon \mathcal{H}o(X) \to \mathcal{H}o$, *such that if* K *is flasque there is a natural isomorphism of* $R\Gamma(U,K)$ *with* $\Gamma(U,K)$.

The naturality here is with respect to maps in $\mathcal{S}(X)$, since $\Gamma(U,-)$ is not a functor on $\mathcal{H}o(X)$.

Proof (cf. [9,§4]): According to the universal property of $\mathcal{H}o(X)$ we need only define a functor $\mathcal{S}(X) \to \mathcal{S}$ (still to be denoted $R\Gamma(U,-)$) and check that it preserves weak equivalences. For each K we choose a *flasque resolution* $i\colon K \to K'$, i.e., K' is flasque and i is a cofibration and a weak equivalence. (For the existence of i, apply the appropriate factorization to the map $K \to *$.) We now define $R\Gamma(U,K)$ to be $\Gamma(U,K')$. In order to make this functorial in K (and to prove independence of the choice of resolution) we need only note that, as an easy consequence of the lifting property, if $i\colon K \to K'$ and $j\colon L \to L'$ are flasque resolutions and $f\colon K \to L$ is a map in $\mathcal{S}(X)$, then there is a unique homotopy class of maps $g\colon K' \to L'$ such that $jf = gi$. To see uniqueness, for example, let g_0, g_1 be two such maps. A homotopy between them is obtained by choosing a lifting in

$$\begin{tikzcd} K \arrow[r] \arrow[d, "i"'] & L'^I \arrow[d, "p"] \\ K' \arrow[r, "(g_0, g_1)"] & L' \times L' \end{tikzcd} ,$$

where the unlabelled map is the constant homotopy of jf and where p is the canonical map "evaluation at the endpoints".

The reader should observe that what we have done so far requires only the trivial part of the proof of Theorem 2, but in order to prove that $R\Gamma(U,-)$ preserves weak equivalences, we need the hard part of Theorem 2. We must, in fact, show that $\Gamma(U,-)$ preserves weak equivalences between flasque sheaves. Let then $f:K \to L$ be a weak equivalence, with K and L flasque. By factoring f, we may assume that f is either a cofibration or a global fibration. [Note that the intermediate sheaf K' that is obtained when f is factored is still flasque, because the composite $K' \to L \to *$ is a global fibration.] In case f is a cofibration, we choose a lifting $r:L \to K$ in the square

$$\begin{tikzcd} K \arrow[r, "id"] \arrow[d, "f"'] & K \arrow[d] \\ L \arrow[r] & * \end{tikzcd} ;$$

r is actually a homotopy inverse for f since a homotopy $fr \simeq id_L$ can be obtained by lifting in

$$\begin{tikzcd} K \arrow[r] \arrow[d, "f"'] & L^I \arrow[d] \\ L \arrow[r, "(fr, id)"] & L \times L \end{tikzcd} ,$$

where the unlabelled horizontal map is the constant homotopy of f. But then $\Gamma(U,r)$ is a homotopy inverse for $\Gamma(U,f)$, so $\Gamma(U,f)$ is a homotopy equivalence. Finally, in case f is a global fibration, we need only note that the proof of the lemma for Theorem 2 shows that $\Gamma(U,f)$ is a weak equivalence. [In the notation of that proof: $f:K \to L$ is $p:E \to B$, and $\Gamma(U,p):\Gamma(U,E) \to \Gamma(U,B)$ is a fibration with contractible fibres $\Gamma(U,F)$.]

REMARK. The proof showed that the functors $R\Gamma(U,-)$ are compatible, in an obvious sense, with inclusions of open sets.

We close this section with some remarks on the case of simplicial sheaves with basepoint, where a basepoint for K is simply a vertex of $\Gamma(X,K)$, or, equivalently, a map $* \to K$ in $\mathcal{S}(X)$. The category of simplicial sheaves with basepoint will be denoted $\mathcal{S}.(X)$. We define weak equivalence, global fibration, and cofibration in $\mathcal{S}.(X)$ by means of the forgetful functor $\mathcal{S}.(X) \to \mathcal{S}(X)$, and it is trivial to check that Theorem 2 holds with $\mathcal{S}(X)$ replaced by $\mathcal{S}.(X)$. [For example, to check the factorization axioms for a map f in $\mathcal{S}.(X)$, we factor f as pi in $\mathcal{S}(X)$ and then observe that there is a unique choice of basepoint in the target of i such that i and p are basepoint preserving.] The homotopy category associated to $\mathcal{S}.(X)$ will be denoted by $\mathcal{H}o.(X)$, and exactly as in Proposition 1 we have functors $R\Gamma(U,-)$: $\mathcal{H}o.(X) \to \mathcal{H}o.$.

2. Generalized sheaf cohomology.

Throughout this section X will continue to be a Noetherian space in which the irreducible closed subsets satisfy the ascending chain condition.

If K is a simplicial sheaf with basepoint, we define the *generalized sheaf cohomology groups* $H^q(X,K)$ by

$$H^q(X,K) = \pi_{-q}R\Gamma(X,K).^{1}$$

Explicitly, if $i:K \to K'$ is a flasque resolution, $H^q(X,K) = \pi_{-q}\Gamma(X,K')$.

EXAMPLE. Let F be an abelian sheaf on X and n a non-negative integer. We define the *Eilenberg-MacLane sheaf* $K(F,n)$ to be the simplicial abelian sheaf which corresponds, under the Dold-Kan correspondence between simplicial objects and non-negative chain complexes over an abelian category [3, §3], to the chain complex consisting of F concentrated in dimension n. Equivalently, we can describe $K(F,n)$ by

$$\Gamma(U,K(F,n)) = K(\Gamma(U,F),n),$$

the latter being the ordinary semi-simplicial Eilenberg-MacLane complex [7, §23].

The following proposition shows that our generalized cohomology groups with coefficients in $K(F,n)$ reduce to ordinary sheaf

[1] Note that in general $H^0(X,K)$ is only a pointed set, but we will ignore this fact because in the examples that occur in this paper either $H^0(X,K) = 0$ or $H^0(X,K)$ has a group structure induced by a group structure on K. Note also that $H^q(X,K) = 0$ for $q > 0$. In a more general treatment, K would be the 0-th sheaf in an Ω-spectrum, and positive-dimensional cohomology groups would be defined using the other sheaves of the spectrum. This generality, however, is not needed for our application to algebraic K-theory, since in this example the positive dimensional groups would vanish anyway. (Cf. Theorem 5 in §3 below.)

cohomology groups, and that as $n \to \infty$ all the cohomology groups $H^*(X,F)$ are obtained.

PROPOSITION 2. *For* $q \leq n$, $H^q(X,F) \approx H^{q-n}(X,K(F,n))$.

Proof: Let

$$0 \to F \xrightarrow{\eta} I^0 \to \dots \to I^n \to 0$$

be an exact sequence of abelian sheaves, where $I^0,\dots,I^{n-1}$ are flasque in the sense of Godement [6]. Let C_* be the non-negative chain complex of sheaves obtained by re-indexing $I^*: C_q = I^{n-q}$ for $q = 0,\dots,n$, $C_q = 0$ otherwise. Finally, let K be the simplicial abelian sheaf corresponding to C_*. Then $H^q(X,F)$ for $q \leq n$ can be computed as $H_{n-q}\Gamma(X,C_*)$, which is the same as $\pi_{n-q}\Gamma(X,K)$. Now K is flasque, because in order that the restriction map $\Gamma(V,K) \to \Gamma(U,K)$ be a fibration, it is necessary and sufficient that the corresponding map of chain complexes $\Gamma(V,C_*) \to \Gamma(U,C_*)$ be surjective in positive dimensions[1], and this is true because C_q is flasque for $q > 0$. Therefore $\pi_{n-q}\Gamma(X,K)$ can be identified with $H^{q-n}(X,K)$. To complete the proof we need only note that the map $\eta: F \to I^*$ induces a weak equivalence $K(F,n) \to K$, so $H^{q-n}(X,K) \approx H^{q-n}(X,K(F,n))$, as required.

For the sake of completeness we point out that the formula

$$\Gamma(U,K(F,n)) = K(\Gamma(U,F),n)$$

also defines $K(F,1)$ for F a sheaf of non-abelian groups and $K(F,0)$ for F a sheaf of pointed sets. These occur in Proposition 3 below, although in our later applications F will always be abelian.

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[1] This is "(i) $\Longleftrightarrow$ (ii)" of [9, Ch.II, §3, Prop. 1], specialized to the abelian case.

If K is a simplicial sheaf with basepoint we define the q-th *homotopy sheaf* $\pi_q K$ to be the sheaf associated to the presheaf $U \mapsto \pi_q \Gamma(U,K)$.

PROPOSITION 3. *Let* K *be a simplicial sheaf with basepoint, and suppose that for some integer* $n \geq 0$, $\pi_q K = 0$ *for* $q \neq n$. *In case* $n = 0$, *suppose further that each connected component of each stalk of* K *has trivial homotopy groups. Then* K *is canonically isomorphic in* Ho. (X) *to* $K(\pi_n K, n)$.

Proof: Replacing K by the n-th Eilenberg subcomplex [7, pp.32-33] of a flasque resolution of K, we may assume that for any open set U, $\Gamma(U,K)$ has a unique simplex in each dimension less than n. Now such a simplicial set always has a canonical normalized n-cocycle with values in π_n; namely, the cocycle which assigns to each n-simplex its homotopy class. [Every n-simplex σ satisfies $d_i \sigma = *$ for all i and therefore is the image of the non-trivial n-simplex of $\Delta^n/\dot{\Delta}^n$ under a unique map; such a map represents an element of π_n. Note that if $n = 0$ or 1 the notion of cocycle needs to be suitably interpreted.] Since normalized cocycles are in 1-1 correspondence with maps into the appropriate Eilenberg-MacLane complex, we obtain a canonical map

$$\Gamma(U,K) \to K(\pi_n \Gamma(U,K), n).$$

These maps are compatible with inclusions of open sets and so define a map of presheaves; passing to associated sheaves yields a map $K \to K(\pi_n K, n)$, which is easily seen to be a weak equivalence by checking that it induces a stalkwise isomorphism on π_n.

We can now prove the main result of this section.

THEOREM 3. *Let* X *be a Noetherian space of finite Krull dimension and let* K *be a simplicial sheaf with basepoint. Assume that*

$\pi_0 K = 0$, *that* $\pi_1 K$ *and* $H^{-1}(X,K)$ *are abelian, and that* $H^p(X,\pi_n K) = 0$ *for* $p \geq n$. *Then there is a fourth quadrant spectral sequence of cohomological type,*

$$E_2^{pq} = H^p(X,\pi_{-q}K) \Longrightarrow H^{p+q}(X,K).$$

REMARKS. 1. Since X has finite cohomological dimension [6, Ch.II, §4.15], there is no problem with convergence.

2. The unnatural assumptions about K can be eliminated if one works with spectra as indicated in the footnote on the first page of this section. A different way of eliminating the assumption that $H^p(X,\pi_n K) = 0$ for $p \geq n$ is indicated in a remark following the proof.

Proof of Theorem 3: We may assume K is flasque. We construct the *Postrikov tower*

$$\cdots \to P_n K \xrightarrow{p_n} P_{n-1}K \to \cdots \to P_0 K \to P_{-1}K = *$$

by defining $P_n K$ to be the sheaf associated to the presheaf $U \mapsto P_n\Gamma(U,K)$, the latter P_n denoting the ordinary n-th Postnikov approximation of a Kan complex [7, pp.32-33]. One has a compatible family of maps $f_n : K \to P_n K$ inducing $\pi_q K \xrightarrow{\approx} \pi_q P_n K$ for $q \leq n$, and $\pi_q P_n K = 0$ for $q > n$. Furthermore, each p_n is stalkwise a fibration and, letting F_n be the fibre of p_n, the inclusion $F_n \hookrightarrow P_n K$ induces $\pi_n F_n \xrightarrow{\approx} \pi_n P_n K \xleftarrow{\approx} \pi_n K$; since $\pi_q F_n = 0$ for $q \neq n$ we deduce from Proposition 3 that $F_n \approx K(\pi_n K, n)$ in $\mathcal{H}o.(X)$.

We will now replace the Postnikov tower by an equivalent tower in which the sheaves are flasque and the maps are global fibrations. We begin by choosing a flasque resolution $i_0 : P_0 K \to L_0$. Assuming inductively that $i_n : P_n K \to L_n$ has been constructed, the next stage is obtained by factoring $i_n p_{n+1} : P_{n+1}K \to L_n$ as a cofibration

$i_{n+1}\colon P_{n+1}K \to L_{n+1}$ followed by a global fibration $q_{n+1}\colon L_{n+1} \to L_n$, with i_{n+1} taken to be a weak equivalence. Applying $\Gamma(X,-)$ to the tower

$$\cdots \to L_{n+1} \xrightarrow{q_{n+1}} L_n \to \cdots$$

we obtain a tower of fibrations of simplicial sets and hence a homotopy exact couple. (It will follow from our computation below of the E_2 term of the spectral sequence that all fibres in the tower are connected; one can then prove inductively that all simplicial sets in the tower are connected, so there is no problem with π_0. This also shows that each fibration in the tower induces a surjection on π_1; our computation of E_∞ will show that for large n, $\pi_1\Gamma(X,L_n) \approx H^{-1}(X,K)$, which is assumed to be abelian, so in fact all fundamental groups in the tower are abelian and there is no problem with π_1.)

To identify the E_2 term, note that the fibre of $\Gamma(X,q_n)$ is $\Gamma(X,G_n)$, where G_n is the fibre of q_n. Since G_n is flasque and is equivalent to $K(\pi_n K,n)$, we know that $\pi_i\Gamma(X,G_n)$ is isomorphic to $H^{-i}(X,K(\pi_n K,n))$, and this in turn (by Proposition 2) is isomorphic to $H^{n-i}(X,\pi_n K)$ for $i \geq 0$. The same result holds for $i < 0$, since then $H^{n-i}(X,\pi_n K) = 0$ by hypothesis. Thus the E_2 term has the required form. (Note that the differential in the exact couple decreases i by 1 and increases n by 1, hence it increases the filtration degree $n-i$ by 2 and the total degree $-i$ by 1, as required.)

Finally, to identify the abutment, let $L = \varprojlim L_n$. Then the spectral sequence abuts to $\pi_i\Gamma(X,L)$, which is $H^{-i}(X,L)$ since L is flasque. To complete the proof it suffices to show that K is equivalent to L. Now we have a map $K \to L$ (induced by the maps $f_n\colon K \to P_n K$), which we claim is a weak equivalence. In order to

check this, it clearly suffices to prove that $\pi_i L \overset{\approx}{\to} \pi_i L_n$ for large n. This is not entirely trivial since the stalk functors do not preserve inverse limits, but we can argue as follows. For any open set U, we see as above that $\pi_i \Gamma(U, G_n) \approx H^{n-i}(U, \pi_n K) = 0$ for $n-i > \dim X$. Therefore $\pi_i \Gamma(U, q_n)$ is an isomorphism for $n > i+1+\dim X$, hence $\pi_i \Gamma(U,L) \overset{\approx}{\to} \pi_i \Gamma(U, L_n)$ for $n > i+2+\dim X$. Passing to the direct limit over the neighborhoods of a point, we find that $\pi_i L \to \pi_i L_n$ is a stalkwise isomorphism for $n > i+2+\dim X$, as required.

REMARK. If we eliminate the assumption that $H^p(X, \pi_n K) = 0$ for $p \geq n$ then the same exact couple[1] yields a "fringed" spectral sequence with $E_2^{pq} = H^p(X, \pi_{-q} K)$ for $p+q \leq 0$ and $E_2^{pq} = 0$ otherwise, with the fringe being in total degree $p+q = 0$. This means that for $p+q = 0$, E_{r+1}^{pq} is a certain subquotient of E_r^{pq}, this subquotient being determined by the exact couple, but (in total degree 0) E_{r+1} is not necessarily the homology of E_r.

We end this section by broadening the notion of flasque sheaf in a way that will be crucial for the application in §3. We will consider now simplicial presheaves P, and for simplicity we will assume that P(U) is a Kan complex for all U[2]. We will call P pseudo-flasque if (a) $P(\emptyset)$ is contractible and (b) for each pair of open sets U, V the square

$$\begin{array}{ccc} P(U \cup V) & \longrightarrow & P(V) \\ \downarrow & & \downarrow \\ P(U) & \longrightarrow & P(U \cup V) \end{array}$$

[1]This only works if we have an abelian group structure on π_0 and π_1 of all the simplicial sets in the tower of fibrations, induced, for example, by an "H-space" structure on K.

[2]The general case can be reduced to this case by replacing P(U) by the singular complex of its geometric realization.

is homotopically cartesian, where a square of Kan complexes

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

is called <u>homotopically cartesian</u> if the natural map $W \to X \overset{h}{\underset{Z}{\times}} Y$ is a weak equivalence, $X \overset{h}{\underset{Z}{\times}} Y$, the <u>homotopy theoretic fibre product</u>, being by definition $X \underset{Z}{\times} Z^I \underset{Z}{\times} Y$. [Note that $X \overset{h}{\underset{Z}{\times}} Y$ is the base extension of g' by f, where g' denotes g converted into a fibration in the standard way, and it is also the base extension of f' by g.]

THEOREM 4. <u>Let P be a pseudo-flasque presheaf and let K be the associated sheaf. Then the natural map</u>

$$\pi_i P(X) \to H^{-i}(X,K)$$

is an isomorphism.

Proof: Let $i\colon K \to K'$ be a flasque resolution. The natural map in question is the map on homotopy groups induced by the composite $f\colon P(X) \to \Gamma(X,K) \to \Gamma(X,K')$. Since $P(X)$ and $\Gamma(X,K')$ are both Kan complexes, one can construct the homotopy theoretic fibre of f over any basepoint $*$ in $\Gamma(X,K')$ as

$$F(X) = P(X) \underset{\Gamma(X,K')}{\times} \Gamma(X,K')^I \underset{\Gamma(X,K')}{\times} * \; .$$

This also makes sense for any open subset of X, and F becomes a presheaf on X. Note that the stalks of F are contractible; furthermore, we claim F is pseudo-flasque. In fact, if U_1 and U_2 are open subsets of X, one sees by checking the definitions that $F(U_1) \overset{h}{\underset{F(U_1\cap U_2)}{\times}} F(U_2)$ is isomorphic to the homotopy theoretic fibre of

$$P(U_1) \overset{h}{\underset{P(U_1\cap U_2)}{\times}} P(U_2) \to \Gamma(U_1,K') \overset{h}{\underset{\Gamma(U_1\cap U_2,K')}{\times}} \Gamma(U_2,K').$$

But this (according to the hypotheses on P and K') is homotopy equivalent to the homotopy theoretic fibre of

$$P(U_1 \cup U_2) \to \Gamma(U_1 \cup U_2, K'),$$

which by definition is $F(U_1 \cup U_2)$, as required. Having verified the claim, we complete the proof by observing that the proof of Theorem 1 of §1, based on Mayer-Vietoris sequences in homotopy, works for pseudo-flasque presheaves exactly as for flasque sheaves. Thus $F(X)$ is contractible and f is a weak equivalence.

3. Application to K-theory.

Let X be a noetherian scheme. For any open subset U of X, let $\mathcal{C}(U)$ be the category of coherent sheaves on U. Let $P(U)$ be the singular complex of the geometric realization of the nerve of $Q(\mathcal{C}(U))$, where "Q" denotes the construction of [8, §2]. Thus $P(U)$ is a connected Kan complex with basepoint and for any integer i, $\pi_{i+1}P(U)$ is by definition $G_i(U)$ [8, §5]. If $i\colon U \to V$ is an inclusion of open sets, the restriction functor $i^*\colon \mathcal{C}(V) \to \mathcal{C}(U)$ induces a map $P(V) \to P(U)$, thus making P a presheaf of Kan complexes.

PROPOSITION 4. The presheaf P is pseudo-flasque. [See the end of §2 for the definition of "pseudo-flasque".]

Proof: Let us first observe that the restriction functor $i^*\colon \mathcal{C}(V) \to \mathcal{C}(U)$ is a localization of abelian categories. (The analogous statement for quasi-coherent sheaves follows from the criterion of [5, Ch. I, Prop. 1.3] together with EGA I, §9.2. The coherent case can be deduced from this, using the fact that a quasi-coherent sheaf is the union of its coherent subsheaves and the fact that the category of coherent sheaves is closed under subobjects and quotient objects in the category of quasi-coherent sheaves.) Consequently [8, Theorem 4], the homotopy theoretic fibre of $P(V) \to P(U)$ is equivalent to the nerve of $Q(\mathcal{C}(V,U))$, where $\mathcal{C}(V,U)$ is the category of coherent sheaves on V which are zero on U.

Consider now an arbitrary pair of open sets U, V, and consider the square

$$\begin{array}{ccc} P(U \cup V) & \longrightarrow & P(V) \\ \downarrow & & \downarrow \\ P(U) & \longrightarrow & P(U \cap V) \end{array} .$$

To prove P is pseudo-flasque, it suffices to show that the induced map from the homotopy theoretic fibre of the left-hand vertical map to that of the right-hand vertical map is an equivalence. But this

map on fibres is equivalent by the preceding paragraph to the nerve of the restriction functor $Q(\mathcal{E}(U \cup V, U)) \to Q(\mathcal{E}(V, U \cap V))$; this functor being an equivalence of categories, the proof is complete.

We now write $P = P_{(1)}$ and we construct a sequence of "deloopings" $P_{(2)}, P_{(3)}, \ldots,$ as follows. The category $Q(\mathcal{E}(U))$ has a coherently commutative and associative operation (induced by direct sum of sheaves). Consequently its nerve has a classifying space $P_{(2)}(U)$ which has a classifying space $P_{(3)}(U)$, etc. These classifying spaces can be constructed, for example, by the method of Anderson [1]; since Anderson's construction is functorial, $P_{(k)}$ is a presheaf for each $k \geq 1$.

THEOREM 5. _Let_ $k \geq 1$ _be an integer and let_ K _be the sheaf associated to the presheaf_ $P_{(k)}$. _Then for any integer_ q, $H^q(X,K) \approx G_{-k-q}(X)$.

Proof: One knows that for any integer i, $\pi_i P_{(k)}(X) \approx \pi_{i-k+1} P(X) = G_{i-k}(X)$. On the other hand, if we verify that $P_{(k)}$ is pseudo-flasque, then by Theorem 4 we can write $\pi_i P_{(k)}(X) \approx H^{-i}(X,K)$. Setting $i = -q$ yields the result. It remains to prove that $P_{(k)}$ is pseudo-flasque. For $k = 1$ this is Proposition 4. Assuming inductively that $P_{(k-1)}$ is pseudo-flasque, consider the natural map

$$P_{(k)}(U \cup V) \xrightarrow{f} P_{(k)}(U) \underset{P_{(k)}(U\cap V)}{\overset{h}{\times}} P_{(k)}(V).$$

Since the source and target of f are connected,[1] f will be an equivalence if Ωf is. But Ωf is equivalent to a map of the same form as f, with k replaced by $k-1$ (since $\Omega P_{(k)} \simeq P_{(k-1)}$), and this is an equivalence by the induction hypothesis.

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[1]For $k > 1$, $P_{(k-1)}(-)$ is connected, by construction. It follows that $P_{(k)}(-)$ is simply connected, which accounts for the connectivity of the target of f.

COROLLARY. Let X be a noetherian scheme of finite Krull dimension. There is a fourth quadrant spectral sequence of cohomological type,

$$E_2^{pq} = H^p(X, \mathcal{G}_{-q}) \Rightarrow G_{-(p+q)}(X),$$

where $\mathcal{G}_{-q}$ is the sheaf associated to the presheaf $U \mapsto G_{-q}(U)$.

Proof: We will see below that the hypotheses of Theorem 3 are satisfied by the sheaf K of Theorem 5, provided that k is sufficiently large. Let, then, $\{{}'E_r^{pq}\}$ be the spectral sequence of Theorem 3. The spectral sequence $\{E_r^{pq}\}$ which we are interested in is obtained from this by re-indexing: $E_r^{pq} = {}'E_r^{p,q-k}$. In order to compute E_2 and to see how large k needs to be so that the hypotheses of Theorem 3 will be satisfied, we compute $\pi_i K$. Now since $P_{(k)} \to K$ is a stalkwise isomorphism, and since π_i commutes with filtered direct limits of simplicial sets, $\pi_i K$ is isomorphic to the sheaf associated to the presheaf $U \mapsto \pi_i P_{(k)}(U)$. But $\pi_i P_{(k)}(U) \approx G_{i-k}(U)$ (see the first sentence of the proof of Theorem 5), so $\pi_i K \approx \mathcal{G}_{i-k}$. Thus Theorem 3 will apply provided $H^p(X, \mathcal{G}_n) = 0$ for $p \geq n+k$. In particular, we can always use any $k > \dim X$; the E_2 term is then given by

$$E_2^{pq} = {}'E_2^{p,q-k} = H^p(X, \pi_{k-q}K) \approx H^p(X, \mathcal{G}_{-q}),$$

and the abutment is $H^{p+q-k}(X,K)$, which is isomorphic to $G_{-(p+q)}(X)$ by Theorem 5.

REMARKS. 1. Since $E_\infty^{pq} = 0$ in positive total degrees (and for other reasons as well) it is reasonable to conjecture[1] that $H^p(X, \mathcal{G}_n) = 0$ for $p > n$. If this is true then we can take $k = 1$ in the above

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[1] For a discussion of this and related conjectures (at least in the regular case), see §7 of S.M. Gersten, "Some exact sequences in the higher K-theory of rings", these proceedings. See also problems 10,11, and 12 in S.M. Gersten, "Problems about higher K-functors", these proceedings.

proof and there is no need to use the de-loopings provided by the theory of Segal and Anderson. (Note, however, that it is sometimes possible to prove the conjecture for low-dimensional X by using the spectral sequence of the Corollary. In such cases, then, one needs to use the existence of the $P_{(k)}$ in order to prove that they are unnecessary.) In any case, the use of the $P_{(k)}$ can be avoided if one is willing to settle for a fringed spectral sequence (see Remark after Theorem 3) with $E_2^{pq} = H^p(X, \mathcal{G}_{-q})$ for $p+q \leq 1$ and $E_2^{pq} = 0$ otherwise, the fringe being in total degree $p+q = 1$.

2. There is a relative version of the spectral sequence for a map $f\colon Y \to X$ of schemes, obtained by using the presheaf on X defined by $U \mapsto P(f^{-1}U)$. The E_2 term is then the cohomology of X with coefficents in the sheaf associated to the presheaf $U \mapsto G_{-q}(f^{-1}U)$, and the abutment is $G_{-(p+q)}(Y)$.

3. In case X is regular and has the property that every coherent sheaf is a quoteint of a locally free sheaf, then $G_n(X) \approx K_n(X)$ and $\mathcal{G}_n \approx \mathcal{K}_n$ (see proof of [8, Theorem 5]), where $K_n(X)$ is K_n (in the sense of [8, §2]) of the category of locally free sheaves, and $\mathcal{K}_n$ is the sheaf associated to the presheaf $U \mapsto K_n(U)$. This applies, for example, if $X = \operatorname{Spec} A$ with A regular (in which case $K_n(X) = K_n(A)$), or if X is a non-singular quasi-projective variety over a field.

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Cornell University
Ithaca, New York

Rice University
Houston, Texas