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Euler Characteristics of Discrete Groups and G-Spaces

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Table of Contents

Introduction

Let Γ be a discrete group. If Γ satisfies suitable finiteness conditions, then one can associate to Γ a rational number $\chi(\Gamma)$, called the *Euler characteristic* of Γ, cf. [18] or §4 below. In this paper we will show how one can obtain information about $\chi(\Gamma)$ by studying the finite subgroups of Γ. Here is a summary of the results:

In §4 we define $\chi(\Gamma)$ under weaker finiteness conditions than in [18] and, at the same time, show that $\chi(\Gamma)$ is an integer if Γ is torsion-free. This answers a question asked by Serre ([18], p. 101).

In §6 we give the main results of the paper, including the following:

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(1) If m is the least common multiple of the orders of the finite subgroups of Γ, then $m \cdot \chi(\Gamma)$ is an integer. This was conjectured by Serre and was proved independently (and earlier) by Verdier [unpublished] for arithmetic groups.

(2) Under suitable finiteness assumptions, the fractional part of $\chi(\Gamma)$ can be computed from an "equivariant Euler characteristic" $\chi_\Gamma(S)$ (see §5), where S is the set of non-trivial finite subgroups of Γ, regarded as an ordered set on which Γ operates. Similarly, if p is a prime number, the p-fractional part of $\chi(\Gamma)$ can be computed from the set of finite subgroups of order divisible by p. This is used in §7 to compute a convenient estimate for the p-fractional part of $\chi(\Gamma)$.

(3) Under suitable finiteness assumptions there is a formula which expresses the difference between $\chi(\Gamma)$ and the "naive" Euler characteristic $\tilde{\chi}(\Gamma)$ (which is always an integer – see §4) as a sum of the form $\sum c(H)/|H|$, where H ranges over the finite subgroups of Γ (up to conjugacy) and $c(H)$ is an integer. (The probable existence of a formula of this type was suggested by Serre.)

In §8 we give some group theoretic applications and in §9 some arithmetic applications of these results. The arithmetic applications concern the values at negative integers of the Dedekind zeta function ζ_k of a totally real number field k. They are obtained by applying the results on $\chi(\Gamma)$ to certain arithmetic groups Γ and using a theorem of Harder's [11] which relates $\chi(\Gamma)$ to the values of ζ_k. For example, we obtain in this way a simple formula for the odd part of the denominator of $\zeta_k(-1)$ (corollary of Proposition 8 of §9.3). For another example, following a suggestion made by Serre, we use the results of §7 to prove in §9.4 a classical theorem of Kummer's and a recent generalization by Greenberg [9]. Results similar to those of §9 have been obtained independently by Gueho [10], using completely different methods.

All of our results on $\chi(\Gamma)$ are obtained by passing to a normal subgroup Γ' of finite index and studying the action of Γ/Γ' on various finite dimensional chain complexes or semi-simplicial complexes. The necessary results on finite group actions are given in §§1 and 2; they are based on a theorem of Swan's concerning projective modules over group rings. The results of §1, which are crucial to the entire paper, are due jointly to I. Berstein and the author. Results similar to those of §2 were obtained independently by Verdier [23].

I wish to thank J-P. Serre for numerous suggestions, without which much of this work would not have been done. I am also grateful to the referee for suggestions which helped improve my exposition. Finally, I thank I. Berstein for permission to include in this paper the results of our joint work (§ 1), and P.J. Kahn for his interest and encouragement.

I. Finite Group Actions

§1. Chain Complexes of Projective $\mathbb{Z}[G]$-Modules

The results of this section represent joint work with I. Berstein.

Let C be a (non-negative) chain complex of abelian groups. If $H_*(C)$ is finitely generated, then we denote by $\chi(C)$ the *Euler characteristic* $\Sigma(-1)^i \operatorname{rank}_{\mathbb{Z}} H_i(C)$. In case a finite group G acts on C (so that C is a complex of $\mathbb{Z}[G]$-modules), we denote by L the alternating sum $\Sigma(-1)^i h_i$, where h_i is the character associated to the $\mathbb{Q}[G]$-module $H_i(C)\otimes\mathbb{Q}$. Thus L is an integer-valued function on G, and $L(s)$ for $s\in G$ is the *Lefschetz number* of the action of s on C. Finally, we denote by R the character of the regular representation of G, i.e., $R(1)=|G|$ and $R(s)=0$ for $s\neq 1$.

Theorem 1. *Let C be a finite dimensional chain complex of projective $\mathbb{Z}[G]$-modules, where G is a finite group, and assume that $H_*(C)$ is finitely generated. Then the complex $C_G=\mathbb{Z}\underset{\mathbb{Z}[G]}{\otimes} C$ has finitely generated homology and the virtual character L defined above is equal to $\chi(C_G)\cdot R$. In other words,*

(a) $\chi(C)=|G|\cdot\chi(C_G)$ *and*

(b) $L(s)=0$ *for* $s\neq 1$.

(C is called *finite dimensional* if $C_i=0$ for sufficiently large i.)

The proof of Theorem 1 will make use of the following lemma, which is implicit in the work of Wall ([25, 26]):

Lemma. *Let C be a chain complex of projective R-modules, where R is a noetherian ring. If $H_i(C)$ is finitely generated for each i, then C is homotopy equivalent to a complex D of finitely generated projectives. If, in addition, there is an integer n such that $H^i(\operatorname{Hom}(C_\bullet, M))=0$ for $i>n$ and all finitely generated R-modules M, then we can choose D so that $D_i=0$ for $i>n$.*

For the first part of the lemma, one simply constructs by induction on k the truncations $D^{(k)}=(D_i)_{i\leq k}$ of the desired D, together with (compatible) maps $f^{(k)}\colon D^{(k)}\to C$, such that $f^{(k)}$ induces a homology isomorphism in dimensions less than k and a homology epimorphism in dimension k. There is no difficulty in carrying out the inductive step, and each D_i can, in fact, be taken to be finitely generated and free. We obtain in this way a complex D and a map $f\colon D\to C$ which induces a homology isomorphism in all dimensions; D and C being dimension-wise projective, f is a homotopy equivalence.

Suppose now that $H^i(\operatorname{Hom}(C_\bullet, M))=0$ for all $i>n$ and all finitely generated M. Then the same is true with C replaced by the complex D just constructed; in other words, any homomorphism $\varphi\colon D_i\to M$ (for

$i>n$) which vanishes on the submodule B_i of i-dimensional boundaries of D factors through D_{i-1}. Taking $M=D_i/B_i$ and φ equal to the projection $D_i \to D_i/B_i$, we conclude that $H_i(D)=0$ and that B_{i-1} is a direct summand of D_{i-1}. It follows at once that we may replace D by its quotient $\bar{D}$ defined by

$$\bar{D}_i = \begin{cases} D_i & i<n \\ D_n/B_n & i=n \\ 0 & i>n, \end{cases}$$

which proves the second part of the lemma.

Proof of Theorem 1. In view of the lemma (applied with $R=\mathbb{Z}[G]$), we may assume that each C_i is finitely generated. Then L can be computed on the chain complex level from the complex $\mathbb{Q} \underset{\mathbb{Z}}{\otimes} C$, and the result follows from the following theorem of Swan's:

Swan's Theorem. *If G is a finite group and P is a finitely generated projective $\mathbb{Z}[G]$-module, then $\mathbb{Q} \underset{\mathbb{Z}}{\otimes} P$ is a free $\mathbb{Q}[G]$-module.*

(See [22], Theorem 4.2, or [1], Chapter XI, Theorem 5.2.)

There is also a "mod p" version of Theorem 1:

Theorem 1′. *Let C be a finite dimensional chain complex of projective $\mathbb{F}_p[G]$-modules, where p is a prime number and G is a finite p-group, and assume that $H_*(C)$ is finitely generated. Then C_G has finitely generated homology and*

$$\chi(C) = |G| \cdot \chi(C_G).$$

(Here $\chi(-)$ is to be understood as the alternating sum of the $\mathbb{F}_p$-ranks of the homology groups.)

As in the proof of Theorem 1, we may assume that each C_i is finitely generated and we may therefore compute Euler characteristics on the chain level. The result now follows from the fact that, since $\mathbb{F}_p[G]$ is a local ring, projective $\mathbb{F}_p[G]$-modules are free (cf. [22], Corollary 4.4, or [20], Chapter IX, § 1).

Remark. There is a topological analogue of Theorem 1: If G acts freely on a paracompact space X of finite cohomological dimension, and if $H^*(X, \mathbb{Z})$ is finitely generated, then the virtual character L associated to the action of G on $H^*(X, \mathbb{Z})$ is equal to $\chi(X/G) \cdot R$. (Here cohomology and cohomological dimension are to be interpreted in the sense of sheaf theory, cf. [8].) A proof of this can be found in Zarelua [27]. Similarly, there is a topological analogue of Theorem 1′, cf. [2], Chapter III.

In case the assumption on $H^*(X, \mathbb{Z})$ is replaced by the assumption that $H^*(X, \mathbb{Q})$ be finite dimensional, the topological analogue of Theorem 1 is no longer valid. In fact, one can construct for any finite group G

a 3-dimensional CW complex X which is $\mathbb{Q}$-acyclic (so $\chi(X)=1$) but which admits a free G-action.

§2. Semi-Simplicial G-Complexes[1]

Notation. X denotes a *finite dimensional* semi-simplicial complex on which a finite group G operates. ("Finite dimensional" means that all simplices of sufficiently high dimension are degenerate.) For any subgroup H of G we write X^H (resp. X_H) for the set of simplices fixed by H (resp. the set of simplices whose isotropy group is H). We denote by Y the orbit complex X/G and by Y_H the image in Y of X_H. Finally, we denote by a_H the character of G induced from the "augmentation character" of H, i.e., $a_H=R-b_H$, where R is the regular character of G and b_H is the character of the $\mathbb{Q}[G]$-module $\mathbb{Q}[G/H]$. (Explicitly, $b_H(s)$ is simply the number of fixed points of s acting on G/H.)

We need a preliminary remark on the homology of semi-simplicial complexes. It will be convenient below to be able to talk about $H_*(A)$, where A is a graded subset of a semi-simplicial complex K which is not a subcomplex but which is a difference $L-L'$ of subcomplexes (which can always be taken with $L'\subset L$). This homology, which is isomorphic to $H_*(L,L')$, is defined directly in terms of A as the homology of a chain complex $C_*(A)$ with one generator for each simplex of A; the differential, as usual, is the alternating sum of the faces, where we set $d_i\sigma=0$ in $C_*(A)$ if, in K, $d_i\sigma\notin A$. As usual, we can normalize $C_*(A)$ (by dividing out by the subgroup generated by the degenerate simplices) without affecting the homology. (Note that A is closed under degeneracy operators.) This discussion applies, in particular, to the subsets X_H of X, for

$$X_H=X^H-\bigcup_{H'\supsetneqq H} X^{H'}.$$

Similarly, Y_H is a difference of subcomplexes of Y.

Theorem 2. *With X and G as above, assume that each fixed-point set X^H has finitely generated integral homology. Then Y and each Y_H have finitely generated homology and the virtual character L associated to the action of G on $H_*(X)$ is given by*

$$L=\chi(Y)\cdot R-\sum_{H\in\Phi}\chi(Y_H)\cdot a_H,$$

[1] The interested reader can translate the results of this section into the topological context. Instead of finite dimensional semi-simplicial complexes, one considers paracompact spaces of finite cohomological dimension.

where Φ is a set of representatives for the conjugacy classes of nontrivial subgroups of G which occur as isotropy groups in X. In other words,

(a) $\chi(X)=|G|\cdot\chi(Y)-\sum_{H\in\Phi}(|G|-(G:H))\chi(Y_H)$ *and*

(b) $L(s)=\sum_{H\in\Phi}\chi(Y_H)b_H(s)$ *for* $s\neq 1$.

(As in §1, L is the alternating sum of the characters of the $\mathbb{Q}[G]$-modules $H_i(X,\mathbb{Q})$ and R is the regular character of G.)

There is also a mod p version of Theorem 2 (for which the hypotheses can be weakened) which is due to Floyd in case $G=\mathbb{Z}/p\mathbb{Z}$ (cf. [6], Chapter III, Theorem 4.3):

Theorem 2′. *Assume that G is a p-group for some prime p and that X has finite* mod *p homology. Then Y and each Y_H have finite* mod *p homology and the formula of Theorem* 2(a) *holds.*

(It is understood here that all Euler characteristics are to be computed from mod p homology.)

Corollary. *Let X be a finite dimensional G-complex (with G an arbitrary finite group) and X′ a G-invariant subcomplex, and assume that X and X′ have finitely generated integral homology. If d is an integer which divides the cardinality of every orbit which occurs in $X-X'$, then d divides $\chi(X)-\chi(X')$.*

Assuming that the formula of Theorem 2(a) applies to both X and X', the corollary follows from the fact that, by hypothesis, d can only fail to divide $(G:H)$ if $X_H=X'_H$. In particular, this proves the corollary if G is a p-group for some prime p, since Theorem 2′ applies; the general case follows easily by passage to the Sylow subgroups of G.

Proof of Theorem 2. The proof is based on the decomposition of X according to orbit type:

$$X=\bigcup_{H\in\Phi^+}G\cdot X_H, \tag{1}$$

where $\Phi^+=\Phi\cup\{1\}$. We can now use the additivity of the Euler characteristic to deduce

$$\chi(X)=\sum_{H\in\Phi^+}\chi(G\cdot X_H), \tag{2}$$

provided each $G\cdot X_H$ has finitely generated homology. Similarly, we find that

$$L=\sum_{H\in\Phi^+}L_H, \tag{2'}$$

where L_H is the virtual character of G associated to the action of G on $H_*(G\cdot X_H)$.

(To justify the additivity assertion, it suffices to show that the subsets $G\cdot X_H$ are the successive differences F_i-F_{i-1} in a filtration of X by

G-invariant subcomplexes,

$$\emptyset = F_0 \subset F_1 \subset \cdots \subset F_n = X.$$

Such a filtration is easy to find: We order the subgroups $H_1, \ldots, H_n$ in Φ^+ in such a way that $|H_i| \geqq |H_{i+1}|$, and we set

$$F_i = \bigcup_{j \leqq i} G \cdot X^{H_j}.)$$

We now verify that $G \cdot X_H$ has finitely generated homology and we compute $\chi(G \cdot X_H)$ and L_H. In the first place, note that $G \cdot X_H$ is the G-set *induced* from the $N(H)$-set X_H, where $N(H)$ is the normalizer of H in G, i.e.,

$$G \cdot X_H = \coprod_{\alpha \in G/N(H)} \alpha \cdot X_H.$$

Moreover, since $X_H = X^H \cap G \cdot X_H$, X_H is a relative subcomplex of $G \cdot X_H$, in the sense that if a simplex of X_H has a face in $G \cdot X_H$, then that face is also in X_H. It follows easily that

$$H_*(G \cdot X_H) = \bigoplus_{\alpha \in G/N(H)} \alpha \cdot H_*(X_H). \tag{3}$$

Now $H_*(X_H)$ is clearly finitely generated, since

$$H_*(X_H) = H_*(X^H, \bigcup_{H' \supsetneqq H} X^{H'})$$

and since each fixed-point set has finitely generated homology. (Note that the family $\{X^{H'}\}$ is closed under intersection.) Therefore $H_*(G \cdot X_H)$ is indeed finitely generated and (3) implies that

$$\chi(G \cdot X_H) = (G : N(H)) \cdot \chi(X_H)$$

and that L_H is the virtual character of G induced from the virtual character of $N(H)$ associated to the action of $N(H)$ on $H_*(X_H)$.

Observe now that $N(H)/H$ acts freely on X_H, so we may apply Theorem 1 (§ 1) to the normalized chain complex of X_H. We deduce that Y_H has finitely generated homology, that $\chi(X_H) = (N(H):H) \cdot \chi(Y_H)$, and that the virtual character of the action of $N(H)$ on $H_*(X_H)$ is $\chi(Y_H) \cdot R_{N(H)/H}$, where $R_{N(H)/H}$ is the regular character of $N(H)/H$, regarded as a character of $N(H)$. Since the character of G induced from $R_{N(H)/H}$ is precisely b_H, the results of the preceding paragraph can be restated as $\chi(G \cdot X_H) = (G:H) \cdot \chi(Y_H)$ and $L_H = \chi(Y_H) \cdot b_H$, so that (2) and (2′) become

$$\chi(X) = \sum_{H \in \Phi^+} (G:H) \cdot \chi(Y_H) \tag{4}$$

and

$$L = \sum_{H \in \Phi^+} \chi(Y_H) \cdot b_H. \tag{4'}$$

Evaluating (4′) at any $s \neq 1$, we obtain the formula (b) of the theorem. In order to obtain (a), we need to eliminate from (4) the term corresponding to $H=1$. Consider the decomposition of Y induced by (1):

$$Y = \bigcup_{H \in \Phi^+} Y_H .$$

Arguing as in the derivation of (2), and using the already established result that $H_*(Y_H)$ is finitely generated, we find that $H_*(Y)$ is finitely generated and that

$$\chi(Y) = \sum_{H \in \Phi^+} \chi(Y_H). \tag{5}$$

Multiplying (5) by $|G|$ and subtracting the result from (4), we obtain (a).

Proof of Theorem 2′. The proof is identical to that of Theorem 2(a), with two modifications: (i) All homology groups are to be understood as having coefficients in $\mathbb{Z}/p\mathbb{Z}$, and Theorem 1′ is used instead of Theorem 1. (ii) Since we no longer *assume* that each X^H has finite (mod p) homology, we need to observe that this follows from the hypothesis; this is a well-known consequence of Smith theory (cf. [6], Chapter III). [It is also a consequence of equivariant homology theory, cf. [17], Corollary 4.3.]

Remarks. 1. Verdier [23] has pointed out that in case G is cyclic, the characters R and a_H $(H \neq 1)$ are linearly independent, so that Theorem 2 enables one to determine $\chi(Y)$ and each $\chi(Y_H)$, provided one knows L.

2. The chain complex analogue of the corollary to Theorems 2 and 2′ is false. More precisely, suppose C is a finite dimensional chain complex of $\mathbb{Z}[G]$-modules such that $H_*(C)$ is finitely generated and each C_i is a direct sum of modules of the form $\mathbb{Z}[G/H]$, with $(G:H)$ divisible by an integer d. Then $\chi(C)$ need not be divisible by d. Counter-examples exist with $G = \mathbb{Z}/4\mathbb{Z}$, for example.

II. Euler Characteristics of Groups

§ 3. Preliminaries: Finiteness Conditions

Let Γ be a group. Γ is said to have *finite cohomological dimension* (written $\operatorname{cd}\Gamma < \infty$) if the $\mathbb{Z}[\Gamma]$-module $\mathbb{Z}$ (with trivial Γ-action) admits a resolution

$$0 \to P_n \to \cdots \to P_0 \to \mathbb{Z} \to 0,$$

where each P_i is a projective $\mathbb{Z}[\Gamma]$-module. In case each P_i can be taken to be finitely generated (resp. finitely generated and free), we say that Γ is of *type* (FP) (resp. (FL)).

If Γ has a subgroup Γ' of finite index such that $\operatorname{cd}\Gamma' < \infty$, then we say that Γ has *virtually finite cohomological dimension* (written $\operatorname{vcd}\Gamma < \infty$); in this case, according to a result of Serre's ([18], No. 1.7, Theorem 1) the subgroups of Γ with finite cohomological dimension are precisely the *torsion-free* subgroups.

Similarly, if Γ has a subgroup of finite index which is of type (FP) (resp. (FL)), then we say that Γ is of *type* (VFP) (resp. (VFL)). In case Γ is of type (VFP), it is easy to see that every torsion-free subgroup of finite index has type (FP). It is not known whether groups of type (VFL) have the analogous property.

For most of the results of the present paper, it will be sufficient to consider the following finiteness condition, which is certainly satisfied by all groups of type (VFP):

Definition. Γ is said to have *finite homological type* if

(i) $\operatorname{vcd} \Gamma < \infty$ and

(ii) every torsion-free subgroup of finite index has finitely generated integral homology.

Example. If Γ is an arithmetic subgroup of $G(k)$, where G is a linear algebraic group over a number field k, then Γ has finite homological type, and, in fact, every torsion-free subgroup of finite index has type (FL). Special cases of this result are given in [18] and [19], and the general case follows from the Borel-Serre [4] construction of a contractible differentiable manifold with corners on which Γ acts properly, with compact quotient. In case G is reductive, S-arithmetic subgroups of $G(k)$ also have the property that every torsion-free subgroup of finite index is of type (FL).

For future reference we state an elementary fact, which follows easily from the Hochschild-Serre spectral sequence:

Proposition 1. *Let Γ be a group and Γ' a subgroup of finite index. Then Γ has finite homological type if and only if Γ' has finite homological type.*

§ 4. Definition and Multiplicative Properties of $\chi(\Gamma)$

If Γ is a group such that $H_*(\Gamma, \mathbb{Q})$ has finite rank over $\mathbb{Q}$, we set

$$\tilde{\chi}(\Gamma) = \sum (-1)^i \dim_{\mathbb{Q}} H_i(\Gamma, \mathbb{Q}).$$

Theorem 3. *Let Γ be a group of finite cohomological dimension and let Γ' be a normal subgroup of finite index with $H_*(\Gamma', \mathbb{Z})$ finitely generated. Then*

$$\tilde{\chi}(\Gamma') = (\Gamma : \Gamma') \cdot \tilde{\chi}(\Gamma).$$

Moreover, the virtual character associated to the action of Γ/Γ' on $H_*(\Gamma')$ is equal to $\tilde{\chi}(\Gamma) \cdot R$.

Let (P_i) be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[\Gamma]$ with $P_i=0$ for large i, and let C be the complex of $\mathbb{Z}[\Gamma/\Gamma']$-modules defined by

$$C_i=(P_i)_{\Gamma'}=\mathbb{Z}\bigotimes_{\mathbb{Z}[\Gamma']} P_i.$$

Then $H_*(\Gamma')=H_*(C)$ and $H_*(\Gamma)=H_*(C_G)$, where $G=\Gamma/\Gamma'$; the theorem is now immediate from Theorem 1 (§ 1).

Recall from [18], No. 1.8, that the *Euler characteristic* $\chi(\Gamma)$ is defined for groups Γ of type (VFL), and is characterized by the following two properties:

(i) *If Γ' is a subgroup of Γ of finite index, then*

$$\chi(\Gamma')=(\Gamma:\Gamma')\cdot\chi(\Gamma).$$

(ii) *If Γ is of type* (FL) *then*

$$\chi(\Gamma)=\sum(-1)^i \operatorname{rank}_{\mathbb{Z}[\Gamma]} F_i,$$

where (F_i) is a finite free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\Gamma]$.

(Note that we also have $\chi(\Gamma)=\tilde{\chi}(\Gamma)$ in the situation of (ii).)

I claim now that we can extend the definition of $\chi(\Gamma)$ to groups Γ of finite homological type in such a way that property (i) continues to hold, as well as:

(ii′) *If Γ is torsion-free then $\chi(\Gamma)=\tilde{\chi}(\Gamma)$.*

In fact, if Γ is a group of finite homological type and Γ' is a torsion-free subgroup of finite index, then we set

$$\chi(\Gamma)=\frac{\tilde{\chi}(\Gamma')}{(\Gamma:\Gamma')},$$

and it follows from Theorem 3 that this is independent of the choice of Γ'. It is clear that (i) and (ii′) hold and that this definition agrees with that of [18] in case Γ is of type (VFL). In particular, *we have obtained an affirmative answer to Serre's question* ([18], p. 101) *as to whether $\chi(\Gamma)$ is integral if Γ is torsion-free* and of type (VFL).

We turn now to a related question asked in [18], p. 101. Let Γ be a group of type (FP) and suppose that Γ operates on a finite dimensional vector space V over a field K, which, for simplicity, we assume to be of characteristic zero. If Γ' is a normal subgroup of finite index, then Γ/Γ' acts on $H^i(\Gamma', V)$ and we may form the alternating sum of characters, which we denote L_V. (The (FP) property guarantees that $H^*(\Gamma', V)$ is finite dimensional.) Serre asks whether L_V is a multiple of the regular character R of Γ/Γ', and again the answer is yes:

Theorem 4. *With the above hypotheses and notation,*

$$L_V = \chi(\Gamma)\cdot \dim_K(V)\cdot R.$$

Let (P_i) be a finite projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[\Gamma]$; then L_V can be computed on the chain complex level from the complex $\mathrm{Hom}_{\Gamma'}(P_{\bullet}, V)$ of $K[\Gamma/\Gamma']$-modules. The theorem now follows from Theorem 7 of Appendix A, provided we verify that Γ is finitely generated. Now the property (FP) implies that the augmentation ideal I of $\mathbb{Z}[\Gamma]$ is a finitely generated left ideal; consequently, it is generated by finitely many elements of the form $s-1$ $(s\in\Gamma)$, and I claim that the elements s which arise in this way generate Γ. In fact, let Γ_0 be the subgroup which they generate, and let $x\in\Gamma/\Gamma_0$ be the coset of the identity. Then Γ_0 fixes x, hence I annihilates x in the $\mathbb{Z}[\Gamma]$-module $\mathbb{Z}[\Gamma/\Gamma_0]$; but this implies that Γ fixes x, so $\Gamma=\Gamma_0$ and the claim is verified.

Corollary 1. *Let Γ be a group of type* (FP) *and let V be a $K[\Gamma]$-module as above. Then*

$$\sum(-1)^i \dim_K H^i(\Gamma, V) = \chi(\Gamma)\cdot\dim_K(V).$$

(This is the case $\Gamma=\Gamma'$ of Theorem 4.)

Corollary 2. *Let*

$$1\to\Gamma'\to\Gamma\to\Gamma''\to 1$$

be a short exact sequence of groups. If Γ' and Γ'' are of type (VFP) *and Γ is virtually torsion-free, then Γ is of type* (VFP) *and*

$$\chi(\Gamma)=\chi(\Gamma')\cdot\chi(\Gamma'').$$

We may assume that Γ is torsion-free, in which case Γ' is of type (FP); we may also assume that Γ'' is of type (FP). Using an argument similar to (but slightly more complicated than) that of [18], No. 1.3, proof of Proposition 6, one can then show that Γ is of type (FP), and the Euler characteristic formula follows from the Hochschild-Serre spectral sequence

$$H^p(\Gamma'', H^q(\Gamma', \mathbb{Q}))\Rightarrow H^{p+q}(\Gamma, \mathbb{Q}),$$

together with Corollary 1.

§ 5. Equivariant Euler Characteristics

Let Γ be a group which operates on a simplicial complex K, and assume (a) that K has only finitely many simplices modulo the action of Γ, and (b) that the isotropy group Γ_σ of every simplex σ of K has finite homological type. We can then choose a (finite) set Σ of representatives for the simplices of K modulo Γ, and we set

$$\chi_\Gamma(K)=\sum_{\sigma\in\Sigma}(-1)^{\dim\sigma}\chi(\Gamma_\sigma).$$

We call $\chi_\Gamma(K)$ the *equivariant Euler characteristic* of the Γ-complex K. [This concept is introduced implicitly by Serre ([18], Proposition 14(b)), where it is shown that $\chi_\Gamma(K)=\chi(\Gamma)$ if K is acyclic.] In case K has a subcomplex K' invariant under Γ, we define the *relative equivariant Euler characteristic* $\chi_\Gamma(K, K')$ by a similar sum, where Σ is now taken to be a set of representatives for the simplices of $K-K'$ modulo Γ; note that

$$\chi_\Gamma(K, K')=\chi_\Gamma(K)-\chi_\Gamma(K').$$

We will be interested in the case where K arises from an action of Γ on a partially ordered set S, i.e., K is the complex $K(S)$ whose vertices are the elements of S and whose n-simplices are the linearly ordered subsets of S of cardinality $n+1$. We set

$$\chi_\Gamma(S)=\chi_\Gamma(K(S)),$$

provided the latter is defined. Similarly, if S' is a Γ-invariant subset of S, we set

$$\chi_\Gamma(S, S')=\chi_\Gamma(K(S), K(S')).$$

Throughout the remainder of this section, we will assume that S is a partially ordered Γ-set satisfying the following three finiteness conditions:

(i) *Every element of S has only finitely many predecessors.*

(ii) *The orbit set S/Γ is finite.*

(iii) *The isotropy group Γ_s of any element s of S has finite homological type.*

The following lemma shows that $\chi_\Gamma(S)$ is defined under these hypotheses:

Lemma. (a) *$K(S)$ has only finitely many simplices modulo the action of Γ.*

(b) *For each simplex σ of $K(S)$, the isotropy group Γ_σ has finite homological type.*

By (ii) we may choose a finite set F of representatives for the elements of S modulo Γ, and we note that every simplex of $K(S)$ is equivalent to one whose largest vertex is in F; but there are only finitely many such simplices by (i), whence (a). Assertion (b) follows from (iii) together with the observation that, by (i), Γ_σ has finite index in Γ_s, where s is the largest vertex of σ.

Here are some elementary properties of the equivariant Euler characteristic:

Proposition 2. (i) *If Γ' is a subgroup of Γ of finite index, then*

$$\chi_{\Gamma'}(S)=(\Gamma:\Gamma')\,\chi_\Gamma(S).$$

(ii) *The equivariant Euler characteristic is compatible with induction. In other words, if S has a subset T invariant under the action of a subgroup N of Γ, and if*

$$S = \coprod_{\alpha \in \Gamma/N} \alpha \cdot T$$

(as an ordered set), then $\chi_N(T)$ is defined and

$$\chi_\Gamma(S) = \chi_N(T).$$

(iii) *If S has a largest or smallest element then*

$$\chi_\Gamma(S) = \chi(\Gamma).$$

(iv) *If Γ has a finite normal subgroup H which acts trivially on S, then $\chi_{\Gamma/H}(S)$ is defined and is equal to $|H| \cdot \chi_\Gamma(S)$.*

A proof of (i) is contained in the proof of Proposition 14(b) of No. 1.8 of [18]. Property (ii) is immediate from the definition. Property (iii) follows from the fact that, in the sum defining $\chi_\Gamma(S)$, all the terms cancel in pairs except for the term $\chi(\Gamma)$ (which corresponds to the extreme element of S, regarded as a vertex of $K(S)$). Finally, for (iv) it suffices to show that for each simplex σ of $K(S)$, Γ_σ/H has finite homological type and

$$\chi(\Gamma_\sigma/H) = |H| \cdot \chi(\Gamma_\sigma).$$

Both facts follow easily from the observation that any torsion-free subgroup of Γ_σ of finite index is isomorphic to its image in Γ_σ/H. (Note: One needs to use Proposition 1 of §3.)

It will be convenient to have a formula which expresses $\chi_\Gamma(S)$ as a sum indexed by the elements of S/Γ. For any $s \in S$, we define subsets $S(s)$ and $S'(s)$ of S by

$$S(s) = \{t \in S\colon t \geqq s\},$$

$$S'(s) = \{t \in S\colon t > s\}.$$

Note that Γ_s acts on $S(s)$ and $S'(s)$.

Proposition 3. *For each $s \in S$, the finiteness conditions* (i), (ii), (iii) *are satisfied for the action of Γ_s on $S(s)$, and*

$$\chi_\Gamma(S) = \sum_{s \in \Phi} \chi_{\Gamma_s}(S(s), S'(s)),$$

where Φ is a set of representatives for S modulo Γ.

Condition (i) holds trivially, and (ii) and (iii) follow from the above lemma, applied to the action of Γ on the set of 1-simplices of $K(S)$. To prove the formula, we choose for each s a set $\Sigma(s)$ of representatives of $K(S(s)) - K(S'(s))$ modulo Γ_s; then we can take $\Sigma = \bigcup_{s \in \Phi} \Sigma(s)$ as a set of representatives for $K(S)$ modulo Γ, and the formula follows at once from the definitions.

Finally, we give a homological interpretation of the equivariant Euler characteristic. Let (S, S') be a pair of ordered Γ-sets such that $\chi_\Gamma(S, S')$ is defined, and let $C(S, S')$ be the complex of (ordered) simplicial chains of the pair $(K(S), K(S'))$.

Proposition 4. *Suppose C is a chain complex of projective $\mathbb{Z}[\Gamma]$-modules which is weakly equivalent*[2] *to $C(S, S')$. Assume either that Γ is torsion-free or that Γ is virtually torsion-free and C is finite dimensional. Then the complex $C_\Gamma = \mathbb{Z} \otimes_{\mathbb{Z}[\Gamma]} C$ of abelian groups has finitely generated homology and*

$$\chi_\Gamma(S, S') = \chi(C_\Gamma).$$

In case Γ is virtually torsion-free and C is finite dimensional, let Γ' be a torsion-free normal subgroup of finite index. Then Theorem 1 of §1 implies that C_Γ will have finitely generated homology if $C_{\Gamma'}$ does and that $\chi(C_{\Gamma'}) = (\Gamma : \Gamma')\,\chi(C_\Gamma)$. Comparing this with Proposition 2(i), we see that we may replace Γ by Γ', i.e., we may assume Γ is torsion-free. Furthermore, we may replace C by any other complex satisfying the hypotheses, since any two such complexes are homotopy equivalent. In particular, we may assume that C is the total tensor product $P \otimes_{\mathbb{Z}} C(S, S')$, where $P = (P_i)$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[\Gamma]$. Then $C_\Gamma = P \otimes_{\mathbb{Z}[\Gamma]} C(S, S')$ and hence the spectral sequences of the double complex $P_\cdot \otimes_{\mathbb{Z}[\Gamma]} C_\cdot(S, S')$ converge to $H_*(C_\Gamma)$. One of these spectral sequences has $E^1_{pq} = H_q(\Gamma, C_p(S, S'))$. If Σ_p is a set of representatives for the p-simplices of $K(S) - K(S')$ modulo Γ, then

$$C_p(S, S') = \bigoplus_{\sigma \in \Sigma_p} \mathbb{Z}[\Gamma/\Gamma_\sigma],$$

so we have

$$E^1_{pq} = \bigoplus_{\sigma \in \Sigma_p} H_q(\Gamma_\sigma, \mathbb{Z}).$$

Since each Γ_σ is torsion-free and of finite homological type, the proposition follows immediately.

Remark. The spectral sequence used in the proof is, essentially, the spectral sequence given by Quillen in [17], I, (1.11).

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[2] Two chain complexes are called *weakly equivalent* if there is a third complex which maps to both of them by maps inducing homology isomorphisms. (In the present case, since C is assumed to be a complex of projectives, this implies that there is actually a map $C \to C(S, S')$ inducing an isomorphism in homology.)

§6. *The Finite Subgroups of Γ*

Let Γ be a group of finite homological type and let S be a set of non-trivial finite subgroups. We assume that S is invariant under conjugation, so that S is an ordered Γ-set (ordered by inclusion). Assume further that the following three conditions are satisfied:

(i) *S contains only finitely many conjugacy classes of subgroups.*

(ii) *For each $H \in S$ the normalizer $N(H)$ of H in Γ has finite homological type.*

(Note that (i) and (ii) imply that $\chi_\Gamma(S)$ is defined.)

(iii) *If two members H, H' of S generate a finite subgroup of Γ, then they have an upper bound in S.*

Under these conditions we have:

Theorem 5. *Let m be the least common multiple of the orders of the finite subgroups of Γ which do not contain any member of S as a subgroup. Then $m \cdot (\chi(\Gamma) - \chi_\Gamma(S))$ is an integer.*

Taking $S = \emptyset$, we obtain:

Corollary 1. *If Γ is a group of finite homological type and m is the least common multiple of the orders of the finite subgroups of Γ, then $m \cdot \chi(\Gamma)$ is an integer. Equivalently, if q is a prime power which divides the denominator of $\chi(\Gamma)$, then Γ has a subgroup of order q.*

We remark that Corollary 1 makes precise a result of Serre's ([18], No. 1.8, Proposition 13), which says that $m^n \cdot \chi(\Gamma) \in \mathbb{Z}$ for some integer n, or, equivalently, that Γ has p-torsion for every prime p which divides the denominator of $\chi(\Gamma)$.

Considering now the other extreme, where S is the set of *all* non-trivial finite subgroups, we obtain:

Corollary 2. *If Γ is a group of finite homological type such that conditions* (i) *and* (ii) *hold for the set S of all non-trivial finite subgroups of Γ, then $\chi(\Gamma) - \chi_\Gamma(S)$ is an integer.*

Thus $\chi(\Gamma)$ and $\chi_\Gamma(S)$ have the same "fractional part".

Finally, we apply Theorem 5 to the set S_p of all finite subgroups of order divisible by a prime p:

Corollary 3. *If Γ is a group of finite homological type such that conditions* (i) *and* (ii) *hold for the set S_p, then $\chi(\Gamma) - \chi_\Gamma(S_p)$ is p-integral.*

Thus the "p-fractional part" of $\chi(\Gamma)$ (i.e., that part of the partial fractions decomposition of $\chi(\Gamma)$ with denominator a power of p) is the same as the p-fractional part of $\chi_\Gamma(S_p)$.

Remark. Conditions (i) and (ii) hold for any S if Γ is as in the example of §3. In fact, (i) follows from the Borel-Serre [4] action with compact quotient, or, in the arithmetic case, from results of Borel and Harish-Chandra, cf. [3], p. 14; and (ii) follows from the fact that if Γ is a group of one of the two types mentioned in the example of §3, then each $N(H)$ is of the same type. (The point is that if G is a reductive group over k then the centralizer of any finite subgroup of $G(k)$ is again reductive.)

The proof of Theorem 5 is based on a construction due to Serre ([18], No. 1.7):

Lemma. *Let Γ be a group of virtually finite cohomological dimension. Then there exists an acyclic finite dimensional semi-simplicial complex Z on which Γ operates, such that the following two conditions are satisfied:*

(i) *The isotropy group of every simplex is finite.*

(ii) *For every finite subgroup H of Γ, the fixed-point subcomplex Z^H is acyclic.*

We briefly recall the construction. Let Γ' be a subgroup of finite index such that $\operatorname{cd}\Gamma' < \infty$, and let W be an acyclic finite dimensional semi-simplicial complex on which Γ' acts freely. The Γ-complex Z is then obtained from the Γ'-complex W by a "multiplicative induction" (see [18], No. 1.7, or [17], II, §16). Property (i) is proven in [18], and property (ii) follows from the fact that Z, as an H-complex, is simply the cartesian product of $(\Gamma:\Gamma')$ copies of W, with H acting by permuting the factors according to the (free) action of H on Γ/Γ'; thus Z^H is isomorphic to the product of $(\Gamma:\Gamma')/|H|$ copies of W, hence is acyclic.

Proof of Theorem 5. Let Z be as in the lemma and let $Z' = \bigcup_{H\in S} Z^H$; Z' is a Γ-invariant subcomplex of Z. Let Γ' be a torsion-free normal subgroup of Γ of finite index, let $X = Z/\Gamma'$ and $X' = Z'/\Gamma'$, and consider the action of $G = \Gamma/\Gamma'$ on the pair (X, X'). One checks easily that if $\sigma \in Z$ and $\bar\sigma$ is its image in X, then the isotropy group $G_{\bar\sigma}$ is the image in G of Γ_σ. In particular, if $\bar\sigma \notin X'$, then $G_{\bar\sigma}$ has order a divisor of m, hence the orbit of $\bar\sigma$ has cardinality divisible by $d = |G|/m$. Using the corollary of Theorems 2 and 2′ (§2), we conclude that

$$m\cdot(\chi(X)-\chi(X'))/|G|$$

is an integer, provided X and X' have finitely generated homology. Now $H_*(X) = H_*(\Gamma')$, so $H_*(X)$ is indeed finitely generated and $\chi(X) = \chi(\Gamma')$. Thus $\chi(X)/|G| = \chi(\Gamma)$, and we will complete the proof by showing that X' has finitely generated homology and that $\chi(X')/|G| = \chi_\Gamma(S)$.

Let $C = C(Z')$ be the chain complex of Z'. Since Z' is the union of the acyclic complexes Z^H $(H \in S)$, it is well-known (see Appendix B) that C is weakly equivalent to $C(S)$. (This is where the condition (iii) on S is used.) Note that the equivalence is compatible with the action of Γ. Since $C(X')=C_{\Gamma'}$, and since C is a complex of free $\mathbb{Z}[\Gamma']$-modules, Proposition 4 of §5 implies that $H_*(X')$ is finitely generated and that $\chi(X')=\chi_{\Gamma'}(S)$; in view of Proposition 2 (i) of §5, this is precisely the desired result.

We end this section by examining the situation of Corollary 2 in more detail. Using Proposition 3 of §5, we can rewrite the conclusion of that corollary as follows:

$$\chi(\Gamma)=\sum_{H\in\Phi}\chi_{N(H)}(S(H),S'(H))+\text{integer},$$

where Φ is a set of representatives for the conjugacy classes of non-trivial finite subgroups of Γ and $S(H)$ (resp. $S'(H)$) is the set of finite subgroups containing (resp. properly containing) H. Our next result makes this formula more precise.

We need some notation. If H is a finite subgroup of Γ, then H acts trivially on $S(H)$, and we set

$$n_H=\chi_{N(H)/H}(S(H),S'(H)).$$

Note that the term $\chi_{N(H)}(S(H),S'(H))$ which occurs in the above sum is equal to $n_H/|H|$ by Proposition 2 (iv) of §5. Let Γ' be a torsion-free normal subgroup of Γ of finite index; then we can identify H with a subgroup of Γ/Γ', and we denote by a_H the character of Γ/Γ' induced from the augmentation character of H, cf. §2. Finally, let L be the virtual character of Γ/Γ' associated to the action of Γ/Γ' on $H_*(\Gamma')$.

Theorem 6. *Let Γ be as in Corollary 2 of Theorem 5 and let Γ' be a torsion-free normal subgroup of finite index. Then, with the above notation, the rational number n_H is an integer for each H and*

$$L=\tilde{\chi}(\Gamma)\cdot R-\sum_{H\in\Phi}n_H a_H.$$

In particular,

$$\chi(\Gamma)=\tilde{\chi}(\Gamma)-\sum_{H\in\Phi}n_H\left(1-\frac{1}{|H|}\right).$$

The second formula follows from the first by evaluating both sides at 1 and dividing by $(\Gamma:\Gamma')$. To prove the first formula, we will again consider the action of $G=\Gamma/\Gamma'$ on $X=Z/\Gamma'$ with Z as in the lemma. The virtual character L which we are trying to compute is the virtual character of G associated to the action of G on $H_*(X)$, and we will compute this by means of Theorem 2 of §2.

We must check that $H_*(X^D)$ is finitely generated for every subgroup D of G. Letting Φ'_D be a (finite) set of representatives for the Γ'-conjugacy classes of finite subgroups of Γ lying over D, and letting $f\colon Z \to X$ be the projection, one verifies easily that

$$X^D = \coprod_{H \in \Phi'_D} f(Z^H) \quad \text{and} \quad f(Z^H) = Z^H/N'(H),$$

where $N'(H) = N(H) \cap \Gamma'$. (For the purpose of verifying these equations, one can replace Γ by the inverse image of D in Γ; then H is the isotropy group of each $z \in Z^H$, and the assertions are obvious.) It follows that

$$H_*(X^D) = \bigoplus_{H \in \Phi'_D} H_*(N'(H)),$$

which is indeed finitely generated in view of the hypotheses on Γ.

Thus Theorem 2 applies and yields the formula

$$L = \chi(Y) \cdot R - \sum_{D \in \Psi} \chi(Y_D)\, a_D,$$

where $Y = X/G = Z/\Gamma$ and Ψ is a set of representatives for the conjugacy classes of non-trivial subgroups D of G. Let Φ_D be the subset of Φ consisting of those members of Φ whose image in G is conjugate to D. Then one sees easily that

$$Y_D = \coprod_{H \in \Phi_D} Y_H,$$

where Y_H is the image in Y of Z_H, the latter being the set of simplices in Z whose isotropy group is H. Moreover, each Y_H is a relative subcomplex of Y_D, so

$$\chi(Y_D) = \sum_{H \in \Phi_D} \chi(Y_H).$$

Since $\Phi = \coprod_{D \in \Psi} \Phi_D$, the above formula for L becomes

$$L = \chi(Y) \cdot R - \sum_{H \in \Phi} \chi(Y_H)\, a_H,$$

and we will complete the proof by showing that $\chi(Y) = \tilde{\chi}(\Gamma)$ and $\chi(Y_H) = n_H$.

The first equality follows at once from the fact that, since the chain complex $C(Z, \mathbb{Q})$ is a projective resolution of $\mathbb{Q}$ over $\mathbb{Q}[\Gamma]$, $H_*(Y, \mathbb{Q}) = H_*(\Gamma, \mathbb{Q})$. For the second equality, note that Z_H is the difference of two unions of acyclic complexes:

$$Z_H = \bigcup_{H' \in S(H)} Z^{H'} - \bigcup_{H' \in S'(H)} Z^{H'}.$$

We conclude as in the proof of Theorem 5 that the normalized chain complex $C(Z_H)$ is equivalent to $C(S(H), S'(H))$. Now $C(Z_H)$ is a finite dimensional complex of free $\mathbb{Z}[N(H)/H]$-modules and $C(Y_H) = C(Z_H)_{N(H)/H}$, so Proposition 4 of §5 implies that

$$\chi(Y_H) = \chi_{N(H)/H}(S(H), S'(H)) = n_H,$$

as required.

Corollary. *If every non-trivial finite subgroup of Γ is contained in a unique maximal finite subgroup, then*

$$\chi(\Gamma)=\tilde{\chi}(\Gamma)-\sum_{H\in\Phi'}\left(1-\frac{1}{|H|}\right)\chi(N(H)/H),$$

where Φ' is a set of representatives for the conjugacy classes of maximal finite subgroups of Γ.

Note first that $S(H)$ (for any H) has a smallest element, so we can write (by Proposition 2 (iii) of § 5)

$$n_H=\chi(N(H)/H)-\chi_{N(H)/H}(S'(H)).$$

In particular, $n_H=\chi(N(H)/H)$ if H is maximal. If, on the other hand, H is not maximal (and non-trivial), then the hypothesis implies that $S'(H)$ has a largest element, so $n_H=0$ and the corollary follows immediately.

This applies for example, if Γ is a Fuchsian group (finitely generated discrete subgroup of $SL_2(\mathbb{R})/\{\pm 1\}$). In this case, moreover, $N(H)=H$ for H maximal, so we obtain

$$\chi(\Gamma)=\tilde{\chi}(\Gamma)-\sum_{H\in\Phi'}\left(1-\frac{1}{|H|}\right),$$

which is a classical formula. (See, for example, [21], Theorem 2.20, for the case of Fuchsian groups of the first kind.)

Remark. In case it is assumed that each $N(H)$ has type (VFP), it is possible to generalize Theorem 6 to a formula for the virtual character L_V associated to a representation of Γ as in Theorem 4 of § 4. The formula is

$$L_V=\tilde{\chi}(\Gamma)\cdot\dim_K(V)\cdot R-\sum_{H\in\Phi}n_H a_{H,V};$$

here $a_{H,V}$ is the character of the representation of Γ/Γ' induced from the representation $I_H\otimes V$ of H, where I_H is the augmentation ideal of $K[H]$.

§ 7. *The p-Fractional Part of $\chi(\Gamma)$*

Throughout this section p will be a fixed prime number and Γ will be a group satisfying the hypotheses of Corollary 3 of Theorem 5 (§ 6). The purpose of this section is to give an approximation to the p-fractional part of $\chi(\Gamma)$, or, equivalently, to the p-fractional part of $\chi_\Gamma(S)$, where we now denote by S the set of finite subgroups of Γ of order divisible by p. We begin with a special case.

Suppose Γ has the property that every $H\in S$ has a unique subgroup of order p. (Every $H\in S$ might be cyclic, for example.) Then, as an ordered set, $S=\coprod S(P)$, where P ranges over the subgroups of Γ of order p and $S(P)$ is the set of finite subgroups of Γ containing P. We can then compute $\chi_\Gamma(S)$ by means of parts (ii) and (iii) of Proposition 2 of § 5; letting Φ be a set of representatives for the conjugacy classes of subgroups of order p of Γ, we obtain:

$$\chi_\Gamma(S)=\sum_{P\in\Phi}\chi_{N(P)}(S(P))=\sum_{P\in\Phi}\chi(N(P)).$$

Thus the p-fractional part of $\chi(\Gamma)$ is the same, in this special case, as that of $\sum_{P\in\Phi}\chi(N(P))$.

In the general case, we will show that this is still approximately true. More precisely, let q be a positive power of p such that $q\cdot\chi(N(H))$ is p-integral for each $H\in S$; for example, in view of Corollary 1 of Theorem 5 of § 6, we can take q to be the maximal order of a p-subgroup of Γ (except in the trivial case where Γ has no p-torsion). Then we have:

Proposition 5. *With q as above, $q\cdot\chi(\Gamma)$ is p-integral and*

$$q\cdot\chi(\Gamma)\equiv q\cdot\sum_{P\in\Phi}\chi(N(P))\pmod p.$$

(The congruence implies that the term of the form a/q $(0\leqq a<p)$ in the partial fractions decomposition of $\chi(\Gamma)$ is the same as the corresponding term in the decomposition of $\sum_{P\in\Phi}\chi(N(P))$.)

Remark. It is sometimes convenient to rewrite the above sum:

$$\sum_{P\in\Phi}\chi(N(P))=\frac{1}{p-1}\sum_{\alpha\in\Phi'}\chi(C(\alpha)),$$

where Φ' is a set of representatives for the conjugacy classes of elements of Γ of order p and $C(\alpha)$ is the centralizier of α in Γ. The equality of the two sums follows from the following lemma, applied with A' equal to the set of elements of Γ of order p and A equal to the set of subgroups of Γ of order p:

Lemma. *Let A and A' be sets on which Γ operates, and let Φ (resp. Φ') be a set of representatives for A (resp. A') modulo Γ. If $f\colon A'\to A$ is a map of Γ-sets such that $f^{-1}(a)$ is finite for each $a\in A$, then*

$$\sum_{a\in\Phi'}\chi(\Gamma_{a'})=\sum_{a\in\Phi}\operatorname{card}(f^{-1}(a))\cdot\chi(\Gamma_a),$$

provided Φ is finite and each $\chi(\Gamma_a)$ is defined.

We may choose Φ' to be related to Φ as follows: For each $a\in\Phi$, let I_a be a set of representatives for the elements of $f^{-1}(a)$ modulo Γ_a;

we then take $\Phi' = \bigcup_{a \in \Phi} I_a$. Thus

$$\sum_{a' \in \Phi'} \chi(\Gamma_{a'}) = \sum_{a \in \Phi} \sum_{a' \in I_a} \chi(\Gamma_{a'});$$

the lemma is now obtained by writing

$$\chi(\Gamma_{a'}) = (\Gamma_a : \Gamma_{a'})\, \chi(\Gamma_a)$$

and noting that

$$\operatorname{card}\big(f^{-1}(a)\big) = \sum_{a' \in I_a} (\Gamma_a : \Gamma_{a'}).$$

Proof of Proposition 5. The proof consists of measuring the failure of S to equal $\coprod S(P)$. More precisely, we form the ordered set $S' = \coprod S(P)$, which has an obvious action of Γ, and we consider the obvious map $f \colon S' \to S$ of ordered Γ-sets. This induces a simplicial map $K(S') \to K(S)$, and we denote by $m(\sigma)$ (for σ a simplex of $K(S)$) the number of simplices of $K(S')$ whose image in $K(S)$ is σ. Note that each such simplex has the same dimension as σ. We may therefore use the above lemma to compute that $\chi_\Gamma(S')$ is equal to

$$\sum_{\sigma \in \Sigma} (-1)^{\dim \sigma}\, m(\sigma)\, \chi(\Gamma_\sigma),$$

where Σ is a set of representatives for the simplices of $K(S)$ modulo Γ.

Now if H is the smallest vertex of a simplex σ of $K(S)$, then $m(\sigma)$ is simply the number of subgroups of H of order p, hence $m(\sigma) \equiv 1 \pmod p$, cf. [7], Chapter IX, § 121, Theorem II. Consequently, since each $q \cdot \chi(\Gamma_\sigma)$ is p-integral, the above expression for $\chi_\Gamma(S')$ yields

$$q \cdot \chi_\Gamma(S') \equiv q \cdot \chi_\Gamma(S) \pmod p.$$

On the other hand, we computed earlier (in studying the special case $S = S'$) that

$$X_\Gamma(S') = \sum_{P \in \Phi} \chi(N(P));$$

substituting this into the above congruence (and using Corollary 3 of Theorem 5), we obtain the proposition.

III. Applications

§ 8. *Group Theoretic Applications*

(a) Embeddings of Torsion-Free Groups

Proposition 6. *Let Γ be a torsion-free group of finite homological type, and suppose that Γ is a subgroup of finite index of a group Γ'.*

(i) *If Γ' is torsion-free then* $(\Gamma' : \Gamma)$ divides $\chi(\Gamma)$.

(ii) *If Γ is normal in Γ' and $(\Gamma' : \Gamma)$ is a prime power which is relatively prime to $\chi(\Gamma)$, then the extension*

$$1 \to \Gamma \to \Gamma' \to \Gamma'/\Gamma \to 1$$

splits.

Note first that Γ' has finite homological type by Proposition 1 of § 3, so $\chi(\Gamma')$ is defined. Then (i) follows from the fact that $\chi(\Gamma')$ is an integer if Γ' is torsion free, cf. § 4. For (ii), note that the hypothesis implies that the denominator of $\chi(\Gamma')$ is equal to $(\Gamma':\Gamma)$, so Γ' has a subgroup of order $(\Gamma':\Gamma)$ by Corollary 1 of Theorem 5 of § 6; Γ being torsion-free, this implies that the extension splits.

Remarks. 1. Suppose Γ is a torsion-free group of finite homological type such that $\chi(\Gamma) \neq 0$. It is reasonable to ask, in view of (i) above, whether Γ can be embedded with index $|\chi(\Gamma)|$ in a torsion-free group Γ'. Serre has given several examples to show that the answer is no, in general. In fact, such a Γ' would have $\chi(\Gamma') = \pm 1$, but Serre has shown that any torsion-free group commensurable with $SL_2(\mathbb{Z}[\frac{1}{7}])$, for example, has Euler characteristic divisible by 3 (and non-zero).

2. As an interesting example of (ii), take $\Gamma = F_n$, the free group on n generators. Then $\chi(\Gamma) = 1 - n$, so any extension of a p-group by Γ must split, unless p divides $n-1$. In case $n=2$ this can be checked directly by using the fact that the group $\mathrm{Aut}(F_2)/F_2$ of outer automorphisms of F_2 is isomorphic to $GL_2(\mathbb{Z})$ (cf. [15], § 3.5, Corollary N4); one computes the p-subgroups of $GL_2(\mathbb{Z})$ and checks that they all lift to $\mathrm{Aut}(F_2)$. For $n > 2$, however, I know of no proof other than the one given here.

(b) Torsion in Arithmetic Groups[3]

If Γ is an arithmetic group to which Harder's formula applies ([11] – see also [18], No. 3.7), then one can compute $\chi(\Gamma)$ and thereby obtain from Corollary 1 of Theorem 5 a lower bound on the amount of torsion in Γ. For example, if Γ is the group $E_7(\mathbb{Z})$ of $\mathbb{Z}$-points of a split group scheme over $\mathbb{Z}$ of type E_7, then Harder's formula yields

$$\chi(\Gamma) = -\frac{691 \cdot 43\,867}{2^{21} \cdot 3^9 \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 19},$$

so we deduce that Γ has subgroups of order 2^{21}, 3^9, etc.

§ 9. *Arithmetic Applications*

Throughout § 9, k will denote a totally real number field with ring of integers $\mathcal{O}$, and ζ_k will denote the Dedekind zeta function associated to k.

[3] This application was pointed out to me by Serre when he conjectured Corollary 1 of Theorem 5.

9.1. A Formula for $\zeta_k(-1)$[4]. The formula referred to in the title is the following:

$$\zeta_k(-1)=\frac{1}{2}\,\tilde{\chi}(\mathrm{SL}_2(\mathcal{O}))-\sum_H\left(\frac{1}{2}-\frac{1}{|H|}\right), \tag{$*$}$$

where H ranges over the maximal finite subgroups of $\mathrm{SL}_2(\mathcal{O})$, up to conjugacy.

The formula $(*)$ follows from the Corollary of Theorem 6 of § 6, applied to the group $\Gamma=\mathrm{SL}_2(\mathcal{O})/\{\pm 1\}$, whose Euler characteristic is $2\zeta_k(-1)$ by Harder's theorem [11] (see also [18], No. 3.7, Example (iv)). In order to obtain $(*)$ from that corollary, we need only show that every non-trivial finite subgroup of Γ is contained in a unique maximal finite subgroup, and that each maximal finite subgroup is equal to its own normalizer. Both of these facts follow from:

Lemma. *If H is a non-trivial finite subgroup of* $\mathrm{SL}_2(\mathcal{O})$ *distinct from* $\{\pm 1\}$, *then the centralizer $C(H)$ of H in* $\mathrm{SL}_2(\mathcal{O})$ *is finite and is the unique maximal finite subgroup of* $\mathrm{SL}_2(\mathcal{O})$ *containing H; furthermore, $C(H)=N(H)$, the normalizer of H in* $\mathrm{SL}_2(\mathcal{O})$.

Note first that every finite subgroup of $\mathrm{SL}_2(\mathcal{O})$ is cyclic (hence, in particular, abelian), since $\mathrm{SL}_2(\mathcal{O})$ can be embedded in $\mathrm{SL}_2(\mathbb{R})$. Thus all assertions of the lemma will follow if we show that $C(H)$ is finite (which implies that $N(H)$ is finite). This can be proved, for example, by interpreting $C(H)$ as a group of automorphisms of a suitable $\mathcal{O}[\zeta]$-module, where ζ is a primitive n-th root of unity, n being the order of H. We omit the details. (Note: The reader may want to refer to § 9.2 below, where a similar argument is given in detail.)

Remark. If $\mathrm{SL}_2(\mathcal{O})$ contains a subgroup of order $n>2$ (and hence an element of order n) then the field k_n obtained from k by adjoining the n-th roots of unity is quadratic over k. We therefore recover from $(*)$ (and even from Corollary 1 of Theorem 5) the following estimate of Serre's ([18], No. 3.7) for the denominator of $\zeta_k(-1)$:

Let m be the least common multiple of the integers n such that $[k_n:k]=2$. Then $m\cdot\zeta_k(-1)$ is an integer.

9.2. A Computation in the Symplectic Group. Throughout this section p will denote a fixed prime number. We will need some notation, all of which is defined relative to p:

F denotes the field $k(\zeta)$, where ζ is a primitive p-th root of unity if p is odd and a primitive fourth root of unity if $p=2$, and E denotes the maximal real subfield of F.

[4] The formula given in this section was found independently by Hirzebruch ([13], § 1).

We set $n=[E:k]$ (thus $[F:k]=2n$).

We denote by A the ring $\mathcal{O}[\frac{1}{p}]$; note that A is the ring of S-integers of k, where S is the union of the infinite primes of k and the finite primes dividing p.

B (resp. C) denotes the integral closure of A in E (resp. F).

We define integers a, h_S^-, and w, as follows: 2^a is the order of the cokernel of the norm map $N: C^* \to B^*$; h_S^- is the order of the kernel of the norm map from the ideal class group of C to that of B; and w is the number of roots of unity in F.

Finally, we denote by Γ the symplectic group $\mathrm{Sp}_{2n}(A)$, where n is as above; in particular, if $n=1$ then $\Gamma=\mathrm{SL}_2(A)$.

The following result will be needed in its full generality in § 9.4. In § 9.3 we will require only the case $n=1$, in which case the proof can be considerably simplified.

Proposition 7. *If p is odd (resp. if $p=2$), then the number of conjugacy classes of elements of Γ of order p (resp. of order 4) is $(p-1)2^{a-1}h_S^-/n$ (resp. $2^a h_S^-$). Moreover, the centralizer of any such element is a finitely generated abelian group whose torsion subgroup is cyclic of order w and whose rank is equal to the number of primes of E lying over p which split in F.*

We will give the proof of Proposition 7 in the case where p is odd; the case $p=2$ requires only trivial modifications. Let α be an element of Γ of order p. Since the polynomial X^p-1 splits into one linear factor and $(p-1)/2n$ irreducible factors of degree $2n$ over k, it is clear that the minimal polynomial of α must be one of the irreducible factors, say $q(X)$. Letting ζ be a root of $q(X)$ in F, we construct an $A[\zeta]$-module M whose underlying A-module is A^{2n}, with the action of ζ defined by α. Now since p is invertible in A, one knows that $A[\zeta]=C$ (cf. [22], Corollary 4.11, for example)[5]; thus M is a C-module, and, as such, it is torsion-free and of rank 1. Moreover, α being symplectic, M has a non-degenerate skew-symmetric A-bilinear form $\langle\cdot,\cdot\rangle$, relative to which ζ acts as an isometry. The fact that ζ acts as an isometry can be expressed by the formula

$$\langle cx, y\rangle = \langle x, \bar{c}y\rangle, \tag{$*$}$$

where $x, y \in M$, $c \in C$, and $\bar{c}$ is the complex conjugate of c relative to any complex embedding of F.

It is now easy to compute the centralizer of α, which is isomorphic to the group of C-module automorphisms of M which preserve $\langle\cdot,\cdot\rangle$. In fact, any C-module automorphism of M is given by multiplication

[5] This is our only reason for dealing with the ring A instead of with $\mathcal{O}$.

by a unit u of C, and $(*)$ shows that u preserves $\langle\cdot,\cdot\rangle$ if and only if $\bar{u}=u^{-1}$, i.e., if and only if u is in the kernel of $N\colon C^*\to B^*$; thus $C(\alpha)\approx\ker N$. It follows that $C(\alpha)$ is a finitely generated abelian group whose torsion subgroup is isomorphic to the group of roots of unity in F, hence is cyclic of order w, and whose rank is equal to rank C^*-rank B^*; this difference is easily seen (using the Dirichlet unit theorem) to equal the number of primes of E lying over p which split in F, whence the second part of the proposition.

Turning now to the first part of the proposition, I claim that the construction given in the first paragraph of the proof (using a fixed ζ) sets up a 1-1 correspondence between conjugacy classes of elements of Γ with minimal polynomial $q(X)$ and isomorphism classes of pairs $(M,\langle\cdot,\cdot\rangle)$, where M is a (finitely generated) torsion-free C-module of rank 1 and $\langle\cdot,\cdot\rangle$ is a non-degenerate skew-symmetric A-bilinear form on M satisfying $(*)$. This claim follows easily from the following lemma:

Lemma. *Let A be a Dedekind ring and M a finitely-generated projective A-module with a non-degenerate skew-symmetric bilinear form. Then M is free and admits a symplectic basis,* i.e., *a basis $\{e_1,\dots,e_n,f_1,\dots,f_n\}$ with*

$$\langle e_i,e_j\rangle=0=\langle f_i,f_j\rangle,\qquad \langle e_i,f_j\rangle=\delta_{ij}.$$

(Note: "Non-degenerate" means that the form induces an isomorphism of M onto its dual, $\operatorname{Hom}(M,A)$.)

The lemma is proved by induction on the rank of M. If rank $M>0$, then we must have rank $M>1$, since otherwise the skew-symmetric form would be zero. Then M has a direct summand which is free of rank 1 (see, for example, [16], Chapter 1), and we take e_1 to be a generator of this summand. Using non-degeneracy, we can find $f_1\in M$ with $\langle e_1,f_1\rangle=1$, and it follows easily that e_1 and f_1 form a basis for the submodule N which they generate. We then have $M=N\oplus N^{\perp}$, and the proof is completed by using the induction hypothesis to find a symplectic basis $\{e_2,\dots,e_n,f_2,\dots,f_n\}$ for $N^{\perp}$.

Returning now to the proof of Proposition 7, we proceed to count isomorphism classes of pairs $(M,\langle\cdot,\cdot\rangle)$. Consider, first, for fixed M, all possible non-degenerate A-bilinear forms (not necessarily skew-symmetric) satisfying $(*)$. I claim that C^* acts freely and transitively on this set of forms; one sees this by observing that such forms are in 1-1 correspondence with C-module isomorphisms $\overline{M}\to M^*$, where $\overline{M}$ is the C-module whose underlying abelian group is the same as that of M, with the C-action transformed by complex conjugation, and where M^* is the A-module dual of M. (Note that $M^*=\operatorname{Hom}_A(M,A)$ inherits a C-module structure from M.) This action of C^* can be described explicitly, as follows: Given a bilinear form $\langle\cdot,\cdot\rangle$ and a unit u of C, the transform $\langle\cdot,\cdot\rangle_u$ of $\langle\cdot,\cdot\rangle$ by u is defined by $\langle x,y\rangle_u=\langle x,uy\rangle$.

Suppose now that $\langle\cdot,\cdot\rangle$ is skew-symmetric; then $\langle\cdot,\cdot\rangle_u$ is also skew-symmetric if and only if $u=\bar{u}$, i.e., $u\in B^*$; thus the skew-symmetric non-degenerate forms satisfying $(*)$ are acted on freely and transitively by B^*. Moreover, one verifies easily that if $\langle\cdot,\cdot\rangle$ is skew-symmetric and $u\in B^*$, then $(M,\langle\cdot,\cdot\rangle)\approx(M,\langle\cdot,\cdot\rangle_u)$ if and only if $u\in N(C^*)$. Thus the number of isomorphism classes of $(M,\langle\cdot,\cdot\rangle)$ for a fixed M is $(B^*:N(C^*))=2^a$, assuming, of course, that M admits at least one bilinear form of the type we are considering.

Next, we count the number of isomorphism classes of C-modules M for which such a form exists. We may assume that all C-modules M being considered are fractionary ideals of C. Then $\overline{M}$ and M^* can also be identified with fractionary ideals; namely, $\overline{M}$ is identified with the set of complex conjugates of elements of M, and M^* is identified with

$$\{x\in F\colon \operatorname{tr}_{F/k}(xM)\subset A\}.$$

It follows that a non-degenerate A-bilinear form on M satisfying $(*)$ necessarily has the form $\langle x,y\rangle=\operatorname{tr}(r\bar{x}y)$, where r is an element of F such that $r\overline{M}=M^*$; moreover, the form is skew-symmetric if and only if r is pure imaginary, i.e., $\bar{r}=-r$. What we are trying to compute, then, is the number of classes of ideals M such that $M^{*-1}\overline{M}$ is principal and is generated by a pure imaginary element of F.

I now assert that there is a pure imaginary element d of C such that $M^{*-1}=dM$. Accepting this for the moment, we see that our condition on M is equivalent to the condition that $M\overline{M}$ is principal and is generated by an element of E, or, equivalently, that $N_{C/B}(M)$ is principal. Thus there are precisely h_S^- such ideal classes; combining this with the result of the earlier part of the proof, we see that there are precisely $2^a h_S^-$ isomorphism classes of pairs $(M,\langle\cdot,\cdot\rangle)$, i.e., $2^a h_S^-$ conjugacy classes of elements of Γ of order p with given minimal polynomial. Finally, since there are $(p-1)/2n$ possibilities for the minimal polynomial, there are precisely $(p-1)2^{a-1}h_S^-/n$ conjugacy classes of elements of order p, as required.

It remains to verify the assertion about the existence of d. In the first place, note that $M^*=D^{-1}M^{-1}$, where D is the *different* of C over A; in fact,

$$x\in M^*\Leftrightarrow \operatorname{tr}\langle xM\rangle\subset A\Leftrightarrow xM\subset D^{-1}\Leftrightarrow x\in D^{-1}M^{-1}.$$

Thus $M^{*-1}=DM$, so the proof will be complete if we show that D is principal and has a pure imaginary generator d. Computing D by the formula of [20], Chapter III, § 6, Proposition 11, we find that D is generated by $(\zeta-1)^{2n-1}$, hence also by $\zeta^m(\zeta-1)^{2n-1}$ for any integer m;

taking $m=(2n-1)(p-1)/2$, for example, we obtain a pure imaginary generator for D. [Note: In case $p=2$, D is generated by 2ζ, which is pure imaginary.]

9.3. The Fractional Part of $\zeta_k(-1)$. In order to compute the fractional part of $\zeta_k(-1)$, we will compute the p-fractional part of $\zeta_k(-1)$ for each prime number p, i.e., the part of the partial fractions decomposition of $\zeta_k(-1)$ with denominator a power of p. (For $p=2$, we will actually compute somewhat more than this.)

Let p be a fixed prime number and let the field F and the integers n, a, h_S^-, and w be as in § 9.2. *We will assume that* $n=1$, since otherwise $\mathrm{SL}_2(\mathcal{O})$ has no p-torsion and $\zeta_k(-1)=\chi(\mathrm{SL}_2(\mathcal{O}))$ is p-integral by [18], No. 1.8, Proposition 13 (or by § 9.1 above).

Let $\zeta_{k,S}$ denote the function obtained from ζ_k by deleting the factors in the Euler product expansion of the latter corresponding to primes of k lying over p, cf. [18], No. 3.7. Then for any positive integer i,

$$\zeta_{k,S}(-i)=\zeta_k(-i)\prod_{\mathfrak{p}\mid p}(1-N\mathfrak{p}^i),$$

where $N\mathfrak{p}$ is the cardinality of the residue field $\mathcal{O}/\mathfrak{p}$. In particular, $\zeta_{k,S}(-1)$ differs from $\zeta_k(-1)$ by a factor which is relatively prime to p, so the knowledge of the p-fractional part of $\zeta_k(-1)$ is equivalent to the knowledge of the p-fractional part of $\zeta_{k,S}(-1)$, which we will now compute.

We consider first the case where p is odd. Recall that, by Serre's estimate of the denominator of $\zeta_k(-1)$ which we reproved in § 9.1, $w\cdot\zeta_k(-1)$ is p-integral.

Proposition 8. *Assume p is odd.*

(i) *If some prime of k lying over p splits in F, then $\zeta_k(-1)$ is p-integral.*

(ii) *If no prime of k lying over p splits in F, then the p-fractional part of $\zeta_{k,S}(-1)$ is the same as that of $2^{a-1}h_S^-/w$; in other words, $w\cdot\zeta_{k,S}(-1)$ is congruent to $2^{a-1}h_S^-$ modulo the highest power of p dividing w.*

Corollary. *Let $h(k)$ (resp. $h(F)$) be the class number of k (resp. F), and let $h^-=h(F)/h(k)$. Then, in the situation of Proposition* 8 (ii) *the power of p dividing the denominator of $\zeta_k(-1)$ is the same as the power of p dividing the denominator of h^-/w.*

The corollary follows immediately from the proposition, once one observes that, under the hypothesis of (ii), h^- differs from h_S^- by a factor which is a power of 2. More generally, if we return temporarily to the situation of §9.2 (including the possibility $n>1$), we have:

Lemma 1. *Let $h^-=h(F)/h(E)$. If no prime of E lying over p splits in F, then h^-/h_S^- is a power of* 2.

In fact, if we ignore 2-torsion, then the ideal class group of F decomposes as the direct sum of its ± 1 eigenspaces under complex conjugation, with the $+1$ eigenspace being isomorphic to the class group of E and the -1 eigenspace being the kernel of the norm map. [This follows from the formal properties of the norm map and the map from the class group of E to that of F induced by the inclusion of rings of integers.] Thus h^- differs from the order of the -1 eigenspace only by a factor which is a power of 2. Similarly, h_S^- is essentially the same as the order of the -1 eigenspace in the S-class group of F, i.e., the class group of C. Now the S-class group of F is obtained from the class group of F by dividing out by the subgroup generated by the classes of the primes of F lying over p, and the hypothesis implies that these classes are all in the $+1$ eigenspace. Thus (still ignoring 2-torsion) the -1 eigenspace is unaffected by the passage to the S-class group, whence the lemma.

Proof of Proposition 8. Let $\Gamma = \mathrm{SL}_2(A)$; one knows ([18], No. 3.7) that $\chi(\Gamma) = \zeta_{k,S}(-1)$. Consequently, since every finite subgroup of Γ is cyclic [recall that $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$], we can determine the p-fractional part of $\zeta_{k,S}(-1)$ by means of the result of the second paragraph of §7. Rewriting that result as in the remark following Proposition 5 of §7, we find that the p-fractional part of $\zeta_{k,S}(-1)$ is the same as that of

$$\frac{1}{p-1} \sum \chi(C(\alpha)), \qquad (*)$$

where α ranges over the elements of order p of Γ, up to conjugacy. Using Proposition 7 of §9.2, we compute $(*)$ to be 0 under the hypothesis of (i) and $2^{a-1} h_S^-/w$ under the hypothesis of (ii), which completes the proof.

We consider now the case $p=2$. In this case the results obtainable by the method of proof of Proposition 8 can be substantially improved by using, instead of Γ, a group Γ'' introduced by Serre in [18], No. 3.7, proof of Proposition 30. The relevant properties of Γ'' are given in Lemma 2 below.

Let $(A^*:A^{*2}) = 2^e$ and let 2^c be the order of the group of ideal classes of A whose square is trivial.

Lemma 2. *There exists a group Γ'' with the following properties:*

(a) *Γ'' contains $\Gamma' = \mathrm{SL}_2(A)/\{\pm 1\}$ as a normal subgroup of index 2^{e+c}.*

(b) *If H is any finite subgroup of Γ'', then $(H : H \cap \Gamma')$ is equal to* 1, 2, *or* 4.

Let $L = A^2$ be the standard lattice in k^2 and let $\tilde{\Gamma}'' \subset \mathrm{GL}_2(k)$ be the group of matrices which transform L to $I \cdot L$ for some fractionary ideal I

of A; then $\tilde{\Gamma}''$ contains the group k^* of scalar multiples of the identity, and Γ'' is defined to be $\tilde{\Gamma}''/k^*$.

The determinant map $\tilde{\Gamma}'' \to k^*$ defines by passage to the quotient a homomorphism $\Gamma'' \to k^*/k^{*2}$ whose kernel is easily seen to be $\mathrm{SL}_2(A)/\{\pm 1\}$; since the index was calculated in [18], *loc. cit.*, to be 2^{e+c}, (a) is proved. Note also that Γ''/Γ' embeds in k^*/k^{*2}, hence has exponent 2.

Suppose now that H is a finite subgroup of Γ''. In case H is cyclic, the image of H in Γ''/Γ' is also cyclic, so $(H\colon H \cap \Gamma')$ is 1 or 2. To obtain (b) in the general case, we need only observe that H contains a cyclic subgroup of index 1 or 2. In fact, by choosing an embedding of k in $\mathbb{R}$, we can regard H as a subgroup of $\mathrm{PGL}_2(\mathbb{R})$; then $H \cap \mathrm{PSL}_2(\mathbb{R})$ is cyclic and has index 1 or 2 in H.

Serre ([18], No. 3.7, Proposition 30) used the group Γ'' to show that $w_0 \cdot \zeta_k(-1)/2^{e+c-3}$ is 2-integral, where $w_0 = w/2$. We will now reprove and sharpen this result.

Proposition 9. *Assume* $p=2$.

(i) *If some prime of* k *lying over* 2 *splits in* F *then* $\zeta_k(-1)/2^{e+c-3}$ *is 2-integral.*

(ii) *Suppose no prime of* k *lying over* 2 *splits in* F *and let* $q = h_S^-/2^{e-a+c-1}$. *Then* q *is an integer and the 2-fractional part of* $\zeta_{k,S}(-1)/2^{e+c-3}$ *is the same as that of* q/w_0; *in other words,* $w_0 \cdot \zeta_{k,S}(-1)/2^{e+c-3}$ *is 2-integral and is congruent to* q *modulo the highest power of* 2 *dividing* w_0.

It is clear from Proposition 7 of §9.2 that any 2-subgroup of Γ' has order dividing w_0, so Lemma 2(b) implies that any 2-subgroup of Γ'' has order dividing $4w_0$. Therefore, by Corollary 1 of Theorem 5 of §6, $4w_0\chi(\Gamma'')$ is 2-integral; since

$$4\chi(\Gamma'') = \zeta_{k,S}(-1)/2^{e+c-3},$$

we immediately recover Serre's result that $w_0 \cdot \zeta_{k,S}(-1)/2^{e+c-3}$ is 2-integral.

We will now prove (i) and (ii) by computing the 2-fractional part of $4\chi(\Gamma'')$. Let S be the ordered set of finite subgroups of Γ'' whose intersection with Γ' has even order. Lemma 2(b) shows that 4 is the largest power of 2 which can divide the order of a finite subgroup of Γ'' not in S, so Theorem 5 of §6 implies that the 2-fractional part of $4\chi(\Gamma'')$ is the same as that of $4\chi_{\Gamma''}(S)$. But $\chi_{\Gamma''}(S)$ is easy to compute:

Since every finite subgroup of Γ' is cyclic, S decomposes as $\coprod S_\alpha$, where α ranges over the elements of Γ' of order 2 and S_α is the set of finite subgroups of Γ'' which contain α. We now apply parts (i), (ii), and (iii) of Proposition 2 of §5 and we obtain:

$$\chi_{\Gamma''}(S)=\frac{1}{2^{e+c}}\chi_{\Gamma'}(S)=\frac{1}{2^{e+c}}\sum_{\alpha\in\Psi}\chi_{C'(\alpha)}(S_\alpha)=\frac{1}{2^{e+c}}\sum_{\alpha\in\Psi}\chi(C'(\alpha)),$$

where Ψ is a set of representatives for the conjugacy classes of elements of order 2 of Γ' and $C'(\alpha)$ is the centralizer of α in Γ'. Using Proposition 7 of §9.2 to compute the last sum, we find that $\chi_{\Gamma''}(S)$ is 0 under the hypothesis of (i) and is equal to $2^{a-1}h_S^-/2^{e+c}w_0=q/4w_0$ under the hypothesis of (ii). Therefore, the 2-fractional part of $4\chi(\Gamma'')$ is 0 under the hypothesis of (i) and is equal to q/w_0 under the hypothesis of (ii). This proves all assertions of the proposition except for the integrality of q, which follows now from the already established fact that $4w_0\chi(\Gamma'')$ is 2-integral.

Remark. The integrality of q, as well as an analogous fact in the situation of (i), can be proved as a consequence of some facts about Γ'', independent of the theory of Euler characteristics. Specifically, by computing the centralizer in $\tilde{\Gamma}''$ of an element of order 4 of $\mathrm{SL}_2(A)$, one can show that Γ'' contains a subgroup D with the following two properties:

(a) D is an abelian group with $m+1$ generators, where m is the number of primes of k lying over 2 which split in F.

(b) $(D\colon D\cap\Gamma')=2^{e-a+b}$, where 2^b is the order of the kernel of the map on ideal class groups induced by the inclusion $A\hookrightarrow C$.

Now (a) implies that $(D\colon D\cap\Gamma')$ divides 2^{m+1}, since Γ''/Γ' has exponent 2, so we obtain from (b) the inequality

$$e-a+b\leqq m+1.$$

This inequality is of interest in itself, and, in particular, it can be used to show that h_S^- is divisible by $2^{e-a+c-1-m}$. [First show that the kernel of the norm map from the ideal class group of C to that of A contains a subgroup of order 2^{c-b}, so that 2^{c-b} divides h_S^-.]

9.4. Theorems of Kummer and Greenberg. The application to be given in this section was suggested by Serre.

Let p be an *odd* prime number, let F, E, n, and w be as in §9.2 and let $h^-=h(F)/h(E)$, as in Lemma 1 of the preceding section.

Part (ii) of the following proposition was proved by Greenberg [9] under the additional hypothesis that p does not divide $[k:\mathbb{Q}]$.

Proposition 10. (i) (Serre [18], No. 3.7) *The rational number*

$$w\cdot\prod_{i=1}^{n}\zeta_k(1-2i)$$

is p-integral.

(ii) *If no prime of E lying over p splits in F, then p divides h^- if and only if p divides the numerator of* $w \cdot \prod_{i=1}^{n} \zeta_k(1-2i)$. *If some prime of E lying over p splits in F, then p necessarily divides the numerator of* $w \cdot \prod_{i=1}^{n} \zeta_k(1-2i)$.

Remarks. 1. The proof of Proposition 10 will yield a congruence mod p, making more precise the relationship between $w \cdot \prod_{i=1}^{n} \zeta_k(1-2i)$ and h^-. In case $n=1$, this result can be improved to a congruence modulo the highest power of p dividing w, cf. Proposition 8 of §9.3. I do not know if such an improvement is possible in the general case.

2. Suppose $k=\mathbb{Q}$. Then we have

$$n=\frac{p-1}{2}, \quad w=2p, \quad \text{and} \quad \zeta(1-2i)=-\frac{B_{2i}}{2i},$$

where $B_2, B_4, \ldots$ are the Bernoulli numbers. By von Staudt's Theorem (cf. [5], Chapter 5, §8, Theorem 4), $B_2, \ldots, B_{p-3}$ are p-integral and pB_{p-1} is p-integral and not divisible by p. The proposition thus implies that p divides h^- if and only if p divides the numerator of one of the Bernoulli numbers $B_2, \ldots, B_{p-3}$; this result is the theorem of Kummer's referred to in the title, cf. [5], Chapter 5, §5, Theorem 4.

Proof of Proposition 10. Let $\Gamma=\mathrm{Sp}_{2n}(A)$ as in §9.2. According to Harder and Serre ([18], No. 3.7, Example (iii)),

$$\chi(\Gamma)=\prod_{i=1}^{n} \zeta_{k,S}(1-2i),$$

where $\zeta_{k,S}$ has the same meaning as in the previous section. In particular, except for a factor which is relatively prime to p, $\chi(\Gamma)$ is the same as $\prod_{i=1}^{n} \zeta_k(1-2i)$.

Observe, now, that any finite p-subgroup of Γ has order dividing w; in fact, such a subgroup must centralize an element of order p, so the assertion follows from Proposition 7 of §9.2. Consequently, Corollary 1 of Theorem 5 implies that $w \cdot \chi(\Gamma)$ is p-integral, whence (i).

Next we apply Proposition 5 of §7 (and the remark following its statement), with q equal to the highest power of p dividing w. We find

$$w \cdot \chi(\Gamma) \equiv \frac{w}{p-1} \sum \chi(C(\alpha)) \pmod p,$$

where α ranges over the elements of order p of Γ, up to conjugacy. Using Proposition 7 again, we compute the right-hand side to be 0 if some prime of E lying over p splits in F, and $2^{a-1} h_S^- /n$ otherwise. This proves the second part of (ii) and shows under the hypothesis of the first part of (ii) that p divides $w \cdot \chi(\Gamma)$ if and only if p divides h_S^-. (Note that n and 2^{a-1} are both prime to p.) The first part of (ii) now follows from Lemma 1 of §9.3.

Appendix A. Projective Modules over Infinite Group Rings

Theorem 7. *Let Γ be a finitely generated group, let P be a finitely generated projective $\mathbb{Z}[\Gamma]$-module, let K be a field, and let V be a $K[\Gamma]$-module of finite rank over K. Then for any normal subgroup Γ' of finite index in Γ, the $K[\Gamma/\Gamma']$-module $\mathrm{Hom}_{\mathbb{Z}[\Gamma']}(P, V)$ is free of rank $r(P) \cdot \dim_K(V)$, where $r(P)$ is the rank of the (free) abelian group $P_\Gamma = \mathbb{Z} \underset{\mathbb{Z}[\Gamma]}{\otimes} P$.*

(Recall the definition of the Γ/Γ'-action on $\mathrm{Hom}_{\mathbb{Z}[\Gamma']}(P, V)$: let $\sigma \in \Gamma$, let s be its image in Γ/Γ', and let $f \in \mathrm{Hom}_{\mathbb{Z}[\Gamma']}(P, V)$; then $s \cdot f$ is defined by $(s \cdot f)(x) = \sigma \cdot f(\sigma^{-1} x)$.)

The proof of Theorem 7 will require two lemmas, both of which are elementary and well-known.

Lemma 1. *Let Γ be a group, Γ' a normal subgroup of finite index, R a commutative ring, P a finitely generated projective $\mathbb{Z}[\Gamma]$-module, and V an $R[\Gamma]$-module which is projective and finitely generated as an R-module. Then $\mathrm{Hom}_{\mathbb{Z}[\Gamma']}(P, V)$ is a projective, finitely generated $R[\Gamma/\Gamma']$-module.*

It suffices to consider the case where $P = \mathbb{Z}[\Gamma]$, in which case one sees easily that $\mathrm{Hom}_{\mathbb{Z}[\Gamma']}(P, V)$ is the $R[\Gamma/\Gamma']$-module induced from the R-module V, and the result follows at once.

Lemma 2. *Let Γ be a finitely generated group, let K be a field, and let $\rho\colon \Gamma \to \mathrm{GL}_n(K)$ be a representation of Γ over K. Then K contains a local ring R with finite residue field, such that $\rho(\Gamma) \subset \mathrm{GL}_n(R)$.*

Let $\{s_i\}$ be a finite set of generators of Γ and let S be the subring of K generated by the entries of all the matrices $\rho(s_i)$ and $\rho(s_i^{-1})$. Then $\rho(\Gamma) \subset \mathrm{GL}_n(S)$. Now S is a finitely generated ring, so any residue field of S, being a field which is finitely generated as a ring, is finite. Therefore we may take R to be the localization of S at any maximal ideal.

We will also require the following refinement of the version of Swan's Theorem stated in §1:

If G is a finite group, P is a finitely generated projective $\mathbb{Z}[G]$-module, and R is the localization of $\mathbb{Z}$ at any prime ideal, then $R \bigotimes_{\mathbb{Z}} P$ is a free $R[G]$-module.

This refinement can be deduced from the version stated in §1 by using [22], Theorem 1.10, or [1], Chapter XI, Proposition 5.1.

Proof of Theorem 7.

Case 1, Γ is finite. The homomorphism $\mathbb{Z} \to K$ factors through a local ring R of $\mathbb{Z}$, and we have:

$$\mathrm{Hom}_{\mathbb{Z}[\Gamma']}(P, V) \approx \mathrm{Hom}_{R[\Gamma']}(R \otimes P, V).$$

Now $R \otimes P$ is free (of rank $r(P)$) over $R[\Gamma]$ by the refinement of Swan's Theorem, and it follows easily that $\mathrm{Hom}_{R[\Gamma']}(R \otimes P, V)$ is free of rank $r(P) \cdot \dim_K(V)$ over $K[\Gamma/\Gamma']$, as required.

Case 2, K is finite. Since $\mathrm{GL}_n(K)$ is finite, the action of Γ on V factors through a finite quotient $G = \Gamma/\Gamma''$, where we may take $\Gamma'' \subset \Gamma'$. Let $G' = \Gamma'/\Gamma''$ and let $\bar{P} = \mathbb{Z}[G] \bigotimes_{\mathbb{Z}[\Gamma]} P$. Then $\Gamma/\Gamma' \approx G/G'$ and

$$\mathrm{Hom}_{\mathbb{Z}[\Gamma']}(P, V) \approx \mathrm{Hom}_{\mathbb{Z}[G']}(\bar{P}, V);$$

the theorem for Γ is thus reduced to the theorem for G, i.e., Case 2 is reduced to Case 1.

Case 3, general case. Identifying V with K^n, we can apply Lemma 2 to find a local ring $R \subset K$ with finite residue field, such that R^n is invariant under Γ. Writing $R^n = M$, an $R[\Gamma]$-module, we have $V \approx K \bigotimes_R M$, and it is easily seen that the natural map

$$K \bigotimes_R \mathrm{Hom}_{\mathbb{Z}[\Gamma']}(P, M) \to \mathrm{Hom}_{\mathbb{Z}[\Gamma']}(P, V)$$

is an isomorphism. (In fact, this need only be checked for P free, in which case it is obvious.) It will therefore suffice to prove that $\mathrm{Hom}_{\mathbb{Z}[\Gamma']}(P, M)$ is free of rank $r = r(P) \cdot \mathrm{rank}_R(M)$ over $R[\Gamma/\Gamma']$.

Now we know from Lemma 1 that $\mathrm{Hom}_{\mathbb{Z}[\Gamma']}(P, M)$ is projective and finitely generated over $R[\Gamma/\Gamma']$, so by a standard argument based on Nakayama's Lemma (see, for example, [1], Chapter III, Proposition 2.12) we need only show that $k \bigotimes_R \mathrm{Hom}_{\mathbb{Z}[\Gamma']}(P, M)$ is free of rank r over $k[\Gamma/\Gamma']$, where k is the residue field of R. But

$$k \bigotimes_R \mathrm{Hom}_{\mathbb{Z}[\Gamma']}(P, M) \xrightarrow{\approx} \mathrm{Hom}_{\mathbb{Z}[\Gamma']}(P, k \bigotimes_R M),$$

so we are reduced to proving the theorem with K replaced by k, i.e., Case 3 is reduced to Case 2 and the proof is complete.

Appendix B. Acyclic Covers

The following result is a variant of a well-known result in Čech cohomology theory, cf. [8], Chapter II, § 5, Corollary of Theorem 5.4.1. [See also [24], § 1, Lemma A.]

Proposition 11. *Let W be a semi-simplicial complex which is the union of acyclic subcomplexes X_s indexed by a partially ordered set S, and assume:*

(i) *if $s \leqq t$ then $X_s \supset X_t$; and*

(ii) *for any s, t, if $X_s \cap X_t \neq \emptyset$ then there is a u in S with $u \geqq s, t$.*

Then the chain complex $C(W)$ is canonically weakly equivalent to $C(S)$.

(See Proposition 4 of §5 for the definitions of $C(S)$ and weak equivalence.)

The proof is based on a "simplicial resolution"

$$\dots U_2 \substack{\rightarrow\\\rightarrow\\\rightarrow} U_1 \rightrightarrows U_0 \to W$$

of W (without degeneracy maps), constructed from the cover $\{X_s\}$ as follows: For any simplex σ of $K(S)$, let s be the largest vertex of σ and let $X_\sigma = X_s$. We then define U_n to be the semi-simplicial complex

$$\coprod_{\sigma \in K_n} X_\sigma,$$

where K_n is the set of n-simplices of $K(S)$. The augmentation $U_0 \to W$ is defined by means of the inclusions $X_s \hookrightarrow W$, and the i-th face map $U_n \to U_{n-1}$ $(i=0, \dots, n)$ is defined by means of the inclusions $X_\sigma \hookrightarrow X_{d_i\sigma}$, where $d_i\sigma$ is the i-th face of the simplex σ.

Applying the (unnormalized) chain complex functor C to each U_n and taking the alternating sum of the face maps, we obtain a double complex $C.(U.)$, and hence a total complex T and two spectral sequences converging to $H_*(T)$. There are obvious maps $T \to C(S)$ and $T \to C(W)$, and we will complete the proof by using the spectral sequences to show that both maps induce homology isomorphisms.

One of the spectral sequences has $E^1_{pq} = H_q(U_p)$; since each X_σ is acyclic, we obtain

$$E^1_{pq} = \begin{cases} 0 & q > 0 \\ C_p(S) & q = 0, \end{cases}$$

from which it follows that the map $T \to C(S)$ induces an isomorphism in homology.

Turning now to the other spectral sequence, one checks the definitions and finds that

$$E^1_{pq} = \bigoplus_{w \in W_p} H_q(K(S_w)),$$

where $S_w = (s \in S\colon w \in X_s\}$. But the hypothesis (ii) implies that S_w is a directed set, hence is a directed union of ordered sets with a largest element; therefore $K(S_w)$ is a directed union of cones and thus is acyclic. (Note that we need the hypothesis $W = \bigcup X_s$ here to guarantee $S_w \neq \emptyset$.) Thus

$$E^1_{pq} = \begin{cases} 0 & q > 0 \\ C_p(W) & q = 0, \end{cases}$$

hence the map $T \to C(W)$ also induces an isomorphism in homology, as required.

References

1. Bass, H.: Algebraic K-theory. New York: Benjamin 1968
2. Borel, A. *et al.*: Seminar on transformation groups, Annals of Math. Studies **46**. Princeton University Press 1960
3. Borel, A.: Arithmetic properties of linear algebraic groups. Proc. Int. Congress, 10–22 Stockholm 1962
4. Borel, A., Serre, J-P.: Corners and arithmetic groups. Comm. Math. Helv. **4**, 436–491 (1973)
5. Borevich, Z. I., Shafarevich, I. R.: Number theory. New York: Academic Press 1966
6. Bredon, G.E.: Introduction to compact transformations groups. New York: Academic Press 1972
7. Burnside, W.: Theory of groups of finite order, 2nd ed. (1911), reprinted by Dover 1955
8. Godement, R.: Topologie algébrique et théorie des faisceaux. Paris: Hermann 1958
9. Greenberg, R.: A generalization of Kummer's criterion. Inventiones Math. **21**, 247–254 (1973)
10. Gueho, M.-F.: Ordres d'Eichler et $\zeta_k(-1)$. Preprint
11. Harder, G.: A Gauss-Bonnet formula for discrete arithmetically defined groups. Ann. Sci. ENS **4**, 409–455 (1971)
12. Hirzebruch, F.: The Hilbert modular group, resolution of the singularities at the cusps and related problems, Sém. Bourbaki, exp. 396, June 1971. Lecture Notes in Mathematics **244**. Berlin-Heidelberg-New York: Springer 1971
13. Hirzebruch, F.: The Hilbert modular group and some algebraic surfaces. International Symposium in Number Theory, Moscow 1971 (to appear)
14. Hirzebruch, F.: Hilbert modular surfaces. L'Enseignement Mathématique **19**, 183–281 (1973)
15. Magnus, W., Karrass, A., Solitar, D.: Combinatorial group theory. New York: Wiley 1966
16. Milnor, J.: Introduction to algebraic K-theory. Annals of Math. Studies **72**, Princeton University Press 1971
17. Quillen, D.: The spectrum of an equivariant cohomology ring, I, II, Annals of Math. **94**, 549–572 and 573–602 (1971)
18. Serre, J-P.: Cohomologie des groupes discrets. Annals of Math. Studies **70**, 77–169, Princeton University Press 1971
19. Serre, J-P.: Cohomologie des groupes discrets, Sem. Bourbaki, exp. 399, June 1971. Lecture Notes in Mathematics **244**. Berlin-Heidelberg-New York: Springer 1971
20. Serre, J-P.: Corps locaux. Paris: Hermann 1968
21. Shimura, G.: Introduction to the arithmetic theory of automorphic functions. Publ. Math. Soc. Japan **11**, Iwanami Shoten, Publishers, and Princeton University Press 1971

22. Swan, R. G.: K-theory of finite groups and orders. Lecture Notes in Mathematics **149**. Berlin-Heidelberg-New York: Springer 1970
23. Verdier, J.-L.: Caractéristique d'Euler-Poincaré. Unpublished mimeographed notes
24. Wagoner, J.: Buildings, stratifications, and higher K-theory, Lecture Notes in Mathematics **341**, pp. 148–165. Berlin-Heidelberg-New York: Springer 1973
25. Wall, C. T. C.: Finiteness conditions for CW-complexes. Annals of Math. **81**, 56–69 (1965)
26. Wall, C. T. C.: Finiteness conditions for CW-complexes II, Proc. Royal Soc. A, **295**, 129–139 (1966)
27. Zarelua, A.: On finite groups of transformations, Proceedings of the international sympoium on topology and its applications, Herceg-Novi, 1968, 334–339

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