THE ARTIN-REES PROPERTY AND HOMOLOGY

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ABSTRACT
The Artin-Rees property for a finitely generated nilpotent group $G$ is used to prove that $H_\ast(G, M) \cong H_\ast(G, \hat{M})$ for any finitely generated $G$-module $M$, where $\hat{M}$ is the completion of $M$ with respect to the augmentation ideal of $\mathbb{Z}[G]$. Applications to topology are given.

Nouazé and Gabriel ([15], 2.7 and 2.8) have shown that the classical Artin-Rees lemma for commutative noetherian rings admits a generalization which applies, for example, to the group ring of a finitely generated nilpotent group. The purpose of the present paper is to give some applications to homological algebra and topology of this generalized Artin-Ress lemma. These applications concern maps (of modules or spaces) which induce homology isomorphisms.

In Section 1 we describe the Artin-Rees property and give the proof that it is satisfied by finitely generated nilpotent groups.

Section 2 contains the applications to homological algebra. The main result, which includes a theorem of Dwyer's [8] as a special case, is the following (2.2, Th. 3): If $G$ is a finitely generated nilpotent group and $M$ is a finitely generated $G$-module, then $H_\ast(G, M) \cong H_\ast(G, \hat{M})$, where $\hat{M}$ is the completion of $M$ with respect to the augmentation ideal of $\mathbb{Z}[G]$. This result is used in 2.3 to show that for a large class of groups, including all finite groups and all finitely generated nilpotent groups, $\hat{M}$ is equal to the $HZ$-localization of $M$ in the sense of Bousfield [2], for any finitely generated $G$-module $M$.

In Section 3 we illustrate how the results of Sections 1 and 2 can be used in topology by proving (a) a vanishing theorem for certain homology groups associated to a prenilpotent space (3.1, Th. 5) and (b) a theorem concerning the homotopy groups of a (higher dimensional) knot complement (3.2, Th. 6).

Finally, an appendix contains a result needed in 3.1 concerning the homology (mod $\mathcal{C}$) of a regular covering space, where $\mathcal{C}$ is a Serre class of abelian groups; as an immediate consequence, we obtain a generalization to nilpotent spaces of Serre's mod $\mathcal{C}$ Hurewicz theorem for simply connected spaces.

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1. The Artin-Rees property

Let $R$ be a left noetherian ring and $I$ a two-sided ideal. If $M$ is a (left) $R$-module, the $I$-adic topology on $M$ is the unique topology which is compatible with the group structure and in which $\{I^nM\}_{n \geq 0}$ is a fundamental system of neighborhoods of $0$. (Equivalently, a neighborhood base at 0 is formed by the submodules $M'$ of $M$ such that $M/M'$ is $I$-nilpotent, i.e., is annihilated by a power of $I$.) We will say that $I$ has the (left) Artin-Rees property if for every finitely generated (left) $R$-module $M$ and every submodule $N$, the $I$-adic topology on $N$ coincides with the restriction to $N$ of the $I$-adic topology on $M$.

The following reformulation of the definition is essentially due to Gabriel (cf. [10], V, §5, Prop. 9):

**Proposition 1.** The following conditions on $I$ are equivalent:

(i) $I$ has the Artin-Rees property.
(ii) If $M$ is a finitely generated $R$-module which contains an essential $I$-nilpotent submodule, then $M$ is $I$-nilpotent.
(iii) If $M$ is a finitely generated $R$-module which contains an essential submodule $N$ such that $IN = 0$, then $M$ is $I$-nilpotent.

(i) $\Leftrightarrow$ (ii): The proof is identical with Gabriel's proof (loc. cit.), so we omit it. [Take $C$, in the notation of [10], to be the category of $I$-nilpotent modules.]

(ii) $\Rightarrow$ (iii): Trivial.

(iii) $\Rightarrow$ (ii): Let $M$ be as in (ii) and let $N = \{ x \in M : Ix = 0 \}$. Then it is easy to verify that $N$ is an essential submodule of $M$, so (ii) follows from (iii).

We now specialize to the case where $R$ is the integral group ring $\mathbb{Z}[G]$ of a group $G$, and $I$ is the augmentation ideal. We will say that a $G$-module is nilpotent if it is $I$-nilpotent, and we will say that $G$ has the Artin-Rees property if $\mathbb{Z}[G]$ is noetherian and $I$ has the Artin-Rees property. (Note that there is no need to distinguish here between the left and right Artin-Rees properties, since $\mathbb{Z}[G]$ has an anti-automorphism which takes $I$ onto itself.)

The following theorem is a special case of a result due to Nouazé and Gabriel ([15], 2.7 and 2.8); we will give the proof for the convenience of the reader.

*Recall that $N$ is said to be an essential submodule of $M$ if every non-zero submodule of $M$ intersects $N$ non-trivially.*
Theorem 1. If $G$ is finitely generated and nilpotent, then $G$ has the Artin-Rees property.

The proof is based on the well-known fact (cf. [12], proof of Theorem 10.2.4) that $G$ has a central series
$$G = G_1 \supset G_2 \supset \cdots \supset G_n = \{1\}$$
such that each quotient $G_i / G_{i+1}$ is cyclic. In particular, it follows easily that $\mathbb{Z}[G]$ is noetherian (cf. [16], p. 136). Theorem 1 now follows, by induction on the minimal length $n$ of such a series, from:

**Proposition 2.** Let $G$ be a group such that $\mathbb{Z}[G]$ is noetherian. If $G$ has a central cyclic subgroup $C$ such that $G/C$ has the Artin-Rees property, then $G$ has the Artin-Rees property.

We will verify condition (iii) of Proposition 1. Thus we must show that if $M$ is a finitely generated $G$-module which contains an essential submodule $N$ on which $G$ acts trivially, then $M$ is nilpotent. We will do this by showing (a) that $M$ is nilpotent as a $C$-module and (b) that $M^C$, the set of elements of $M$ fixed by $C$, is nilpotent as a $G/C$-module (and hence as a $G$-module). Assuming for the moment that (a) and (b) have been established, we complete the proof as follows. Let $r = 1 - t$, were $t$ is a generator of $C$. Then $r$ is a central element of $\mathbb{Z}[G]$ and multiplication by $r$ is a $G$-module endomorphism of $M$ whose kernel is $M^C$. Using the exact sequences
$$0 \rightarrow M^C \rightarrow \ker r^n \rightarrow \ker r^n / r^n - 1,$$
we conclude from (b) (by induction on $n$) that $\ker r^n$ is a nilpotent $G$-module for each $n \geq 1$. Since $\ker r^n = M$ for large $n$ by (a), $M$ is indeed nilpotent.

It remains to prove (a) and (b). For (b) we need only note that $M^C$ is a finitely generated $G/C$-module which contains $N$ as an essential submodule, hence $M^C$ is nilpotent by the assumption on $G/C$. To prove (a) we consider the ascending chain $\{\ker r^n\}_{n \geq 1}$ of submodules of $M$. This chain must stabilize since $M$ is finitely generated, and it follows easily that $\ker r^n \cap \im r^n = 0$ for large $n$. Since $N \subset \ker r^n$ and $N$ is essential, we conclude that $\im r^n = 0$, as required.

**Remark.** If $G$ is only assumed to be polycyclic instead of finitely generated nilpotent, then $G$ need not have the Artin-Rees property. For example, let $k$ be the field $\mathbb{Z}/p\mathbb{Z}$, where $p$ is an odd prime, let $M$ be a two-dimensional vector space over $k$, let $M'$ be a one-dimensional subspace, and let $G$ be the group of automorphisms of $M$ which act as the identity on $M'$. (Thus $G$ is the matrix
group \((e, \cdot)\). Then one verifies easily that \(M'\) is an essential \(G\)-submodule of \(M\), but that \(M\) is not a nilpotent \(G\)-module (in fact, \(IM = M\)), so the Artin-Rees property fails. (Note that \(G\) is polycyclic, being the semi-direct product of the additive and multiplicative groups of \(k\), both of which are cyclic.)

2. Applications to homological algebra

2.1. \textit{Tor and Completion}

Let \(R\) be a ring and \(I\) a two-sided ideal. If \(M\) is a left \(R\)-module, we denote by \(\hat{M}\) the completion of \(M\) with respect to the \(I\)-adic topology:

\[
\hat{M} = \lim \leftarrow M/I^nM.
\]

We denote by \(\alpha\) the canonical map \(M \to \hat{M}\). In case \(M = R\), the completion \(\hat{R}\) is a ring and \(\alpha : R \to \hat{R}\) is a ring homomorphism, by means of which we regard \(\hat{R}\) as an \(R\)-bimodule.

For any (left) \(R\)-module \(M\) we denote by \(\bar{M}\) the \(\hat{R}\)-module obtained from \(M\) by extension of scalars:

\[
\bar{M} = \hat{R} \otimes_R M.
\]

There is a canonical map \(\beta : M \to \bar{M}\) and, since \(\hat{M}\) has an obvious \(\hat{R}\)-module structure, there is a unique \(\hat{R}\)-module homomorphism \(\gamma : \overline{M} \to \hat{M}\) such that \(\gamma \beta = \alpha\):

\[
\begin{array}{c}
\beta \\
M \rightarrow \bar{M} \\
\alpha \downarrow \gamma \\
\hat{M}
\end{array}
\]

**Proposition 3.** Assume that \(R\) is left noetherian and that \(I\) satisfies the left Artin-Rees property.

(i) The functor \(M \mapsto \hat{M}\) is exact on the category of finitely generated left \(R\)-modules.

(ii) If \(M\) finitely generated then \(\gamma : \overline{M} \to \hat{M}\) is an isomorphism.

(iii) \(\hat{R}\) is flat as a right \(R\)-module.

(iv) Let \(M\) be a finitely generated left \(R\)-module and let \(I^nM\) be the kernel of \(\alpha : M \to \hat{M}\), i.e. \(I^nM = \bigcap_{n \geq 1} I^nM\). Then \(I \cdot I^nM = I^nM\), and \(I^nM\) is the largest submodule of \(M\) with this property.

These consequences of Artin-Rees property are proved exactly as in the commutative case. See, for example, [1], Props. 10.12, 10.13 and 10.14, and the proof of Prop. 10.17.
NOTE. Proposition 3 has an obvious analogue for a right noetherian ring and an ideal with the right Artin-Rees property. This analogue will be referred to as Proposition 3. 

We can now prove the main result of this section:

**Theorem 2.** Assume that $R$ is left and right noetherian and that $I$ satisfies the left and right Artin-Rees properties. For any left $R$-module $M$, $\beta$ induces an isomorphism

$$\text{Tor}_n^R(R/I, M) \cong \text{Tor}_n^R(R/I, \tilde{M});$$

if $M$ is finitely generated, then $\alpha$ induces an isomorphism

$$\text{Tor}_n^R(R/I, M) \cong \text{Tor}_n^R(R/I, \tilde{M}).$$

In view of Proposition 3 (ii), it suffices to prove the assertion about $\beta$. Since $\hat{R}$ is a flat right $R$-module (Prop. 3(iii)), the functor $M \mapsto \tilde{M}$ is exact, and hence the functors $\text{Tor}_n^R(R/I, \tilde{M})$, as functors of the variable $M$, form a connected exact sequence of functors, in the sense of [4], Chap. V, §4. It therefore suffices to show (loc. cit., Prop. 4.4) (a) that $\beta$ induces an isomorphism $R/I \otimes_R M \cong R/I \otimes_R \hat{M}$, and (b) that $\text{Tor}_i(R/I, \tilde{M}) = 0$ if $i > 0$ and $M$ is free.

To prove (a), consider the commutative square

$$
\begin{array}{ccc}
R/I \otimes_R M & \xrightarrow{\delta \otimes_R M} & (R/I \otimes_R \hat{R}) \otimes_R M \\
R/I \otimes_R \hat{R} \beta \downarrow & & \downarrow \\
R/I \otimes_R \hat{M} & = & R/I \otimes_R (\hat{R} \otimes_R M),
\end{array}
$$

where $\delta: R/I \rightarrow R/I \otimes_R \hat{R}$ is the canonical map, $x \mapsto x \otimes 1$. Now Prop. 3,(ii) implies that $\delta$ can be identified with the canonical map of $R/I$ to its $I$-adic completion as a right $R$-module; but $R/I$ is complete, so $\delta$ is an isomorphism and (a) follows at once. To prove (b), it suffices to consider the case $M = R$, in which case the result follows from the flatness of $\hat{R}$ as a left $R$-module (Prop. 3,(iii)).

**Corollary 1.** The following are equivalent for a map $f: M \rightarrow N$ of finitely generated $R$-modules:

(i) The map $f_0: \text{Tor}_i^R(R/I, M) \rightarrow \text{Tor}_i^R(R/I, N)$ induced by $f$ is an isomorphism for all $i \geq 0$.

(ii) $f_0$ is an isomorphism and $f_1$ is an epimorphism.

(iii) $f$ induces an isomorphism $\hat{f}: \hat{M} \rightarrow \hat{N}$. 

In fact, (i) \(\Rightarrow\) (ii) trivially; (ii) \(\Rightarrow\) (iii) by the argument of [5], proof of Prop. 5.2 (which does not require the Artin-Rees property or finiteness of \(M\) and \(N\)); and (iii) \(\Rightarrow\) (i) by Theorem 2.

In case \(N = 0\), the implication (ii) \(\Rightarrow\) (i) of Corollary 1 yields:

**Corollary 2.** If \(M\) is a finitely generated left \(R\)-module such that \(\text{Tor}^R_0(R/I, M) = 0\), then \(\text{Tor}^R_i(R/I, M) = 0\) for all \(i\).

**Remark.** The isomorphisms

\[
R/I \otimes_R M \overset{\cong}{\longrightarrow} R/I \otimes_R \hat{M}
\]

and

\[
R/I \otimes_R M \overset{\cong}{\longrightarrow} R/I \otimes_R \hat{M}
\]

of Theorem 2 can be proved under much weaker hypotheses than those of the theorem, namely, we need only assume that \(I\) is finitely presented as a right \(R\)-module (no hypotheses on \(M\)). One proves the first isomorphism by computing \(R/I \otimes_R \hat{M}\) by means of the short exact sequence

\[
0 \rightarrow \lim^{\leftarrow} \text{Tor}^R_1(N, M_i) \rightarrow N \otimes_R \lim_i M_i \rightarrow \lim_i N \otimes_R M_i \rightarrow 0
\]

(cf. [17], Th. 2), valid for any tower of \(R\)-modules \(\{M_i\}\) such that \(\lim^{\leftarrow} M_i = 0\) and for any right \(R\)-module \(N\) such that there exists an exact sequence \(F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow N \rightarrow 0\) with \(F_1\) finitely generated and free. The second isomorphism can be deduced from the first, applied with \(M = R\).

### 2.2. The homology of a finitely generated nilpotent group

Let \(G\) be a group with the Artin-Rees property, e.g., a finitely generated nilpotent group (Section 1, Th. 1). Then all of the results of 2.1 apply with \(R = \mathbb{Z}[G]\) and \(I\) equal to the augmentation ideal. In particular, for any \(G\)-module \(M\) we have \(G\)-modules \(\hat{M}\) and \(\hat{M}\) and maps \(\alpha: M \rightarrow \hat{M}\) and \(\beta: M \rightarrow \hat{M}\), and Theorem 2 and its corollaries yield:

**Theorem 3.** For any \(G\)-module \(M\), \(\beta\) induces an isomorphism \(H_*(G, M) \overset{\cong}{\rightarrow} H_*(G, \hat{M})\); if \(M\) is finitely generated, then \(\alpha\) induces an isomorphism \(H_*(G, M) \overset{\cong}{\rightarrow} H_*(G, \hat{M})\).

**Corollary 1.** The following are equivalent for a map \(f: M \rightarrow N\) of finitely generated \(G\)-modules:

(i) The map \(f_i: H_i(G, M) \rightarrow H_i(G, N)\) induced by \(f\) is an isomorphism for all \(i\).
(ii) \(f_0\) is an isomorphism and \(f_i\) is an epimorphism.
(iii) \(f\) induces an isomorphism \(\hat{f}: \hat{M} \rightarrow \hat{N}\).
COROLLARY 2. (Dwyer [8]). If $M$ is a finitely generated $G$-module such that $H_0(G, M) = 0$, then $H_i(G, M) = 0$ for all $i$.

2.3. $HZ$-localization for modules over prenilpotent groups

Let $G$ be a group. We recall some terminology from [2]. A map $f: M \to N$ of $G$-modules is called an $HZ$-map if the induced map $f_*: H_i(G, M) \to H_i(G, N)$ is an isomorphism for $i = 0$ and an epimorphism for $i = 1$. A $G$-module $M$ is said to be $HZ$-local if every $HZ$-map $f: N_1 \to N_2$ induces an isomorphism $\text{Hom}(N_2, M) \cong \text{Hom}(N_1, M)$. Finally, an $HZ$-localization of a $G$-module $M$ is an $HZ$-map $f: M \to M'$ with $M'$ $HZ$-local. It is easy to see any two $HZ$-localizations of $M$ are canonically isomorphic. Moreover, it is proved in [2] that every $G$-module $M$ admits an $HZ$-localization, but we will not need to use this fact.

We call a group $G$ prenilpotent if the lower central series $\{\Gamma_i G\}_{i \geq 1}$ stabilizes, i.e., if $\Gamma_i G = \Gamma_{i+1} G$ for large $i$. (Here $\Gamma_i G = G$ and $\Gamma_{i+1} G = (G, \Gamma_i G)$, cf. [12], Chap. 10.) For example, every finite group is prenilpotent. The purpose of this section is to prove:

THEOREM 4. If $G$ is a finitely generated prenilpotent group and $M$ is a finitely generated $G$-module, then the canonical map $\alpha: M \to \hat{M}$ is the $HZ$-localization of $M$.

(As in 2.2, $\hat{M}$ is the $I$-adic completion of $M$, where $I$ is the augmentation ideal of $\mathbb{Z}[G]$.)

We will need the following three lemmas:

LEMMA 1. For any group $G$ and any $G$-module $M$, the $I$-adic completion $\hat{M}$ is $HZ$-local.

In fact, it is immediate from the definition that an inverse limit of local modules is local, so it suffices to show that any nilpotent module $M$ is local. Assume, then that $I^n M = 0$ and let $f: N_1 \to N_2$ be an $HZ$-map. Then $f$ induces an isomorphism $N_1/I_1^n \cong N_2/I_2^n$ ([5], Prop. 5.2), and the lemma follows from the diagram

$$
\begin{array}{ccc}
\text{Hom}(N_2, M) & \cong & \text{Hom}(N_1, M) \\
\downarrow & & \downarrow \\
\text{Hom}(N_2/I_1^n N_2, M) & \cong & \text{Hom}(N_1/I_1^n N_1, M).
\end{array}
$$
Lemma 2. Let $G$ be a prenilpotent group and let $\Gamma = \Gamma_i G$ for large $i$. If $M$ is a $G$-module, then $\Gamma$ acts trivially on $M/I^n M$ for all $n$.

This follows from the elementary fact that if $\gamma \in \Gamma_n G$ then $\gamma - 1 \in I^n$. (See, for example, [14], Chap. I, §3, Th. 3.2; alternatively, prove this fact by induction on $n$, using the first lemma on p. 138 of [16].)

Lemma 3. If $G$ is finitely generated and prenilpotent, then the abelianization $\Gamma_{ab}$ of $\Gamma$ is a finitely generated $G/\Gamma$-module, where $\Gamma$ is as in Lemma 2.

(The action of $G/\Gamma$ on $\Gamma_{ab}$ is induced by the conjugation action of $G$ on $\Gamma$.)

In fact, if $S$ is a finite set of generators of $G$, then it is easy to see that $\Gamma_i G$ is the normal subgroup of $G$ generated by the $i$-fold commutators of elements of $S$. Thus $\Gamma$ is finitely generated as a normal subgroup of $G$, and the lemma follows at once.

Proof of Theorem 4. In view of Lemma 1, it suffices to show that $\alpha : M \to \hat{M}$ is an $HZ$-map. Let $\Gamma$ be as in Lemmas 2 and 3 and let $\nu = G/\Gamma$. Note that $\nu$ is nilpotent. Let $N$ be the $\nu$-module $H_0(\Gamma, M) = M/I_1 M$, where $I_1$ is the augmentation ideal of $Z[\Gamma]$. It follows easily from Lemma 2 that $\hat{M}$ is a $\nu$-module and that, moreover, $\hat{M}$ can be identified with $\hat{N}$. We therefore obtain a commutative diagram

$$
\begin{array}{ccc}
H_i(G, M) & \xrightarrow{\alpha_i} & H_i(G, \hat{M}) = H_i(G, \hat{N}) \\
\downarrow \phi_i & & \downarrow \psi_i \\
H_i(\nu, N) & \xrightarrow{\alpha'_i} & H_i(\nu, \hat{N}),
\end{array}
$$

where $\phi_i$ and $\psi_i$ are induced by the projections $G \to \nu$ and $M \to N$ and $\alpha_i$ (resp. $\alpha'_i$) is induced by the canonical map $\alpha : M \to \hat{M}$ (resp. $\alpha' : N \to \hat{N}$). Since $\alpha'_i$ is an isomorphism by Theorem 3, the proof will be complete if we show that $\phi_0$, $\psi_0$, and $\psi_i$ are isomorphisms and that $\phi_i$ is an epimorphism.

Now it is trivial to verify that $\phi_0$ and $\psi_0$ are isomorphisms, and $\phi_i$ is easily seen to be an epimorphism by means of the Lyndon-Hochschild-Serre spectral sequence

$$
E^2_{pq} = H_p(\nu, H_q(\Gamma, M)) \Rightarrow H_{p+q}(G, M).
$$
Finally, to see that \( \psi_t \) is an isomorphism, we again use the spectral sequence, but with coefficient module \( \hat{N} \), and we obtain an exact sequence
\[
H_0(\nu, \Gamma_{ab} \otimes \hat{N}) \rightarrow H_1(G, \hat{N}) \xrightarrow{\cdot \psi_t} H_1(\nu, \hat{N}) \rightarrow 0.
\]
But \( H_0(\nu, \Gamma_{ab} \otimes \hat{N}) = \Gamma_{ab} \otimes \nu \hat{N} \) (this follows at once from the definition of tensor product), so it suffices to show that \( \Gamma_{ab} \otimes \nu \hat{A} = 0 \) for any finitely generated \( \nu \)-module \( A \). Now \( A \rightarrow \Gamma_{ab} \otimes \nu \hat{A} \) is right exact (2.1, Prop. 3(i)), so we may assume \( A = \mathbb{Z}[\nu] \). But then \( \Gamma_{ab} \otimes \nu \hat{A} \) is simply the completion of the right \( R \)-module \( \Gamma_{ab} \) (Prop. 3(ii) and Lemma 3); since \( (G, \Gamma) = \Gamma \), we conclude that \( \Gamma_{ab}I = \Gamma_{ab} \), so \( \hat{\Gamma}_{ab} = 0 \), as required. [An alternative proof that \( \Gamma_{ab} \otimes \nu \hat{A} = 0 \), which does not depend on the Artin-Rees property or on the finiteness of \( A \), can be based on the short exact sequence given in the remark at end of Section 2.1.]

3. Applications to topology

The applications we will give concern the structure of homology equivalences, i.e., of maps which induce isomorphisms on integral homology.

3.1. Prenilpotent spaces

Recall that a CW-complex \( X \) is said to be nilpotent if \( X \) is connected, \( \pi_nX \) is nilpotent, and \( \pi_0X \) is a nilpotent \( \pi_1X \)-module for \( n > 1 \). (Thus \( \pi_nX \) for \( n > 1 \) is annihilated by some power of the augmentation ideal of \( \mathbb{Z}[\pi_1X] \), cf. Section 1.) A CW-complex \( X \) is called prenilpotent if there is a homology equivalence \( f: X \rightarrow Y \) with \( Y \) nilpotent, or, equivalently, if the \( H_n(-, \mathbb{Z}) \)-localization of \( X \) in the sense of Bousfield [2] is nilpotent. Prenilpotent spaces are studied in [7], where it is shown that, for CW-complexes of finite type (i.e. with finitely many cells in each dimension), one can give an intrinsic characterization of prenilpotence. See also [9], where some examples of prenilpotent spaces are discussed.

In case \( Y \) is the circle \( S^1 \), \( X \) is called a homology circle. The analysis of homology circles [6] depends heavily on the fact that (for trivial reasons) the Serre spectral sequence of \( f: X \rightarrow S^1 \) collapses, i.e. \( H_p(S^1, H_q(F)) = 0 \) for \( q > 0 \), where \( F \) is the homotopy fibre of \( f \). (Note: We are dealing here with homology with local coefficients.) The purpose of this section is to prove an analogous collapsing theorem in the general case:

**Theorem 5.** Let \( X \) be a prenilpotent space of finite type and let \( F \) be the homotopy fibre of a homology equivalence \( f: X \rightarrow Y \) with \( Y \) nilpotent. Then for all \( q > 0 \) and all \( p \),
\[
E^2_{pq} = H_p(Y, H_q(F)) = 0.
\]
The proof will use the following topological analogue of Theorem 3 (Section 2.2):

**Proposition 4.** Let $Y$ be a nilpotent space with $\pi_1 Y$ finitely generated. If $M$ is any $\pi_1 Y$-module, then $H_\ast(Y, M) \xrightarrow{\cong} H_\ast(Y, \tilde{M})$; if $M$ is finitely generated then $H_\ast(Y, M) \xrightarrow{\cong} H_\ast(Y, \tilde{M})$.

(See Section 2.2 for the definitions of $\tilde{M}$ and $\tilde{M}$.)

The second assertion of proposition follows from the first, in view of 2.1, Prop. 3(ii). To prove the first assertion we use the refined Postnikov tower of $Y$ ([3], Chap. II, §4, Prop. 4.7):

$$\cdots \rightarrow Y_i \xrightarrow{p_i} Y_{i-1} \rightarrow \cdots \rightarrow Y_1 = K(\nu, 1),$$

where $\nu = \pi_1 Y$. Here $p_i$ is a principal fibration with fibre $F_i$ of the form $K(A_i, n_i)$ where $2 \leq n_i \rightarrow \infty$, and $Y = \varprojlim Y_i$. It suffices to prove by induction on $i$ that $H_\ast(Y_i, M) \xrightarrow{\cong} H_\ast(Y_i, \tilde{M})$ for any $\nu$-module $M$. The case $i = 1$ being true by Theorem 3, we may assume that $i > 1$ and that the result is known for $Y_{i-1}$. Consider the map of Serre spectral sequences (with local coefficients) induced by the coefficient homomorphism $M \rightarrow \tilde{M}$:

$$H_\ast(Y_{i-1}, H_q(F_i, M)) \Rightarrow H_{\ast+q}(Y_i, M)$$

$$H_\ast(Y_{i-1}, H_q(F_i, \tilde{M})) \Rightarrow H_{\ast+q}(Y_i, \tilde{M}).$$

Note that the groups $H_q(F_i, \_\_\_\_\_)$ which occur here are ordinary homology groups with constant coefficients [$F_i$ is simply connected]; note further that the action of $\nu = \pi_1 Y_{i-1}$ on $H_q(F_i, \_\_\_\_)$ comes entirely from the action of $\nu$ on the coefficient module. [The action of $\pi_1 Y_{i-1}$ on $F_i$ (in the homotopy category) is trivial, since $p_i$ is a principal fibration with connected fibre.] Therefore, in view of the flatness of $\hat{R}$ over $R = \mathbb{Z}[\nu]$ (2.1, Prop. 3(iii)), we have isomorphisms of $\nu$-modules

$$H_q(F_i, \tilde{M}) = H_q(F_i, \hat{R} \otimes_R M) \cong \hat{R} \otimes_H q(F_i, M) = \overline{H_q(F_i, M)}.$$

The induction hypothesis now implies that the above map of spectral sequences is an isomorphism on $E^2$, hence on $E^\infty$, which completes the proof.

We will also need the following lemma:

**Lemma.** Let $G$ be a group such that $\mathbb{Z}[G]$ is noetherian and let $M$ and $N$ be finitely generated $G$-modules, one of which is finitely generated over $\mathbb{Z}$. Then $M \otimes N$ and $M \ast N$ are finitely generated $G$-modules.
(Here $M \otimes N$ and $M \ast N$ are the tensor and torsion products over $\mathbb{Z}$, with the usual (diagonal) action of $G$.)

Assume, for example, that $M$ is finitely generated over $\mathbb{Z}$, and let $(F_i)_{i \geq 0}$ be a free resolution of $N$ over $\mathbb{Z}[G]$, with each $F_i$ finitely generated. Then $M \otimes N$ and $M \ast N$ can be computed as homology groups of the complex $M \otimes_\mathbb{Z} F$; the lemma therefore follows from the easily verified fact that $M \otimes_\mathbb{Z} \mathbb{Z}[G]$ is finitely generated over $G$.

**Proof of Theorem 5.** We begin with two preliminary observations:

(a) $\pi_1 f: \pi_1 X \to \pi_1 Y = \nu$ is surjective; hence, in particular, $\nu$ is finitely generated and $F$ is connected. This follows from the surjectivity of $H_1 f: H_1 X \to H_1 Y$, by an argument analogous to that of Lemma 1 (a) of the appendix. [In fact, one actually knows ([5], Prop. 5.1 and first paragraph of Section 6) that $\pi_1 X$ is prenilpotent and that $\nu = \pi_1 X / \Gamma$, in the notation of Section 2.3.]

(b) $H_n (F)$ is a finitely generated $\nu$-module for each $n$. In fact, let $\bar{\pi}; \bar{Y} \to Y$ be the universal cover of $Y$, and consider the pull back $\bar{f}$ of $f$ to a map over $\bar{Y}$:

$$
\begin{array}{c}
\bar{X} \to X \\
\bar{f} \downarrow \quad \downarrow f \\
\bar{Y} \to Y.
\end{array}
$$

(thus $\bar{X}$ is a regular covering space of $X$, with covering group $\nu$.) Then $F$ is also the homotopy fibre of $\bar{f}$. Moreover, since $\nu$ acts as a group of automorphisms of the map $\bar{f}$, the Serre spectral sequence

$$E^{2}_{pq} = H_p (\bar{Y}, H_q (F)) \Rightarrow H_{p+q} (\bar{X})$$

is a spectral sequence of $\nu$-modules. Now $H_n (\bar{Y})$ is finitely generated over $\mathbb{Z}$ for all $n$ by Cor. 1 of Prop. 5 of the appendix; and $H_n (\bar{X})$ is finitely generated over $\nu$ for all $n$, since the cellular chain complex of $\bar{X}$ is a complex of finitely generated modules over the noetherian ring $\mathbb{Z}[\nu]$.

Assertion (b) therefore follows at once by a standard mod $\mathcal{C}$ spectral sequence argument (cf. [18]), where $\mathcal{C}$ is the class of finitely generated $\nu$-modules. [Note: the crucial point here is that if $N$ is a finitely generated $\nu$-module, then $H_p (\bar{Y}, N)$, which is a $\nu$-module via the action of $\nu$ on $\bar{Y}$ and on $N$, is finitely generated. This follows from the universal coefficient theorem and the above lemma.]

We can now prove Theorem 5 by induction on $q$. Thus assume that $E^{2}_{pq'} = H_p (Y, H_q (F)) = 0$ for $0 < q' < q$ and all $p$. Then the edge isomorphism
$H_*(X) \Rightarrow H_*(Y)$ implies that $E^{2}_{0q} = 0$. (We are using here the fact that, by (a), $F$ is connected.) But $E^{2}_{0q} = H_0(\nu, H_q(F))$, so $H_q(F)^{\wedge} = 0$, and hence (by (b) and Prop. 4) $H_p(Y, H_q(F)) = 0$ for all $p$, as required.

3.2. Homology circles and knot complements

Let $X$ be a homology circle of finite type, and assume that $\pi = \pi_1 X \cong \mathbb{Z}$. Let $\alpha = \pi_n X$ ($n \geq 2$) be the first non-zero higher homotopy group of $X$. Then $\alpha$ is a finitely generated $\pi$-module and is perfect, i.e., $H_0(\pi, \alpha) = 0$; moreover, these are the only conditions on $\alpha$, since Kervaire [13] has shown that any finitely generated perfect $\pi$-module can arise in this way from a homology circle. Our purpose in this section is to give a similar analysis of the module $\beta = \pi_{n+1} X$. Under suitable finiteness assumptions we will show that the $I$-adic completion $\hat{\beta}$ is determined by $\alpha$ and that there are no further conditions on $\beta$ (see below for a precise statement). In particular, the ‘perfect part’ $I^m \beta$ (cf. 2.1, Prop. 3 (iv)) can be arbitrary.

We remark that the results of this section can easily be translated into results about (higher) knot complements. In fact, it is well-known that every knot complement is a homology circle. Conversely, if $X$ is a homology circle which is a finite complex of dimension $r$, and if $\pi_1 X$ is generated by the conjugates of a single element (e.g., if $\pi_1 X = \mathbb{Z}$), then $X$ is $(m-r)$-equivalent to the complement of an $(m-2)$-sphere in $S^m$ for sufficiently large $m$ ([20], p. 17, Th. 1.7).

**Theorem 6.** Assume that $H_{n+2}(\alpha, n)$ is a finitely generated $\pi$-module. Then $\beta = \pi_{n+1} X$ is a finitely generated $\pi$-module and $\hat{\beta} = H_{n+2}(\alpha, n)^{\wedge}$. Moreover, if $\phi : \beta \to \beta'$ is a $\pi$-module homomorphism with $\beta'$ finitely generated and $\hat{\phi} : \hat{\beta} \to \hat{\beta}'$ an isomorphism, then one can attach finitely many cells to $X$ to obtain a homology circle $X'$ such that $f_* : \pi_i X \to \pi_i X'$ is an isomorphism for $i \leq n$ and is equivalent to $\phi$ for $i = n + 1$, where $f : X \to X'$ is the inclusion.

**Remark.** If $n \geq 3$ then the hypothesis on $\alpha$ holds automatically and the conclusion concerning $\beta$ simply says that $\beta$ is perfect. In fact, if $n \geq 3$ then $H_{n+2}(\alpha, n) \cong \alpha/2\alpha$, which is finitely generated and perfect. [More generally, one can show that $H_{n+k}(\alpha, n)$ is finitely generated and perfect for $n > k$.]

**Proof of Theorem 6.** Let $\tilde{X}$ be the universal cover of $X$ and let $p : \tilde{X} \to K(\alpha, n)$ be the canonical map of $\tilde{X}$ to the first non-trivial space in its Postnikov decomposition. From the Serre spectral sequence of $p$ we obtain an exact sequence of $\pi$-modules,

$$H_{n+2}(\tilde{X}) \rightarrow H_{n+2}(\alpha, n) \rightarrow \beta \rightarrow H_{n+1}(\tilde{X}).$$
Since $X$ is a complex of finite type, $H_{n+1}(\hat{H})$ is a finitely generated $\pi$-module; the finite generation of $\beta$ therefore follows from that of $H_{n+2}(\alpha, n)$. Furthermore, it is easy to see that $H_q(\hat{X})$ is perfect for $q > 0$, so $H_{n+1}(\hat{X})' = 0 = H_{n+2}(\hat{X})'$. We therefore obtain an isomorphism $H_{n+2}(\alpha, n) \approx_{\beta} \hat{\beta}$ by applying the completion functor to the above exact sequence (2.1, Prop 3(i)). Finally, let $\phi : \beta \rightarrow \beta'$ be as in the statement of the theorem; then $\phi$ is an $HZ$-map of $\pi$-modules (2.2, Cor. 1 of Th. 3), so Lemma 6.2 of [2] implies that we can attach cells to $X$ to obtain a space $X'$ with the desired properties. It is clear from the proof of that lemma that only finitely many cells are required.

**Appendix. The homology (mod $\mathcal{C}$) of a regular covering space**

In this appendix we will prove a result about covering spaces (Prop. 5 below), the first corollary of which was needed in Section 3.1, and we will show it can be used to extend to nilpotent spaces Serre's mod $\mathcal{C}$ Hurewicz theorem for simply connected spaces (see Prop. 6 below).

Let $R$ be a commutative ring and let $\mathcal{C}$ be a Serre’s class of $R$-modules, i.e., $\mathcal{C}$ contains 0 and is closed under submodules, quotient modules, and extensions. Assume that $\mathcal{C}$ has the following property:

$\dagger$ If $M, N \in \mathcal{C}$, then $\text{Tor}^p(M, N) \in \mathcal{C}$ for all $p \geq 0$.

**Proposition 5.** Let $\hat{X} \rightarrow X$ be a regular covering map of path-connected spaces, let $G$ be the group of covering transformations, and let $n$ be a positive integer. Assume that $H_p(G, R) \in \mathcal{C}$ for each $p > 0$ and that $H_i(\hat{X}, R)$ is a nilpotent $G$-module for $i < n$. Then the following conditions are equivalent:

(i) $H_i(\hat{X}, R) \in \mathcal{C}$ for $1 \leq i < n$.

(ii) $H_i(X, R) \in \mathcal{C}$ for $1 \leq i < n$.

Furthermore, (i) and (ii) imply:

(iii) $p$ induces a $\mathcal{C}$-isomorphism

$$H_n(\hat{X}, R) \approx_{\mathcal{C}} H_n(X, R).$$

(Note: For any $R[G]$-module $M$, we denote by $M_G$ the $R$-module $H_0(G, M) = M/I M$, where $I$ is the augmentation ideal of $R[G]$.)

**Corollary 1.** Let $X$ be a nilpotent space and let $\hat{X}$ be its universal cover. If $H_i(X)$ is finitely generated for each $i$ then so is $H_i(\hat{X})$.

In fact, it is not hard to show that $H_i(\hat{X})$ is a nilpotent $G$-module each $i$, where $G = \pi_1 X$. [Prove inductively that $G$ acts nilpotently on the homology of
the Postnikov approximations to $\tilde{X}$. Alternatively, apply Lemma 5.4 of [3], Chap. II, §5, to the fibration $\tilde{X} \to X \to K(G, 1)$. Moreover, since $H_i(G) = H_i(X)$ is finitely generated, Lemma 1(a) below shows that $\Gamma G / \Gamma_{i+1} G$ is finitely generated for each $i$, and it follows easily that $H_p(G)$ is finitely generated for each $p$. We can therefore apply the proposition to the covering map $\tilde{X} \to X$, with $R = \mathbb{Z}$ and $c$ equal to the class of finitely generated abelian groups, and the corollary follows at once.

**Corollary 2.** Let $f : \tilde{X} \to X$ be a regular covering map of degree a power of a prime $p$. If $H_i(X; \mathbb{Z}/p \mathbb{Z})$ is finite for each $i$, then so is $H_i(\tilde{X}; \mathbb{Z}/p \mathbb{Z})$.

In fact, one knows that if $G$ is a finite $p$-group and $k$ is a field of characteristic $p$, then every $k[G]$-module is nilpotent (cf. [19], Chap. IX, §1, Cor. of Th. 2). The corollary therefore follows from the proposition, applied with $R = \mathbb{Z}/p \mathbb{Z}$ and $c$ equal to the class of finite $R$-modules.

The proof of Proposition 5 requires two lemmas.

**Lemma 1.** Let $G$ be a group such that $H_i(G, R) \in c$.

(a) Letting $\{\Gamma_i G\}_{i=1}^\infty$ be the lower central series of $G$, the $R$-modules $R \otimes_{\mathbb{Z}} (\Gamma_i G / \Gamma_{i+1} G)$ are in $c$ for all $i \geq 1$.

(b) If $M$ is an $R[G]$-module such that $M_0 \in c$, then $I^i M / I^{i+1} M \in c$ for all $i \geq 0$, where $I$ is the augmentation ideal of $R[G]$. In particular, if $M$ is nilpotent, then $M \in c$.

To prove (a), recall that the commutator map $G \times \Gamma_i G \to \Gamma_{i+1} G$ induces by passage to the quotient a $\mathbb{Z}$-bilinear map $G_{ab} \times \Gamma_i G / \Gamma_{i+1} G \to \Gamma_{i+1} G / \Gamma_{i+2} G$, where $G_{ab}$ is the abelianization $\Gamma_1 G / \Gamma_2 G$ of $G$, cf. [12], p. 329, for example. This yields by extension of scalars an $R$-module homomorphism

$$(R \otimes_{\mathbb{Z}} G_{ab}) \otimes_R (R \otimes_{\mathbb{Z}} (\Gamma_i G / \Gamma_{i+1} G)) \to R \otimes_{\mathbb{Z}} (\Gamma_{i+1} G / \Gamma_{i+2} G),$$

which is clearly surjective by the definition of $\Gamma_{i+1} G$. Since $R \otimes_{\mathbb{Z}} G_{ab} = H_i(G, R) \in c$, the result follows by induction on $i$ from the fact that $c$ is closed under tensor products and quotients.

Similarly, to prove (b), note that the multiplication map $I^i M \to I^{i+1} M$ induces an epimorphism
\[(I/I^2) \otimes_R (I^iM/I^{i+1}M) \rightarrow I^{i+1}M/I^{i+2}M.\]

Since \(I/I^2 \simeq H_\cdot(G, R)\) (cf. [4], p. 184, formula (4)), (b) follows at once by induction on \(i\).

**Lemma 2.** Let \(G\) be a group such that \(H_p(G, R) \in \mathcal{C}\) for all \(p > 0\). If \(M\) is a nilpotent \(R[G]\)-module whose underlying \(R\)-module is in \(\mathcal{C}\), then \(H_p(G, M) \in \mathcal{C}\) for all \(p \geq 0\).

Since \(G\) acts trivially on the quotients \(I^iM/I^{i+1}M\), which are in \(\mathcal{C}\), it suffices to consider the case where \(G\) acts trivially on \(M\). In this case the result follows from (*) and the universal coefficient spectral sequence

\[E^2_{pq} = \text{Tor}_p^\mathcal{C}(H_q(G, R), M) \Rightarrow H_{p+q}(G, M)\]

([11], Chap. I, Th. 5.5.1).

**Proof of Proposition 5.** We will use the spectral sequence

\[E^2_{pq} = H_p(G, H_q(X, R)) \Rightarrow H_{p+q}(X, R).\]

Note that the hypotheses imply that \(E^2_{pq} \in \mathcal{C}\) for \(p > 0\). Assuming now that (i) holds, Lemma 2 implies that \(E^2_{pq} \in \mathcal{C}\) for \(1 \leq q < n\) and all \(p \geq 0\), whence, by a standard spectral sequence argument, (ii) and (iii) hold. It remains to prove that (ii) implies (i). Assuming that (ii) holds, and assuming inductively that \(H_i(X, R) \in \mathcal{C}\) for \(1 \leq j < i\) (where \(i\) is fixed, \(i \leq j < n\)), it follows from what we have just proved that there is a \(\mathcal{C}\)-isomorphism

\[H_i(X, R) \simeq H_i(X, R),\]

hence \(H_i(X, R) \in \mathcal{C}\). But \(H_i(X, R)\) is a nilpotent \(R[G]\)-module, so Lemma 1 (b) implies that \(H_i(X, R) \in \mathcal{C}\), as required.

We now specialize to the case \(R = \mathbb{Z}\), and we assume that \(\mathcal{C}\) satisfies, in addition to (*), the following property:

\[\text{(**) If } G \in \mathcal{C} \text{ then } H_p(G) \in \mathcal{C} \text{ for all } p > 0.\]

If \(G\) is a nilpotent group such that \(\Gamma_iG/\Gamma_{i+1}G \in \mathcal{C}\) for each \(i \geq 1\), then we will say, by abuse of language, that \(G \in \mathcal{C}\). It is easy to see that the conclusion of (***) continues to hold for such a \(G\).
If $X$ is a path-connected space, then $\pi_1(X, x_0)$ operates on $\pi_n(X, x_0)$ for $n \geq 2$ and we set

$$\pi_n^*X = \pi(X, x_0)_{\pi_1(X, x_0)}.$$  

(Here $x_0$ is an arbitrary basepoint, but the right-hand side is independent of $x_0$, up to canonical isomorphism.)

**Proposition 6.** Let $X$ be a nilpotent space. For any integer $n \geq 2$, the following conditions are equivalent:

(i) $\pi_iX \in \mathcal{C}$ for $1 \leq i < n$.

(ii) $H_iX \in \mathcal{C}$ for $1 \leq i < n$.

Furthermore, (i) and (ii) imply:

(iii) The Hurewicz map $\pi_n^*X \to H_nX$ is a $\mathcal{C}$-isomorphism.

Assume first that (ii) holds. Then $(\pi_1X)_{ab} \in \mathcal{C}$, so $\pi_1X \in \mathcal{C}$ by Lemma 1 (a). Letting $p : \tilde{X} \to X$ be the universal cover of $X$, it follows that the hypotheses of Proposition 5 are satisfied (cf. proof of Cor. 1 above). We conclude that $H_i\tilde{X} \in \mathcal{C}$ for $1 \leq i < n$ and that $p$ induces a $\mathcal{C}$-isomorphism

$$(H_n\tilde{X})_G \xrightarrow{\sim} H_nX,$$

where $G = \pi_1X$. The mod $\mathcal{C}$ Hurewicz theorem for simply connected spaces [18] now implies that $\pi_i\tilde{X} \in \mathcal{C}$ for $2 \leq i < n$ and that the Hurewicz map is a $\mathcal{C}$-isomorphism

$$\pi_n\tilde{X} \xrightarrow{\sim} H_n\tilde{X}.$$  

Since $\pi_i\tilde{X} \xrightarrow{\sim} \pi_iX$ for $i \geq 2$, (i) follows at once and (iii) follows from (1) and (2) together with the easily verified fact that a $\mathcal{C}$-isomorphism $M \to N$ of nilpotent $G$-modules induces a $\mathcal{C}$-isomorphism $M_G \to N_G$. [This can be deduced from Lemma 2 above.] Thus (ii) implies (i) and (iii). The implication $(i) \Rightarrow (ii)$ is proved similarly.

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