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Euler Characteristics of Groups: The p-Fractional Part

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Introduction

If Γ is a group satisfying suitable finiteness conditions, then one associates to Γ a rational number $\chi(\Gamma)$, called the *Euler characteristic* of Γ, whose failure to be an integer is closely related to the presence of torsion in Γ (cf. [1]). For example, the "fractional part" of $\chi(\Gamma)$ can be computed from the action of Γ by conjugation on the partially ordered set S of non-trivial finite subgroups of Γ. More precisely, if we write $a \sim b$ whenever a and b are rational numbers whose difference is integral, then we have ([1], § 6), under suitable finiteness conditions,

$$\chi(\Gamma) \sim \chi_\Gamma(S). \tag{$*$}$$

(Here $\chi_\Gamma(S)$ is an equivariant Euler characteristic – see § 1 below.)

The purpose of this paper is to prove the following "local" analogue of ($*$): Let p be a prime number and let S_p be the set of non-trivial finite p-subgroups of Γ. If we write $a \overset{p}{\sim} b$ whenever a and b are rational numbers whose difference is p-integral (i.e., has denominator prime to p), then we have, under suitable finiteness conditions,

$$\chi(\Gamma) \overset{p}{\sim} \chi_\Gamma(S_p). \tag{$**$}$$

This confirms a conjecture of Serre's [private communication] that the "p-fractional part" of $\chi(\Gamma)$ should be computable in terms of the p-subgroups of Γ and their normalizers, and it improves one of the results of [1] (§ 6, Cor. 3 of Thm. 5).

In § 1 we summarize some facts about Euler characteristics and equivariant Euler characteristics which are needed later. In § 2 we prove a theorem about the fractional part of $\chi(\Gamma)$. This is then used in § 3 to prove ($**$). Finally, in § 4 we mention an application to number theory.

1. Preliminaries

Recall from [1], § 3, that a group is said to have *finite homological type* if it has subgroups of finite index which have finite cohomological dimension and if every such subgroup has finitely generated integral homology. The *Euler characteristic* $\chi(\Gamma)$ is defined for groups Γ of finite homological type and is characterized by the following two properties ([1], § 4):

(i) If Γ' is a subgroup of Γ of finite index, then $\chi(\Gamma') = (\Gamma : \Gamma') \cdot \chi(\Gamma)$.

(ii) If Γ is torsion-free then $\chi(\Gamma) = \sum (-1)^i \dim_{\mathbf{Q}} H_i(\Gamma, \mathbf{Q})$.

Suppose now that a group Γ operates on a semi-simplicial complex K and assume (a) that K has only finitely many non-degenerate simplices modulo the

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action of Γ and (*b*) that the isotropy group Γ_σ of every simplex σ has finite homological type. Then the *equivariant Euler characteristic* $\chi_\Gamma(K)$ is defined by

$$\chi_\Gamma(K) = \sum_{\sigma \in \Sigma} (-1)^{\dim \sigma} \chi(\Gamma_\sigma),$$

where Σ is a set of representatives for the non-degenerate simplices of K modulo Γ. In case K is the complex $K(S)$ associated to a partially ordered set S on which Γ operates, then we write $\chi_\Gamma(S)$ instead of $\chi_\Gamma(K(S))$. [Here $K(S)$ denotes the semi-simplicial complex whose n-simplices are the increasing sequences $s_0 \leqq \cdots \leqq s_n$ of elements of S, with the obvious face and degeneracy operators.]

Proposition 1. *Let K be a semi-simplicial Γ-complex satisfying* (*a*) *and* (*b*) *above.*

(i) *If Γ' is a subgroup of Γ of finite index, then*

$$\chi_{\Gamma'}(K) = (\Gamma : \Gamma') \cdot \chi_\Gamma(K).$$

(ii) *If Γ is torsion-free and C is a chain complex of projective $\mathbf{Z}[\Gamma]$-modules which is weakly equivalent to the chain complex $C(K)$ (regarded as a complex of $\mathbf{Z}[\Gamma]$-modules), then the complex $C_\Gamma = \mathbf{Z} \otimes_{\mathbf{Z}[\Gamma]} C$ of abelian groups has finitely generated homology and*

$$\chi_\Gamma(K) = \chi(C_\Gamma).$$

(iii) *If Γ is of type* (VFP) *and K has finitely generated rational homology, then*

$$\chi_\Gamma(K) = \chi(\Gamma) \cdot \chi(K).$$

(See [1], § 3, for the definition of "type (VFP)".)

A proof of (i) is contained in [4], no. 1.8, proof of Prop. 14(b). For (ii), see [1], § 5, Prop. 4, where the proof is given for the case where $K = K(S)$; the proof in the general case is identical. To prove (iii) we may assume, by passing to a subgroup of finite index, that Γ is torsion-free. Letting C be as in (ii), one has a spectral sequence

$$E_2^{pq} = H^p(\Gamma, H^q(K, \mathbf{Q})) \Rightarrow H^{p+q}(C_\Gamma, \mathbf{Q}).$$

(This can be obtained, for example, from the double complex

$$\mathrm{Hom}_\Gamma(P_\cdot, \mathrm{Hom}_{\mathbf{Z}}(C_\cdot, \mathbf{Q})),$$

where $P_\cdot$ is a projective resolution of $\mathbf{Z}$ over $\mathbf{Z}[\Gamma]$.) Since

$$\sum_p (-1)^p \dim_{\mathbf{Q}} H^p(\Gamma, H^q(K, \mathbf{Q})) = \chi(\Gamma) \cdot \dim_{\mathbf{Q}} H^q(K, \mathbf{Q})$$

([1], § 4, Cor. 1 of Thm. 4), (iii) follows at once.

2. The Fractional Part of $\chi(\Gamma)$

Let Γ be a group of finite homological type such that $\chi_\Gamma(S)$ is defined, where S, as in the introduction, is the set of non-trivial finite subgroups of Γ. The following result is implicit in [1], § 6, proof of Thm. 5:

Lemma. *Let L be a finite dimensional semi-simplicial Γ-complex such that* (*a*) *each isotropy group Γ_σ ($\sigma \in L$) is finite and* (*b*) *each fixed-point complex L^H ($H \in S$)*

is acyclic. If Γ' is a torsion-free normal subgroup of Γ of finite index such that L/Γ' has finitely generated homology, then

$$\chi_\Gamma(S) \sim \frac{\chi(L/\Gamma')}{(\Gamma:\Gamma')}.$$

Proof. Let $M = \bigcup_{H\in S} L^H$. Then $C(M)$ is weakly equivalent to $C(S)$ ([1], Appendix B). Since $C(M)$ is a complex of free $\mathbf{Z}[\Gamma']$-modules, parts (i) and (ii) of Proposition 1 imply that M/Γ' has finitely generated homology and that

$$\chi_\Gamma(S) = \frac{\chi_{\Gamma'}(S)}{(\Gamma:\Gamma')} = \frac{\chi(M/\Gamma')}{(\Gamma:\Gamma')}.$$

Thus

$$\frac{\chi(L/\Gamma')}{(\Gamma:\Gamma')} - \chi_\Gamma(S) = \frac{\chi(L/\Gamma', M/\Gamma')}{(\Gamma:\Gamma')},$$

which is indeed an integer; for Γ/Γ' acts freely on $L/\Gamma' - M/\Gamma'$, so

$$\chi(L/\Gamma', M/\Gamma') = (\Gamma:\Gamma') \cdot \chi(L/\Gamma, M/\Gamma)$$

by [1], § 1, Thm. 1, applied to the normalized chain complex $C(L/\Gamma', M/\Gamma')$.

Theorem 1. *Let K be a semi-simplicial Γ-complex such that $\chi_\Gamma(K)$ is defined, and assume that K^H is acyclic for each $H\in S$. Then $\chi(\Gamma) \sim \chi_\Gamma(K)$.*

Proof. Let Z be a finite dimensional acyclic Γ-complex satisfying conditions (a) and (b) of the lemma (cf. [1], § 6, Lemma), and let $L = K \times Z$. Then Γ acts on L by the diagonal action, and L also satisfies (a) and (b). Let Γ' be a torsion-free normal subgroup of Γ of finite index. Since Z is acyclic, the projection $L \to K$ induces a weak equivalence $C(L) \to C(K)$, so we may use Proposition 1 as in the proof of the above lemma to deduce that L/Γ' has finitely generated homology and that $\chi_\Gamma(K) = \chi(L/\Gamma')/(\Gamma:\Gamma')$. In view of the lemma, it follows that $\chi_\Gamma(K) \sim \chi_\Gamma(S)$. The theorem now follows from the known fact that $\chi(\Gamma) \sim \chi_\Gamma(S)$ ([1], § 6, Cor. 2 of Thm. 5; alternatively, apply the above lemma to the complex Z).

3. The p-Fractional Part of $\chi(\Gamma)$

Let Γ be a group of finite homological type, and assume that Γ has only finitely many conjugacy classes of finite p-subgroups and that the normalizer of any such subgroup has finite homological type. These hypotheses imply (by [1], § 5, Lemma) that $\chi_\Gamma(S_p)$ is defined, where S_p is, as in the introduction, the set of non-trivial finite p-subgroups of Γ.

Theorem 2. *Under the above hypotheses, $\chi(\Gamma) \overset{p}{\sim} \chi_\Gamma(S_p)$.*

The proof will use the following lemma:

Lemma. *Let T be an ordered set which contains an element t_0 such that for any $t\in T$ the least upper bound $t_0 \vee t$ exists. Then $K(T)$ is contractible, hence, in particular, acyclic.*

Proof. Let $T' = \{t\in T\colon t \geqq t_0\}$ and let $r\colon T \to T'$ be defined by $r(t) = t_0 \vee t$. Then r is an order-preserving retraction of T onto T' and induces a retraction

$K(r)\colon K(T) \to K(T')$. Moreover, since $r(t) \geqq t$ for all t, the composite

$$K(T) \xrightarrow{K(r)} K(T') \hookrightarrow K(T)$$

is homotopic to the identity (cf. [3], § 1, Prop. 2, for example), so $K(T')$ is a deformation retract of $K(T)$. The lemma now follows from the fact that, since T' has a smallest element, $K(T')$ is contractible ([3], § 1, Cor. 2 of Prop. 2).

Proof of Theorem 2. Let Γ' be a torsion-free normal subgroup of Γ of finite index and let Γ_p be the inverse image in Γ of a p-Sylow subgroup of Γ/Γ'. Then $\chi(\Gamma) = \chi(\Gamma_p)/(\Gamma:\Gamma_p)$ and $\chi_\Gamma(S_p) = \chi_{\Gamma_p}(S_p)/(\Gamma:\Gamma_p)$. If therefore suffices, $(\Gamma:\Gamma_p)$ being prime to p, to show that $\chi(\Gamma_p) \sim \chi_{\Gamma_p}(S_p)$, which we will do by means of Theorem 1. (Note that every finite subgroup of Γ_p is a p-group, so Γ_p satisfies the hypotheses of § 2.) Thus we must show that $K(S_p)^H$ is acyclic for every non-trivial finite subgroup H of Γ_p. Now $K(S_p)^H = K(T)$, where T is the set of nontrivial finite p-subgroups of Γ which are normalized by H; since H and P generate a finite p-group for any $P \in T$, the above lemma shows that $K(T)$ is indeed acyclic.

Corollary 1. *Suppose every non-trivial finite p-subgroup of Γ has a unique subgroup of order p. Then*

$$\chi(\Gamma) \overset{p}{\sim} \sum_{P \in \Phi} \chi(N(P)) = \frac{1}{p-1} \sum_{\alpha \in \Psi} \chi(C(\alpha)),$$

where Φ is a set of representatives for the conjugacy classes of subgroups of Γ of order p, Ψ is a set of representatives for the conjugacy classes of elements of Γ of order p, $N(P)$ is the normalizer of P in Γ, and $C(\alpha)$ is the centralizer of α in Γ.

In fact, under the given hypothesis it is easy to see that

$$\chi_\Gamma(S_p) = \sum_{P \in \Phi} \chi(N(P)) = \frac{1}{p-1} \sum_{\alpha \in \Psi} \chi(C(\alpha)),$$

cf. [1], § 7, pp. 247–248.

Remark. The hypothesis of Corollary 1 holds if and only if every finite p-subgroup of Γ is cyclic or generalized quaternion ([2], §§ 104 and 105).

Corollary 2. *If Γ is of type* (VFP) *and $K(S_p)$ has finitely generated rational homology, then $\chi(S_p) \equiv 1$ modulo the highest power of p dividing the denominator of $\chi(\Gamma)$. In particular, if Γ is finite, then $\chi(S_p) \equiv 1$ modulo the highest power of p dividing the order of Γ.*

In fact, under the given hypotheses one has $\chi_\Gamma(S_p) = \chi(\Gamma) \cdot \chi(S_p)$ (Proposition 1 (iii)), so Theorem 2 yields $\chi(\Gamma) \overset{p}{\sim} \chi(\Gamma) \cdot \chi(S_p)$, from which the first assertion follows at once. If Γ is finite then $\chi(\Gamma) = 1/|\Gamma|$, whence the second assertion.

4. Application to Number Theory

Using Corollary 1 of Theorem 2, one can improve the results of [1], § 9.4, and settle a question raised there (*loc. cit.*, Remark 1). We indicate briefly the improvement:

Proposition 2. *With the hypotheses and notation of* [1], § 9.4, *one has:*

(i) *If some prime of* E *lying over* p *splits in* F, *then* $\prod_{i=1}^{n} \zeta_k(1-2i)$ *is* p*-integral.*

(ii) *If no prime of* E *lying over* p *splits in* F, *then*

$$\prod_{i=1}^{n} \zeta_{k,S}(1-2i) \overset{p}{\sim} 2^{a-1} h_{\bar{S}}/n w.$$

This is proved by applying Corollary 1 of Theorem 2 to the group Γ used in [1], § 9.4, proof of Prop. 10. (Note that every p-subgroup of Γ is cyclic by [1], § 9.2, Prop. 7, so Corollary 1 is applicable.) All the necessary computations are done in [1].

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